EQUIVALENT NORMS WITH AN EXTREMELY NONLINEABLE SET OF NORM ATTAINING FUNCTIONALS

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Abstract. We present a construction that enables one to find Banach spaces X whose sets $\mathrm{NA}(X)$ of norm attaining functionals do not contain two-dimensional subspaces and such that, consequently, X does not contain proximinal subspaces of finite codimension greater than one, extending the results recently provided by Read [29] and Rmoutil [30]. Roughly speaking, we construct an equivalent renorming with the requested properties for every Banach space X where the set $\mathrm{NA}(X)$ for the original norm is not "too large". The construction can be applied to every Banach space containing c_0 and having a countable system of norming functionals, in particular, to separable Banach spaces containing c_0. We also provide some geometric properties of the norms we have constructed.

1. Introduction

A subset Y of a (real) Banach space X is said to be proximinal if for every $x \in X$ there is a $y \in Y$ such that $\|x-y\| = \mathrm{dist}(x, Y)$. The classical Bishop-Phelps theorem implies that every infinite-dimensional Banach space contains a one-codimensional proximinal subspace. More than 40 years ago, Ivan Singer [33, Problem 2.1] asked whether every infinite-dimensional Banach space contains proximinal subspaces of codimension 2. Recently Charles J. Read [29] answered this question in the negative. The corresponding space $\mathscr{R}$ is c_0 equipped with a special equivalent norm $|||\cdot|||$ ingeniously constructed by Read.

In [30, Theorem 4.2], Martin Rmoutil demonstrates that the same space $\mathscr{R}$ gives the negative solution to another (at that time open) problem by Gilles Godefroy [19, Problem III]: is it true that for every infinite-dimensional Banach space the set of those functionals in the dual space which attain their norm contains a two-dimensional linear subspace? Recall that a subset S of a vector space is called *lineable* if $S \cup \{0\}$ contains an infinite-dimensional linear subspace; so by Rmoutil's work, the set of norm attaining functionals on $\mathscr{R}$ is extremely nonlineable.

We note that there is a general statement saying that if X contains proximinal subspaces of finite codimension at least two, then the set of norm attaining functionals contains a two-dimensional linear subspace (see [19, Proposition III.4]).

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Motivated by these facts, let us say that an equivalent norm p on a Banach space X is a *Read norm* if the set of norm attaining functionals for this norm does not contain two-dimensional linear subspaces, so the space X endowed with the norm p does not contain proximinal subspaces of finite codimension greater than one. In this paper we present a clear geometric idea which enables us to simplify substantially Read's original construction of a Read norm on c_0, and to extend the construction to some other spaces. In particular, we show that every Banach space having a countable norming system of functionals and containing a copy of c_0 admits an equivalent Read norm. We further provide some geometric properties of the constructed Read norms which extend the ones given in [22] for Read's original space $\mathscr{R}$.

To this end, we introduce the concept of modesty and weak-star modesty of subspaces (see Definition 2.3) and show that a Read norm can be constructed whenever the linear span of the set of norm attaining functionals is weak-star modest.

The outline of the paper is as follows. We finish this introduction with a subsection which collects all the notation and terminology used in the paper. We devote Section 2 to preliminaries: we provide properties of two kinds of renorming of a Banach space which will be used throughout the paper, we introduce the concept of modest and weak-star modest subspace, and we give some needed results. The main part of the paper is contained in Section 3, where we show that a Banach space admits an equivalent Read norm if the linear span of the set of norm attaining functionals for the given norm is weak-star modest in the dual space, recovering in particular the original results of Read and Rmoutil. We also show that the constructed Read norms are always strictly convex. Section 4 contains the main application of the previous result: if a Banach space X has a countable norming system of functionals and contains an isomorphic copy of c_0, then it admits an equivalent Read norm; in particular, this is so if X is separable and contains a copy of c_0. We also show that for every $0 < \varepsilon < 2$, an equivalent Read norm can be chosen in such a way that all convex combinations of slices of its unit ball have diameter greater than $2 - \varepsilon$, so its dual norm is $(2 - \varepsilon)$-rough; in the case when X is separable, it is possible to get a Read norm which is strictly convex and smooth and whose dual norm is strictly convex and rough; if moreover X^* is separable, then in addition to the above properties the bidual norm is strictly convex. Finally, we discuss in Section 5 some limitations of our construction as, for instance, that no Banach space with the Radon-Nikodým property admits an equivalent norm for which the linear span of the norm attaining functionals is weak-star modest.

1.1. **Notation and terminology.** Throughout the paper, the letters X, Y, Z will stand for real Banach spaces. For a Banach space X, X^* denotes its topological dual, B_X and S_X are, respectively, the closed unit ball and the unit sphere of X, and we write $J_X\colon X \longrightarrow X^{**}$ to represent the canonical isometric inclusion of X into its bidual. We write $\mathrm{NA}(X)$ to denote the subset of X^* of all functionals attaining their norm, that is, those functionals $f \in X^*$ such that $\|f\| = |f(x)|$ for some $x \in S_X$. If necessary, we will write $\mathrm{NA}(X, \|\cdot\|)$ to make clear that we are considering the space X endowed with the norm $\|\cdot\|$.

A Banach space X (or its norm) is said to be *strictly convex* if S_X does not contain any non-trivial segment or, equivalently, if $\|x + y\| < 2$ whenever $x, y \in B_X$, $x \neq y$. The space X is said to be *smooth* if its norm is Gâteaux differentiable at every non-zero element. A norm of a Banach space X is said to be *ρ-rough* $(0 < \rho \leqslant 2)$ if

$$\limsup_{\|h\|\to 0} \frac{\|x+h\| + \|x-h\| - 2\|x\|}{\|h\|} \geqslant \rho$$

for every $x \in X$. We refer the reader to the classical books [14] and [16] for more information and background on the geometry of Banach spaces.

Finally, we will denote by $\{e_n\}$ the canonical basis of c_0 or ℓ_1, that is, the k-th coordinate of e_n equals 0 for $n \neq k$ and equals 1 for $n = k$.

2. Preliminaries

Our first goal in this section is to present the properties of two types of equivalent renormings of a Banach space. In the first one, we add to the original norm of each element the norm of its image under the action of a fixed operator. This kind of renorming is well known in Banach space theory, see e.g. [20, Proposition III.2.11], and it was also used by Read to produce his counterexample [29].

Lemma 2.1. *Let X, Y be Banach spaces, and let $R\colon X \longrightarrow Y$ be a bounded linear operator. Define an equivalent norm on X by*

$$|||x||| = \|x\|_X + \|R(x)\|_Y \qquad (x \in X).$$

Then

- (a) $B_{(X,|||\cdot|||)^*} = B_{(X,\|\cdot\|)^*} + R^*(B_{Y^*})$;
- (b) $|||x^{**}||| = \|x^{**}\|_{X^{**}} + \|R^{**}x^{**}\|_{Y^{**}}$ *for every* $x^{**} \in X^{**}$;
- (c) *if* $x \in X$ *and* $x^* \in X^*$ *satisfy* $|||x^*||| = 1$ *and* $x^*(x) = |||x|||$, *then* $x^* = \tilde{x}^* + R^*y^*$ *where* $\tilde{x}^* \in S_{(X,\|\cdot\|)^*}$ *with* $\tilde{x}^*(x) = \|x\|_X$ *and* $y^* \in S_{Y^*}$ *with* $y^*(Rx) = \|Rx\|_Y$;
- (d) *if* $R(X)$ *is strictly convex and* R *is one-to-one, then* $(X, |||\cdot|||)$ *is strictly convex;*
- (e) *if* X *is* ρ*-rough for some* $0 < \rho \leqslant 2$, *then* $(X, |||\cdot|||)$ *is* $\rho(1 + \|R\|)^{-1}$*-rough.*

Proof. (a) Write $D = B_{(X,\|\cdot\|)^*} + R^*(B_{Y^*})$. First, it is clear that

$$\sup_{x^* \in D} x^*(x) = |||x|||$$

for every $x \in X$. This means that $B_{(X,|||\cdot|||)}$ is the polar set of D. Consequently, $B_{(X,|||\cdot|||)^*}$ is the bipolar of D. So, it remains to demonstrate that D is weak-star closed, a fact which follows from the fact that both $B_{(X,\|\cdot\|)^*}$ and $R^*(B_{Y^*})$ are weak-star compact.

(b) This is just [22, Proposition 3]. Remark that this fact can be deduced much more easily directly from (a).

(c) This is immediate from (a).

(d) This fact is widely used in the theory of equivalent renormings; it was first remarked by Victor Klee, see the proof of [15, Ch. 4, § 2, Theorem 1].

Finally, (e) follows immediately from the definition of roughness. □

In the second type of renorming, the new unit ball is the sum of the given unit ball and the image of a weakly compact unit ball by a bounded linear operator. This kind of renorming was used in [13] to study properties of the set of norm attaining functionals.

Lemma 2.2. *Let X be a Banach space, let Z be a reflexive space and let $S\colon Z \longrightarrow X$ be a bounded linear operator. Then there is an equivalent norm $|\cdot|$ on X whose unit ball is the set $B_X + S(B_Z)$, and the following assertions hold:*

(a) $|x^*| = \|x^*\|_{X^*} + \|S^*x^*\|_{Z^*}$ *for every* $x^* \in X^*$*;*
(b) $B_{(X,|\cdot|)^{**}} = B_{(X,\|\cdot\|)^{**}} + J_X(S(B_Z))$*;*
(c) *if X and Z are strictly convex, then $(X, |\cdot|)$ is strictly convex;*
(d) $\mathrm{NA}(X, |\cdot|) = \mathrm{NA}(X, \|\cdot\|)$.

Proof. First, as Z is reflexive and S is weakly continuous, the set $S(B_Z)$ is weakly compact, so $B_X + S(B_Z)$ is closed. As it is also bounded, balanced and solid, it is the unit ball of an equivalent norm $|\cdot|$ on X.

(a) is elementary and it is shown in the proof of [13, Theorem 9(4)].

(b) follows from (a) of this lemma and (a) of the previous Lemma 2.1.

(c) Consider $\widetilde{x}, \widetilde{y} \in S_{(X,|\cdot|)}$ such that $|\widetilde{x} + \widetilde{y}| = 2$. Write $\widetilde{x} = x + T(u)$, $\widetilde{y} = y + T(v)$ with $x, y \in B_X$ and $u, v \in B_Z$ and consider $f \in X^*$ with

$$|f| = 1 \quad \text{and} \quad |f(\widetilde{y} + \widetilde{z})| = 2.$$

As we have that

$$\begin{aligned} 2 = |f(\widetilde{x} + \widetilde{y})| &= |f(x + y) + [T^*f](u + v)| \\ &\leqslant \|f\|_{X^*}\|x + y\| + \|T^*f\|_Y\|u + v\| \\ &\leqslant 2(\|f\|_{X^*} + \|T^*f\|) = 2, \end{aligned}$$

it follows that $\|x + y\| = 2$ and $\|u + v\| = 2$. Since X and Z are both strictly convex, it follows that $x = y$ and $u = v$, so $\widetilde{x} = \widetilde{y}$.

(d) is proved in [13, Theorem 9(4)]: a bounded linear functional attains its supremum on $B_{(X,|\cdot|)}$ if and only if it attains its supremum on both B_X and $S(B_Z)$, but all functionals attain their maxima on the weakly compact set $S(B_Z)$. □

The second goal in this section is to introduce the concepts of modesty and weak-star modesty of subspaces of a Banach space and to present some properties which will be important in our further discussion.

Definition 2.3. A linear subspace Y of a Banach space X is said to be an *operator range* if there is an infinite-dimensional Banach space E and a bounded injective operator $T\colon E \longrightarrow X$ such that $T(E) = Y$. A linear subspace $Z \subset X$ is said to be *modest* if there is a separable dense operator range $Y \subset X$ such that $Z \cap Y = \{0\}$. If X is a dual space, a linear subspace $Z \subset X$ is said to be *weak-star modest* if there is a separable weak-star dense operator range $Y \subset X$ such that $Z \cap Y = \{0\}$.

The study of dense operator ranges in Hilbert spaces goes back to Dixmier, and many results were given by Fillmore and Williams (see [17]). The extension of this study to operator ranges in Banach spaces has attracted the attention of many mathematicians since the domain of a closed operator between Banach spaces is an operator range and every operator range is the domain of some closed linear operator. We refer to the paper [11] (and references therein) for a detailed account of the known results about operator ranges and also for references and background.

We would like to emphasize some remarks. Let X be a Banach space and let Y be a linear subspace. First, Y is an operator range if and only if there is a complete norm on Y which is stronger than the restriction of the given norm of X to Y, see [10, Proposition 2.1]; if Y is dense, Y is contained in a non-closed dense operator range if and only if it is non-barrelled, see [34, Theorem 15.2.1]; finally, the injectivity of T in the definition of operator range can be substituted by the condition $\dim Y = \infty$, because for every non-injective $T\colon E \longrightarrow X$ there is an injective $\widetilde{T}\colon E/\ker T \longrightarrow X$ with the same range.

Next, we would like to make some remarks about modest and weak-star modest subspaces. The first observation is that in the definition of modest (and weak-star modest) subspace, the space E which is the domain of T can be supposed to be separable (just consider the closed linear span of the inverse image of a dense subset of Y). Actually, any infinite-dimensional separable Banach space can be chosen to be the domain of the dense (or weak-star dense) operator range, because every separable infinite-dimensional Banach space can be densely and injectively embedded into any other separable infinite-dimensional Banach space, and we may even suppose that the operator T is nuclear, see [11, Proposition 3.1] for both results. We will often apply this remark in that the (weak-star) modesty of $Z \subset X$ can be witnessed by an operator range $Y = T(\ell_1)$. Remark also that, obviously, if a subspace is modest (or weak-star modest) then all smaller subspaces are also modest (or weak-star modest).

Here is the key example of a modest subspace.

Proposition 2.4. *The subspace* $\operatorname{span}\{e_n\} \subset \ell_1$ *consisting of all sequences with finite support is modest.*

This results immediately from the following lemma which will also be useful later on.

Lemma 2.5. *There is a dense operator range $Y \subset \ell_1$ such that every non-zero element of Y has a finite number of zero coordinates.*

Proof. Let $\mathbb{D}$ be the closed unit disc, and let $A(\mathbb{D})$ be the disc algebra consisting of all continuous functions on $\mathbb{D}$ that are analytic on the interior of $\mathbb{D}$, viewed as a real Banach space. We let $A_r(\mathbb{D}) \subset A(\mathbb{D})$ be the closed real subspace consisting of those f that take real values on the real axis, and denote $t_n = 2^{-n}$ for every $n \in \mathbb{N}$. We define $T\colon A_r(\mathbb{D}) \longrightarrow \ell_1$ by

$$Tf = \left(f(t_1), \frac{1}{2}f(t_2), \frac{1}{4}f(t_3), \dots\right) \qquad \big(f \in A_r(\mathbb{D})\big).$$

Then, the identity theorem for analytic functions implies that in $Y := T(A_r(\mathbb{D}))$ every non-zero element has a finite number of zero coordinates (if any). It remains to demonstrate the density of Y in ℓ_1. To this end it is sufficient to show that every element e_m of the canonical basis of ℓ_1 belongs to the closure of Y. Indeed, for a fixed $m \in \mathbb{N}$, consider the function $f(z) = \frac{1}{4}(4 - (z - t_m)^2)$ for every $z \in \mathbb{D}$. This $f \in A_r(\mathbb{D})$ takes the value 1 at t_m and $0 < f(t_k) < 1$ for all $k \neq m$. Denote $f_n = f^n \in A_r(\mathbb{D})$. Then, $\lim_{n\to\infty} f_n(t_m) = 1$, and $\lim_{n\to\infty} f_n(t_k) = 0$ for $k \neq m$, indeed $\lim_{n\to\infty} \sup_{k\neq m} |f_n(t_k)| = 0$, hence $\lim_{n\to\infty} Tf_n = e_m/2^{m-1}$, and e_m is in the closure of Y. $\square$

In fact, Proposition 2.4 can be generalised.

Proposition 2.6. *For every separable Banach space X every subspace with a countable Hamel basis is modest.*

Proof. For every subspace W with a countable Hamel basis, there is a dense subspace $W_1 \supset W$ with a countable Hamel basis, say $\{w_n : n \in \mathbb{N}\}$. The construction of [25, Proposition 1.f.3] provides us with sequences (v_n) in X and (v_n^*) in X^* such that $\operatorname{span}\{v_1, \ldots, v_n\} = \operatorname{span}\{w_1, \ldots, w_n\}$ and $v_n^*(v_m) = \delta_{m,n}$ for all m and n. Upon replacing v_n by $v_n/\|v_n\|$ and v_n^* by $\|v_n\|v_n^*$ we may assume that (v_n) is bounded.

Consider now the bounded linear operator $S\colon \ell_1 \longrightarrow X$ defined by $S(x) = \sum_{n=1}^{\infty} x(n)v_n$ for every $x \in \ell_1$ and let $T\colon A_r(\mathbb{D}) \longrightarrow \ell_1$ be the operator defined in Lemma 2.5. Then, $\widetilde{T} = S \circ T\colon A_r(\mathbb{D}) \longrightarrow X$ is bounded, has dense range since $T(A_r(\mathbb{D}))$ is dense in ℓ_1 and $S(\ell_1) \supset W_1$ is dense in X. Finally, $\widetilde{T}(A_r(\mathbb{D})) \cap W_1 = \{0\}$. Indeed, if $w = \sum_{k=1}^{n} \alpha_k v_k \in W_1$ has the form $\sum_{k=1}^{\infty} \frac{f(t_k)}{2^{k-1}} v_k$, then apply v_l^* to these series to see that $\alpha_l = f(t_l)/2^{l-1}$ for $l \leqslant n$ and $0 = f(t_l)/2^{l-1}$ for $l > n$. The latter implies $f = 0$ since f is analytic, therefore $w = 0$.

This shows that W_1 is modest and so is the smaller subspace W. $\square$

We would like to comment that the gist of the construction of a Markushevich basis in [25, Proposition 1.f.3] alluded to above is the Gram-Schmidt orthogonalisation. Indeed, let $J\colon X \longrightarrow H$ be an injective bounded linear operator into a Hilbert space with dense range; for example, embed X isometrically into $C[0,1]$ and further continuously into $L_2[0,1]$, and let $H \subset L_2[0,1]$ be the closure of the image of X in $L_2[0,1]$. Then perform the Gram-Schmidt procedure on the linearly independent sequence $(J(w_n))$ to obtain an orthogonal basis $(h_n) \subset J(X)$ for $H = H^*$. Finally, put $v_n = J^{-1}(h_n)$ and $v_n^* = J^*(h_n)$, i.e., $v_n^*(x) = \langle Jx, h_n\rangle_H$.

We next present a known result about operator ranges which we will use later on.

Proposition 2.7 ([10, Proposition 2.6])**.** *In every separable infinite-dimensional Banach space X there are two dense operator ranges with trivial intersection.*

The main property of operator ranges which we will need in the paper is the following one.

Proposition 2.8. *Let $Y \subset X$ be a separable operator range. Then, there is an injective norm-one linear operator $T\colon \ell_1 \longrightarrow Y$ such that the set $\left\{\frac{Te_n}{\|Te_n\|} : n \in \mathbb{N}\right\}$ is dense in S_Y.*

We need the following technical result to provide the proof of the proposition.

Lemma 2.9. *Let X be a Banach space and let $Y \subset X$ be an operator range. Then, for every sequence $\{u_n\}$ in Y, there is a sequence of positive reals $\{s_n\}$ in $(0,1]$ such that for every $x = (x_1, x_2, \ldots) \in \ell_1$ we have $\sum_n s_n x_n u_n \in Y$.*

Proof. By definition, there is a Banach space E and a bounded bijective linear operator $T\colon E \longrightarrow Y$. To complete the proof it is sufficient to take $s_n = \min\{1, \|T^{-1}u_n\|^{-1}\}$ and remark that the series $\sum_n s_n x_n T^{-1}u_n$ converges absolutely for each $x \in \ell_1$, say to $e \in E$, so $\sum_n s_n x_n u_n = T(e) \in Y$. □

Proof of Proposition 2.8. By the remarks after Definition 2.3, there is an infinite-dimensional separable Banach space E and a bounded injective linear operator $T_1\colon E \longrightarrow Y$ with dense range. Applying Proposition 2.7, we can find two dense operator ranges $E_1, E_2 \subset E$ with trivial intersection. Without loss of generality, we may assume the existence of injective $U_1, U_2\colon \ell_1 \longrightarrow E$ such that $U_i(\ell_1) = E_i$, $i = 1, 2$ (see the remarks following Definition 2.3). Fix a countable dense subset $\{w_n\}_{n\in\mathbb{N}} \subset S_{T_1(E_1)}$, then $\{w_n\}_{n\in\mathbb{N}}$ is dense in S_Y as well. Denote $u_n = \frac{T_1^{-1}(w_n)}{\|T_1^{-1}(w_n)\|} \in E_1$, select the corresponding sequence $\{s_n\}$ from Lemma 2.9 and define the requested operator $T\colon \ell_1 \longrightarrow Y$ as follows:

$$T(e_n) = s_n T_1\left(u_n + \varepsilon_n U_2 e_n\right),$$

where the $\varepsilon_n \in (0,1)$ are small enough to ensure that $\left\|T_1^{-1}(w_n)\right\| \varepsilon_n \longrightarrow 0$. Then

$$\begin{aligned}\left\|\frac{Te_n}{\|Te_n\|} - w_n\right\| &= \left\|\frac{T_1\left(u_n + \varepsilon_n U_2 e_n\right)}{\|T_1\left(u_n + \varepsilon_n U_2 e_n\right)\|} - w_n\right\| \\ &= \left\|\frac{w_n + \varepsilon_n \|T_1^{-1}(w_n)\| T_1(U_2 e_n)}{\left\|w_n + \varepsilon_n \|T_1^{-1}(w_n)\| T_1(U_2 e_n)\right\|} - w_n\right\| \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

So, $\left\{\frac{Te_n}{\|Te_n\|}\colon n \in \mathbb{N}\right\}$ is dense in S_Y. It remains to demonstrate that T is injective. Assume that for some $x = (x_1, x_2, \ldots) \in \ell_1$

$$Tx = \sum_{n\in\mathbb{N}} x_n T e_n = \sum_{n\in\mathbb{N}} T_1\left(x_n s_n u_n\right) + T_1 U_2 \left(\sum_{n\in\mathbb{N}} x_n s_n \varepsilon_n e_n\right) = 0.$$

Then,

$$\sum_{n\in\mathbb{N}} T_1\left(x_n s_n u_n\right) = -T_1 U_2 \left(\sum_{n\in\mathbb{N}} x_n s_n \varepsilon_n e_n\right),$$

and by the injectivity of T_1

$$\sum_{n\in\mathbb{N}} x_n s_n u_n = -U_2 \left(\sum_{n\in\mathbb{N}} x_n s_n \varepsilon_n e_n\right).$$

But the left hand side of the last equation belongs to E_1 and the right hand side belongs to E_2, so both of them are equal to 0. Since $\{e_n\}_{n\in\mathbb{N}}$ forms a basis of ℓ_1 and U_2 is injective, this implies that $x = 0$. Finally, the fact that $\|T\| = 1$ can be obtained just by dividing by its norm. □

Our last result in this section allows us to extend a modest subspace from a complemented subspace to the whole space, in some cases.

Proposition 2.10. *Let X be a Banach space such that $X = X_1 \oplus X_2$ for suitable closed subspaces X_1 and X_2. Writing X^* in its canonical form $X^* = X_1^* \oplus X_2^*$ we have the following.*

(a) *If X_1^* is weak-star separable and $F_2 \subset X_2^*$ is weak-star modest in X_2^*, then $X_1^* \oplus F_2$ is weak-star modest in X^*.*
(b) *If X_1^* is norm separable and $F_2 \subset X_2^*$ is modest in X_2^*, then $X_1^* \oplus F_2$ is modest in X^*.*

Proof. Let P_1, P_2 be the natural projections of X^* onto X_1^* and X_2^*, respectively. For (a), take in $S_{X_1^*}$ a countable subset $\{y_n^*\}_{n\in\mathbb{N}}$ whose linear span is weak-star dense in X_1^*; for (b), take in $S_{X_1^*}$ a countable subset $\{y_n^*\}_{n\in\mathbb{N}}$ whose linear span is norm dense in Z_1. Let $T\colon \ell_1 \longrightarrow X_2^*$ be an injective operator whose image is weak-star dense for the case (a) and norm dense for the case (b) in X_2^* and $F_2 \cap T(\ell_1) = \{0\}$. Without loss of generality we may assume that $\|T(e_n)\| \longrightarrow 0$, where e_n are the elements of the canonical basis of ℓ_1 (indeed, if not, just compose T with the operator $T_1\colon \ell_1 \longrightarrow \ell_1$ that maps e_k to e_k/k for $k = 1, 2, \dots$). Also, fix a partition of $\mathbb{N}$, $\mathbb{N} = \bigsqcup_{n\in\mathbb{N}} A_n$, into a countable family of disjoint infinite subsets. Now let us define the requested operator $\widetilde{T}\colon \ell_1 \longrightarrow X^*$ as follows:

$$\widetilde{T}(x) = \sum_{n\in\mathbb{N}} \sum_{k\in A_n} x_k (y_n^* + T(e_k)) \qquad \bigl(x = (x_n)_n\bigr),$$

i.e., $\widetilde{T}(e_k) = y_n^* + T(e_k)$ for all $k \in A_n$. Then the closure of $\widetilde{T}(\ell_1)$ contains all the functionals y_n^*, and consequently it contains also all $T(e_k)$, so $\overline{\widetilde{T}(\ell_1)} \supset \operatorname{span}\{y_n^*\colon n \in \mathbb{N}\} \oplus T(\ell_1)$ which in the case (a) is weak-star dense in X^* and norm dense in the case (b). Injectivity of $\widetilde{T}$ follows from injectivity of T. It remains to demonstrate that $\widetilde{T}(\ell_1)$ has trivial intersection with $Z_1 \oplus F_2$. Indeed, let $x_1^* + f_2 = \sum_{n\in\mathbb{N}} \sum_{k\in A_n} x_k(y_n^* + T(e_k))$ for some $x = (x_1, x_2, \dots) \in \ell_1$ with $x_1^* \in X_1^*$ and $f_2 \in F_2$. Applying P_2, we obtain $f_2 = \sum_{n\in\mathbb{N}} \sum_{k\in A_n} x_k T(e_k) = Tx$, which means that $x = 0$. □

3. The main construction

Our goal here is to present a general argument providing Read norms. We also present some geometric properties of the norms constructed in this way. We denote the dual norm to an equivalent norm p by p^*.

Theorem 3.1. *Let X be a Banach space such that $\operatorname{span}(\mathrm{NA}(X))$ is a weak-star modest subspace of X^*. Then X possesses an equivalent Read norm p. Moreover p can be chosen in such a way that, given two linearly independent functionals $x^*, z^* \in \mathrm{NA}(X, p)$ with $p^*(x^*) = p^*(z^*) = 1$, one has $x^* + z^* \notin \mathrm{NA}(X, p)$ or $x^* - z^* \notin \mathrm{NA}(X, p)$.*

Proof. Let $Y \subset X^*$ be a separable weak-star dense operator range with $\operatorname{span}(\mathrm{NA}(X)) \cap Y = \{0\}$ according to Definition 2.3. By Proposition 2.8, we may assume that $Y = T(\ell_1)$, where $T\colon \ell_1 \longrightarrow X^*$ is an injective bounded linear operator such that the set $\left\{\frac{Te_n}{\|Te_n\|}\colon n \in \mathbb{N}\right\}$ is dense in S_Y. Take a sequence $\{r_n\}$ of positive reals such that $\sum_{k\in\mathbb{N}} r_k < \infty$, denote $v_n^* = T(e_n)$, and consider the operator $R\colon X \longrightarrow \ell_1$ given by $[R(x)](n) = r_n v_n^*(x)$ for every $n \in \mathbb{N}$ and every $x \in X$. Then, we define an equivalent norm on X by

$$p(x) = \|x\| + \|Rx\|_{\ell_1} \qquad (x \in X).$$

The adjoint operator $R^*\colon \ell_\infty \longrightarrow X^*$ acts as follows: $R^*\big(\{t_n\}_{n\in\mathbb{N}}\big) = \sum_{n\in\mathbb{N}} t_n r_n v_n^*$. Consequently, according to Lemma 2.1(a), we have that

$$B_{(X,p)^*} = B_{X^*} + R^*(B_{\ell_\infty}) = B_{X^*} + \sum_{n\in\mathbb{N}} r_n[-v_n^*, v_n^*].$$

Consider two linearly independent functionals $x^*, z^* \in \mathrm{NA}(X,p)$ with $p^*(x^*) = p^*(z^*) = 1$, and let $x, z \in X$ with $p(x) = p(z) = 1$ such that $x^*(x) = z^*(z) = 1$. Due to Lemma 2.1(c), there are representations

$$x^* = x_0^* + \sum_{n\in\mathbb{N}} t_n r_n v_n^*, \quad z^* = z_0^* + \sum_{n\in\mathbb{N}} \tau_n r_n v_n^*. \tag{1}$$

with $t_k, \tau_k \in [-1,1]$ such that $x_0^*, z_0^* \in S_{X^*} \cap \mathrm{NA}(X)$, for every $n\in\mathbb{N}$ where $v_n^*(x) \neq 0$ one has $t_n = \operatorname{sign} v_n^*(x)$, and for every $n\in\mathbb{N}$ where $v_n^*(z)\neq 0$ one has $\tau_n = \operatorname{sign} v_n^*(z)$. Let $\theta = \pm 1$ be a sign such that $x \neq \theta z$. First, remark that, by weak-star density of Y, the set of restrictions of functionals from Y to the linear span of x and z is the whole $(\operatorname{span}\{x,z\})^*$. So, there is $y_0^* \in S_Y$ such that $y_0^*(x) < 0$ and $y_0^*(\theta z) > 0$. Consequently, there is a neighbourhood U_0 of y_0^* in S_Y such that for all $y^* \in U_0$, we have $y^*(x) < 0$ and $y^*(\theta z) > 0$. Then, for all those $n \in \mathbb{N}$ for which $\frac{v_n^*}{\|v_n^*\|} \in U_0$, we have that

$$t_n + \theta\tau_n = \operatorname{sign} v_n^*(x) + \theta \operatorname{sign} v_n^*(z) = 0. \tag{2}$$

We are going to demonstrate that $x^* + \theta z^* \notin \mathrm{NA}(X,p)$. Assume to the contrary that there is $e \in X$ with $p(e) = 1$ at which $x^* + \theta z^*$ attains its norm, that is $(x^* + \theta z^*)(e) = p^*(x^* + \theta z^*)$. Lemma 2.1(c) says that one can write

$$\frac{x^* + \theta z^*}{p^*(x^* + \theta z^*)} = h_0^* + \sum_{n\in\mathbb{N}} s_n r_n v_n^*, \tag{3}$$

with $s_k \in [-1,1]$, $h_0^* \in \mathrm{NA}(X)$, and for every $n\in\mathbb{N}$ where $v_n^*(e) \neq 0$, one has $s_n = \operatorname{sign} v_n^*(e)$.

Since Y is weak-star dense, it cannot be contained in a weak-star closed hyperplane. Consequently, the set $S_Y \cap \{h^* \in X^* \colon h^*(e) = 0\} = S_Y \cap \{h^* \in Y \colon h^*(e) = 0\}$ is nowhere dense in S_Y. This implies that there is a non-empty relatively open subset $U_1 \subset U_0$ of S_Y which does not intersect the hyperplane $\{h^* \in Y \colon h^*(e) = 0\}$. Denote $N_1 = \Big\{ n \in \mathbb{N}\colon \frac{v_n^*}{\|v_n^*\|} \in U_1 \Big\}$, which is non-empty by density of $\{\frac{v_n^*}{\|v_n^*\|} \colon n \in \mathbb{N}\}$ in S_Y. Then, for every $n \in N_1$ the conditions (2) and the fact that $s_n = \operatorname{sign} v_n^*(e)$ hold true at the same time.

Now, from equations (1) and (3) we get

$$\begin{aligned} 0 &= x^* + \theta z^* - p^*(x^* + \theta z^*) \frac{x^* + \theta z^*}{p^*(x^* + \theta z^*)} \\ &= (x_0^* + \theta z_0^* - p^*(x^* + \theta z^*) h_0^*) + \sum_{n\in\mathbb{N}} (t_n + \theta\tau_n - p^*(x^* + \theta z^*) s_n) r_n v_n^*. \end{aligned}$$

In other words,

$$x_0^* + \theta z_0^* - p^*(x^* + \theta z^*) h_0^* = -T\left(\sum_{n\in\mathbb{N}} (t_n + \theta\tau_n - p^*(x^* + \theta z^*) s_n) r_n e_n \right).$$

The left hand side belongs to $\operatorname{span}(\operatorname{NA}(X))$, the right hand side belongs to Y, so both of them are equal to zero. Since T is injective, and $\{e_n\}_{n\in\mathbb{N}}$ forms a basis of ℓ_1, this means that all $t_n+\theta\tau_n-p^*(x^*+\theta z^*)s_n$ are equal to zero. On the other hand, as we remarked before, for every $n\in N_1$ we have $t_n+\theta\tau_n=0$ and $s_n=\operatorname{sign} v_n^*(e)\neq 0$, so $t_n+\theta\tau_n-p^*(x^*+\theta z^*)s_n\neq 0$. This contradiction completes the proof. ☐

Observe that $\operatorname{span}(\operatorname{NA}(c_0))=\operatorname{NA}(c_0)$ consists on those elements of ℓ_1 that have finite support, so it is modest by Proposition 2.4. Therefore, Theorem 3.1 applies, giving Read's [29] and Rmoutil's [30] results.

Corollary 3.2 ([29, 30])**.** *There exists an equivalent norm p on c_0 such that $\operatorname{NA}(c_0,p)$ does not contain two-dimensional subspaces and, therefore, (c_0,p) does not contain finite-codimensional proximinal subspaces of codimension greater than 1.*

Although we extensively use Read's ideas in our construction, his original construction is not a particular case of ours. Namely, Read's norm on c_0 is defined by a very similar formula, but his choice of corresponding functionals v_n^* is quite different; in Read's choice they belong to $\operatorname{NA}(c_0)$ whereas our v_n^* are sort of "orthogonal" to this set.

We will provide further examples in the next section.

Next, we would like to present some geometric properties of the Read norms we have constructed here, extending some of the results of [22]. First, we need to expound in detail the norms constructed in Theorem 3.1.

Remark 3.3. Let X be a Banach space. If $\operatorname{span}(\operatorname{NA}(X))$ is a weak-star modest subspace of X^*, then there is a sequence $\{v_n^*\}_{n\in\mathbb{N}}$ in B_{X^*} for which $\{v_n^*/\|v_n^*\|\}_{n\in\mathbb{N}}$ is weak-star dense in S_{X^*}, such that given a sequence $\{r_n\}_{n\in\mathbb{N}}$ of positive reals with $\rho=\sum_{k\in\mathbb{N}} r_k<\infty$ and defining the bounded linear operator $R\colon X\longrightarrow \ell_1$ by

$$[R(x)](n)=r_n v_n^*(x)\qquad \big(n\in\mathbb{N},\ x\in X\big), \tag{4}$$

the norm

$$p(x)=\|x\|_X+\|R(x)\|_{\ell_1}\qquad (x\in X) \tag{5}$$

is a Read norm. If moreover $\operatorname{span}(\operatorname{NA}(X))$ is modest, we get that the sequence $\{v_n^*/\|v_n^*\|\}_{n\in\mathbb{N}}$ can be selected to be norm-dense in S_{X^*}.

Let us mention that it is clear that $\|R\|\leqslant\rho$ and that R is compact since $\|P_nR-R\|\leqslant \sum_{k>n} r_k\longrightarrow 0$, where P_n projects ℓ_1 onto $\operatorname{span}\{e_1,\dots,e_n\}$.

We are now ready to present some geometric properties of our Read norms.

Proposition 3.4. *Let X be a Banach space. If $\operatorname{span}(\operatorname{NA}(X))$ is a weak-star modest subspace of X^*, then the Read norm p defined in (5) is strictly convex. Moreover, if $\operatorname{span}(\operatorname{NA}(X))$ is actually a modest subspace of X^*, then p can be built in such a way that $(X,p)^{**}$ is strictly convex and so $(X,p)^*$ is smooth.*

Proof. For the first part, we only have to show that the operator R given in (4) is one-to-one and that $R(X)$ is strictly convex, and then apply Lemma 2.1(d). Both assertions are a consequence of the fact that the sequence $\{v_n^*/\|v_n^*\|\}_{n\in\mathbb{N}}$ is weak-star dense in S_{X^*}, the first one being immediate. For the strict convexity of $R(X)$, consider $x, y \in X$ such that $R(x) \neq \alpha R(y)$ for every $\alpha > 0$. Then, $x \neq \alpha y$ for every $\alpha > 0$, so by the Hahn-Banach theorem, there is $x^* \in S_{X^*}$ such that $x^*(x) < 0 < x^*(y)$ and by weak-star density of $\{v_n^*/\|v_n^*\|\}_{n\in\mathbb{N}}$ in S_{X^*}, we get that there is $n \in \mathbb{N}$ such that $v_n^*(x) < 0 < v_n^*(y)$, so $|v_n^*(x+y)| < |v_n^*(x)|+|v_n^*(y)|$. From here, it is immediate that $\|R(x) + R(y)\|_{\ell_1} < \|R(x)\|_{\ell_1} + \|R(y)\|_{\ell_1}$, showing the strict convexity or $R(X)$.

For the moreover part, we first use the modesty of $\operatorname{span}(\mathrm{NA}(X))$ in order to get that $\{v_n^*/\|v_n^*\|\}_{n\in\mathbb{N}}$ is norm-dense in S_{X^*}. By Lemma 2.1(b), we know that the bidual norm of p is given by

$$p(x^{**}) = \|x^{**}\|_{X^{**}} + \|R^{**}(x^{**})\|_{\ell_1^{**}} \qquad (x^{**} \in X^{**}).$$

As R is compact, $R^{**}(X^{**}) \subset J_{\ell_1}(R(X))$, so to get the strict convexity of the bidual norm we only need to show that R^{**} is one-to-one, but this is consequence of the fact that now the sequence $\{v_n^*/\|v_n^*\|\}_{n\in\mathbb{N}}$ is norm-dense in S_{X^*}, as this implies that $R^*(\ell_\infty)$ is norm dense in X^*. □

We do not know if for separable Banach spaces, the result above can be improved to get that the Read norm is actually weakly locally uniformly rotund, as it happens for the original Read norm of c_0 [22, Theorem 9].

We finish the section with the following result which appears in [22, Lemma 11]: given a Read norm on a separable Banach space, there is another equivalent Read norm which is smooth. One obtains this fact just applying the renorming sketched in Lemma 2.2.

4. Applicability of the main construction

The aim of this section is to demonstrate that Theorem 3.1 is applicable (after making an appropriate renorming) to all those Banach spaces that contain an isomorphic copy of c_0 and have a countable norming system of functionals. A *countable norming system of functionals* of a Banach space X is a bounded subset $\{x_n^* \colon n \in \mathbb{N}\}$ of X^* for which there is a constant $K \geqslant 0$ such that

$$\|x\| \leqslant K \sup_{n\in\mathbb{N}} |x_n^*(x)| \qquad (x \in X).$$

Banach spaces with a countable norming system of functionals are those for which there is a bounded subset of the dual with non-empty interior which is weak-star separable or, equivalently, those which are isomorphic to closed subspaces of ℓ_∞, see [12, p. 254] for instance.

Our next result shows that the construction of the previous section is applicable to all Banach spaces which are isomorphic to a closed subspace of ℓ_∞ and contain a copy of c_0; in particular, it is applicable to separable spaces containing a copy of c_0.

Theorem 4.1. *Let X be a Banach space containing an isomorphic copy of c_0 and possessing a countable norming system of functionals. Then X is isomorphic to a space $\widetilde{X}$ such that* $\operatorname{span}(\mathrm{NA}(\widetilde{X}))$ *is weak-star modest in $\widetilde{X}^*$. Therefore, we can apply Theorem 3.1 to get that the norm given by (5) originating from the norm of $\widetilde{X}$ is a Read norm.*

We need a preliminary technical result.

Lemma 4.2. *Let X be a Banach space containing an isomorphic copy of c_0 and possessing a countable norming system of functionals. Then X is isomorphic to a closed subspace X_1 of ℓ_∞ containing the canonical copy of c_0 inside ℓ_∞.*

Proof. As X is isomorphic to a closed subspace of ℓ_∞, we can assume that X itself is a closed subspace of ℓ_∞. Denote by Y_1 a closed subspace of X that is isomorphic to c_0. According to the Lindenstrauss-Rosenthal theorem [25, Theorem 2.f.12(i)], for arbitrary isomorphic closed subspaces Y_1, Y_2 of ℓ_∞ such that both ℓ_∞/Y_1, ℓ_∞/Y_2 are non-reflexive, every bijective isomorphism $T\colon Y_1 \longrightarrow Y_2$ extends to an automorphism $\widetilde{T}\colon \ell_\infty \longrightarrow \ell_\infty$. If we apply this result to our Y_1, to $Y_2 = c_0$, and to an arbitrary bijective isomorphism $T\colon Y_1 \longrightarrow c_0$ (which is possible by [25, Proposition 2.f.13]), the resulting $X_1 = \widetilde{T}(X)$ will be the subspace we are looking for. □

Proof of Theorem 4.1. By Lemma 4.2, we may assume without loss of generality that $c_0 \subset X \subset \ell_\infty$. Consider a non-trivial ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ and denote by u the linear functional on ℓ_∞ that assigns to each $x = (x_n)_{n\in\mathbb{N}} \in \ell_\infty$ the $\mathfrak{U}$-limit of its coordinates:

$$u(x) = \lim_{\mathfrak{U}} x_n.$$

There are two cases: (1) for some non-trivial ultrafilter $\mathfrak{U}$ our space X lies in the corresponding $\ker u$, and (2) $X \not\subset \ker u$ for any $\mathfrak{U}$. Let us demonstrate that the second case can be reduced to the first one. Indeed, in the second case denote by $R_1\colon \ell_\infty \longrightarrow \ell_\infty$ the right shift operator: $R_1((x_1, x_2, \dots)) = (0, x_1, x_2, \dots)$. Then always $R_1(X) \not\subset \ker u$ (otherwise X lies in the kernel of the limit with respect to the shifted ultrafilter). Consequently, $R_1(X) \cap \ker u$ is a one-codimensional subspace of $R_1(X) \cong X$, so $\widetilde{X} := \mathbb{R}e_1 \oplus (R_1(X) \cap \ker u)$ is isomorphic to X. Since $c_0 \subset \widetilde{X} \subset \ker u$, the reduction to the first case is completed.

So the picture that we are considering is $c_0 \subset X \subset \ker u$. Since c_0 forms an M-ideal of ℓ_∞, c_0 is also an M-ideal of X [20, Proposition I.1.17], that is, $X^* = (c_0)^\perp \oplus_1 \ell_1$. Then

$$\mathrm{NA}(X) \subset \big[(c_0)^\perp \cap \mathrm{NA}(X)\big] \oplus_1 \big[\ell_1 \cap \mathrm{NA}(X)\big] \subset (c_0)^\perp \oplus_1 \big[\ell_1 \cap \mathrm{NA}(X)\big],$$

where in the first inclusion we use the elementary fact that if $f + g$ with $\|f + g\| = \|f\| + \|g\|$ attains its norm, then both f and g attain their norms. If a non-zero element $f = (f_1, f_2, \dots) \in \ell_1$ attains its norm at some $x = (x_1, x_2, \dots) \in S_X$, then for all n where $f_n \neq 0$ we have $|x_n| = 1$. Since $\lim_{\mathfrak{U}} x_n = 0$, this means that for every element $f \in \ell_1 \cap \mathrm{NA}(X)$ the set $\{n \in \mathbb{N}\colon f_n = 0\}$ belongs to $\mathfrak{U}$. Any linear combination of elements of $\ell_1 \cap \mathrm{NA}(X)$ will have the same property. Let $Y \subset \ell_1$ be the dense operator range from Lemma 2.5. Since ℓ_1 is weak-star dense in X^*, this Y is also weak-star dense in X^*. Every non-zero element of Y has a finite number of zero coordinates, but for $f \in Y \cap \operatorname{span}(\mathrm{NA}(X))$, the number of zero coordinates is infinite by the above discussion. Consequently $Y \cap \operatorname{span}(\mathrm{NA}(X)) \subset Y \cap \operatorname{span}(\ell_1 \cap \mathrm{NA}(X)) = \{0\}$. This demonstrates that $\operatorname{span}(\mathrm{NA}(X))$ is weak-star modest in X^*. □

If X is actually separable, things may be done in an easier fashion; and in the case when X^* is separable we get a stronger result. We state the result here.

Proposition 4.3. *Let X be a separable Banach space containing c_0. Then, there is an equivalent norm q on X such that, in this norm, $(X, q) = c_0 \oplus_\infty Z$ for some Z and* span(NA((X,q))) $\subset$ NA(c_0) $\oplus Z^*$ *is weak-star modest. If moreover* X^* *is separable, then* span(NA((X,q))) *is actually modest. Therefore, we can apply Theorem 3.1 to get that the norm given by* (5) *is a Read norm.*

This is just a consequence of Sobczyk's theorem (see [4, 2.5.8]) and Proposition 2.10.

Our next aim is to give geometric properties of the Read norms that we have constructed in this section, which extends those results given in [22] for the original Read space.

The first result contains all the geometric properties of the Read norms in Theorem 4.1 and Proposition 4.3 we know about.

Proposition 4.4. *Let* X *be a Banach space containing* c_0 *and having a countable norming system of functionals. Then, for every* $0 < \varepsilon < 2$, *there is an equivalent Read norm* p_ε *on* X *satisfying the following:*

(a) (X, p_ε) *is strictly convex;*
(b) *every convex combination of slices of the unit ball of* (X, p_ε) *has diameter* $\geqslant 2-\varepsilon$, *so every relatively weakly open subset of the unit ball of* (X, p_ε) *has diameter* $\geqslant 2-\varepsilon$;
(c) *the norm of* $(X, p_\varepsilon)^*$ *is* $(2-\varepsilon)$*-rough.*

Moreover, if X^* *is separable, then*

(d) $(X, p_\varepsilon)^{**}$ *is strictly convex, so* $(X, p_\varepsilon)^*$ *is smooth.*

We need a couple of preliminary results for the proof which have their own interest. The first is surely known, but we include an elementary proof for the sake of completeness.

Lemma 4.5.

(a) *Let* X *be a closed subspace of* ℓ_∞ *which contains the canonical copy of* c_0. *Then, given* $x \in B_X$ *there are two sequences* $\{y_n\}$, $\{z_n\}$ *in* S_X *that both converge weakly to* x *and such that* $e_n^*(y_n - z_n) = 2$ *for every* $n \in \mathbb{N}$, *where* e_n^* *denotes the* n*-th coordinate functional on* X.
(b) *Let* X *be a Banach space such that* $X = c_0 \oplus_\infty Z$ *for some closed Banach space* Z. *Then there is a sequence* $\{f_n\}$ *in* S_{X^*} *such that given* $x \in B_X$ *there are two sequences* $\{y_n\}$, $\{z_n\}$ *in* S_X *which converge weakly to* x *and such that* $f_n(y_n - z_n) = 2$ *for every* $n \in \mathbb{N}$.

Proof. For the first part, just define $y_n = x + (1 - x(n))e_n$ and $z_n = x - (1 + x(n))e_n$ for every $n \in \mathbb{N}$, where e_n is the n-th element of the canonical basis of c_0. Then, $\{y_n\}$, $\{z_n\}$ are contained in S_X, both converge weakly to x, and $e_n^*(y_n - z_n) = 2$ for every $n \in \mathbb{N}$.

The second part is equally easy: consider $f_n = (e_n^*, 0) \in X^*$ for every $n \in \mathbb{N}$. Given $x = (u, z)$ with $u \in B_{c_0}$ and $z \in B_Z$, the sequences

$$\{(u + (1 - u(n))e_n, z)\} \qquad \text{and} \qquad \{(u - (1 + u(n))e_n, z)\}$$

fulfill all of our requirements. □

The next preliminary result allows to transfer properties of a given norm to the norm constructed by (5).

Lemma 4.6. *Let X be a Banach space and suppose that there is a sequence $\{f_n\}$ in S_{X^*} such that for every $x \in B_X$, there are two sequences $\{y_n\}$, $\{z_n\}$ in S_X which converge weakly to x and such that $f_n(y_n - z_n) = 2$ for every $n \in \mathbb{N}$. Let $R\colon X \longrightarrow Y$ be a compact operator from X to some Banach space Y and define an equivalent norm on X by*

$$|||x||| = \|x\|_X + \|R(x)\|_Y \qquad (x \in X).$$

Then, there is a sequence $\{g_n\}$ in the unit sphere of $(X, |||\cdot|||)^*$ such that given $x \in X$ with $|||x||| = 1$, there exist two sequences $\{\tilde{y}_n\}$, $\{\tilde{z}_n\}$ in the unit ball of $(X, |||\cdot|||)$ that both converge weakly to x and such that $\lim_n g_n(\tilde{y}_n - \tilde{z}_n) \geqslant 2(1 + \|R\|)^{-1}$.

Proof. We have that

$$1 = |||x||| = \|x\|_X + \|R(x)\|_Y \leqslant (1 + \|R\|)\|x\|_X,$$

so $\|x\|_X \geqslant (1 + \|R\|)^{-1}$. By hypothesis, we may take two sequences $\{y_n\}$, $\{z_n\}$ in X both converging weakly to x and a sequence $\{f_n\}$ in S_{X^*} such that $f_n(y_n - z_n) = 2\|x\|$, $\|y_n\| = \|z_n\| = \|x\|$ and $\|y_n - z_n\| = 2\|x\|$ for every $n \in \mathbb{N}$. As R is compact, we have that $\lim Ry_n = \lim Rz_n = Rx$, so

$$\lim \|Ry_n\| = \lim \|Rz_n\| = \|Rx\| \quad \text{and} \quad \lim \|R(y_n - z_n)\| = 0.$$

Therefore, $\lim |||y_n||| = \lim |||z_n||| = 1$ and $\lim |||y_n - z_n||| \geqslant 2\|x\|$. Also, $|||f_n|||^* \leqslant \|f_n\|^* = 1$. Finally, the sequences $\tilde{y}_n = |||y_n|||^{-1} y_n$, $\tilde{z}_n = |||z_n|||^{-1} z_n$ and $g_n = |||f_n|||^{-1} f_n$ fulfill all of our requirements. ☐

We are now ready to give the proof of Proposition 4.4.

Proof of Proposition 4.4. We start by using Lemma 4.2 and (the proof of) Theorem 4.1 to get an equivalent norm on X such that $c_0 \subset X \subset \ell_\infty$ isometrically, where c_0 is the canonical copy, and such that $\operatorname{span}(\mathrm{NA}(X))$ is weak-star modest. Next, for $0 < \varepsilon < 2$, we consider an operator R_ε defined by (4) from Remark 3.3 with $\|R_\varepsilon\| < \frac{\varepsilon}{2-\varepsilon}$, and consider the norm

$$p_\varepsilon(x) = \|x\| + \|R_\varepsilon(x)\|_{\ell_1} \qquad (x \in X),$$

which is a Read norm. By Proposition 3.4, (X, p_ε) is strictly convex, so this gives (a). To get (b), we just have to apply Lemmas 4.5 and 4.6. Indeed, let $\{g_n\}$ be the sequence in the unit sphere of $(X, p_\varepsilon)^*$ given by Lemma 4.6. Consider $C = \sum_{i=1}^N t_i S_i$, a convex combination of slices in the unit ball of (X, p_ε), and $x_0 \in C$. We write $x_0 = \sum_{i=1}^N t_i x_i$ where $x_i \in S_i$ for every i. There is no loss of generality if we assume that $p_\varepsilon(x_i) \geqslant 1 - \delta$ for every i, where δ is a positive number as small as we want. By using Lemma 4.6 again, we get that for every i there are sequences $\{y_n^i\}$ and $\{z_n^i\}$ in the unit ball of (X, p_ε) both weakly converging to x_i and such that $\lim_n g_n(y_n^i - z_n^i) \geqslant (1 - \delta)(2 - \varepsilon)$. Therefore, for large enough n, we have that $\sum_{i=1}^N t_i y_n^i$, $\sum_{i=1}^N t_i z_n^i$ are elements in C with distance, at least, $(1 - 2\delta)(2 - \varepsilon)$. As δ is arbitrary, we conclude that the diameter of C is, at least, $2 - \varepsilon$. Finally, every relatively weakly open subset of a unit ball contains a convex combination of slices (a result due to Bourgain, see [8, Lemme 5.3]), and this gives the last part of (b).

Item (c) is a consequence of (b) by using [14, Proposition I.1.11].

If X^* is separable, we may suppose that $X = c_0 \oplus_\infty Z$ for some Banach space Z and we use Proposition 4.3 to get that this norm makes $\operatorname{span}(\mathrm{NA}(X))$ modest. Now, for $0 < \varepsilon < 2$, we follow the same process as before to construct the norm p_ε. Again, Proposition 3.4 gives (a) and Lemmas 4.5 and 4.6 give (b), and [14, Proposition I.1.11] gives (c) from (b). Finally, (d) is a consequence of Proposition 3.4 since now $\operatorname{span}(\mathrm{NA}(X))$ is actually modest. □

In the separable case, we may get Read norms with better properties by using a convenient renorming from [13] which was used in [22] for the original Read norm.

Proposition 4.7. *Let X be a separable Banach space containing c_0. Then, for every $0 < \varepsilon < 2$, there is an equivalent Read norm q_ε on X such that:*

(a) *(X, q_ε) is strictly convex;*
(b) *$(X, q_\varepsilon)^*$ is strictly convex, so (X, q_ε) is smooth;*
(c) *$(X, q_\varepsilon)^*$ is $(2-\varepsilon)$-rough, equivalently, every slice of the unit ball of (X, q_ε) has diameter $\geqslant 2-\varepsilon$.*

Moreover, if X^* is separable, then

(d) *$(X, q_\varepsilon)^{**}$ is strictly convex, so $(X, q_\varepsilon)^*$ is smooth.*

Proof. We fix a dense subset $\{x_n : n \in \mathbb{N}\}$ of B_X, and for every $0 < \rho < 2$, we define the bounded linear operator $S_\rho : \ell_2 \longrightarrow X$ by $S_\rho(\{a_n\}) = \rho \sum_{n=1}^{\infty} \frac{a_n}{2^n} x_n$ for every $\{a_n\} \in \ell_2$, which satisfies that $\|S_\rho\| \leqslant \rho$. For $0 < \varepsilon < 2$, we take $0 < \varepsilon' < \varepsilon$ and $\rho > 0$ such that $(2-\varepsilon')(1+\rho)^{-1} > 2-\varepsilon$, we consider the norm $p_{\varepsilon'}$ from Proposition 4.4, and we define the equivalent norm q_ε on X to be the one for which

$$B_{(X,q_\varepsilon)} = B_{(X,p_{\varepsilon'})} + S_\rho(B_{\ell_2}).$$

First, Lemma 2.2(d) gives that $\mathrm{NA}(X, q_\varepsilon) = \mathrm{NA}(X, p_{\varepsilon'})$ and so q_ε is a Read norm. It follows from Lemma 2.2(a) that

$$q_\varepsilon(f) = p_{\varepsilon'}(f) + \|S_\rho^*(f)\|_2$$

for every $f \in X^*$. As ℓ_2 is strictly convex and T^* is one-to-one, it follows from Lemma 2.1(d) that $(X, q_\varepsilon)^*$ is strictly convex, so (X, q_ε) is smooth, giving (b). Lemma 2.2(c) gives that (X, q_ε) is strictly convex since both $(X, p_{\varepsilon'})$ and ℓ_2 are; hence (a) holds. Finally, we know from Proposition 4.4 that $(X, p_{\varepsilon'})^*$ is $(2-\varepsilon')$-rough, and then Lemma 2.1(e) gives that (X, q_ε) is $(2-\varepsilon')(1+\rho)^{-1}$-rough, which gives the first part of (c) due to the way in which we have chosen the constant ε' and ρ. Finally, the second part of (c) is equivalent to the first one by [14, Proposition I.1.11].

If moreover X^* is separable, as $B_{(X,q_\varepsilon)^{**}} = B_{(X,p_{\varepsilon'})^{**}} + J_X(S_\rho(B_{\ell_2}))$ by Lemma 2.2(b) and $(X, p_{\varepsilon'})^{**}$ is strictly convex by Proposition 4.4, the strict convexity of $(X, q_\varepsilon)^{**}$ follows from Lemma 2.2(c). □

5. Limits of our construction

The main open problem related to Read norms is the following one.

Problem 5.1. Does every non-reflexive separable Banach space admit an equivalent norm such that the set of norm attaining functionals contains no linear subspaces of dimension two?

Remark that for non-reflexive non-separable Banach spaces the answer to the above problem is negative. Indeed, every renorming E of $\ell_\infty(\Gamma)$ with uncountable Γ contains an isometric copy of $\ell_\infty(\mathbb{N})$ [27, Corollary on p. 207]. This copy is one-complemented, so $\mathrm{NA}(E) \supset \mathrm{NA}(\ell_\infty(\mathbb{N}))$, which in turn contains an infinite-dimensional linear subspace, viz., $\ell_1(\mathbb{N})$.

Taking into account that $\ell_\infty(\Gamma)$ is a $C(K)$ space, it is natural to ask the following question.

Problem 5.2. What is the description of those compacts K for which the corresponding $C(K)$ admits an equivalent norm in which the set of norm attaining functionals contains no linear subspaces of dimension two?

We do not know whether the answer to Problem 5.1 is positive, but we would like to discuss the reasons why our construction cannot provide such a positive answer.

Observe that the key in our construction is that $X^*\setminus\mathrm{span}(\mathrm{NA}(X))$ is big enough to contain a weak-star dense separable operator range. It is known that this is not possible for Banach spaces with the Radon-Nikodým property or with an almost LUR norm, as the following result of Bandyopadhyay and Godefroy shows.

Proposition 5.3 ([5, Proposition 2.23])**.** *Let X be a Banach space with the Radon-Nikodým property or with an almost LUR norm. Then* $\mathrm{span}(\mathrm{NA}(X)) = X^*$*.*

Therefore, the main open question related to our construction is the following one.

Problem 5.4. Does every Banach space with weak-star separable dual and failing the Radon-Nikodým property admit an equivalent norm for which the linear span of the set of norm attaining functionals is weak-star modest in the dual space?

We don't even know the answer for the space $L_1[0,1]$.

Let us comment that the proof of Proposition 5.3 is a consequence of the fact that $\mathrm{NA}(X)$ contains a dense G_δ subset of X^* when X has the Radon-Nikodým property (see Theorem 8 in [7], for instance) or X has an almost LUR norm ([5]), so $\mathrm{NA}(X)$ is residual in this case. Actually, this latter hypothesis is sufficient to get that $\mathrm{NA}(X) - \mathrm{NA}(X) = X^*$ from the Baire category theorem. We include the next result, which is contained in the proof of [5, Proposition 2.23], for completeness.

Proposition 5.5 ([5, included in the proof of Proposition 2.23])**.** *Let X be a Banach space. If B is a residual subset of X, then $B - B = X$ and so* $\mathrm{span}(B) = X$*. In particular, if* $\mathrm{NA}(X)$ *is residual in* X^* *then* $\mathrm{span}(\mathrm{NA}(X)) = X^*$*.*

Proof. We just have to note that for every $x \in X$, $\big(x+B\big)\cap B$ is not empty since, otherwise, the second category set $x + B$ would be contained in the first category set $X \setminus B$, which is impossible. $\square$

Let us comment that the converse result to the above one is not true: for $X = L_1[0,1]$, $\mathrm{NA}(X)$ is of the Baire first category (so it cannot be residual), but $\mathrm{span}(\mathrm{NA}(X)) = X^*$ (a description of $\mathrm{NA}(L_1[0,1])$ can be found in [3, Lemma 2.6]). Therefore, our construction is not applicable to $X = L_1[0,1]$ in its usual norm, but we do not know whether it could be the case in some renorming. Actually, it is known that every separable Banach space failing the Radon-Nikodým property can be renormed in such a way that the set of norm attaining functionals is of the first Baire category (see the proof of [9, Theorem 3.5.5]) but, as the previous example shows, this does not imply that the linear span of the set of norm attaining functionals is also of the first Baire category.

Remark also that a similar argument to the proof of Proposition 5.5 can give us the following curious result.

Proposition 5.6. *Let X be an infinite-dimensional Banach space. If B is a residual subset of X such that $tx \in B$ for every $x \in B$ and $t \in \mathbb{Q}$, then B contains an infinite sequence of linearly independent elements whose linear span over the field $\mathbb{Q}$ lies in B. In particular, if $\mathrm{NA}(X)$ is residual in X^*, then $\mathrm{NA}(X)$ contains the $\mathbb{Q}$-linear span of an $\mathbb{R}$-linearly independent infinite sequence.*

Proof. Take $0 \neq x_1 \in B$. Assume, inductively, that linearly independent elements $x_1, \ldots, x_n \in B$ have been constructed so that the set $\{x_1, \ldots, x_n\}$ is linearly independent and the $\mathbb{Q}$-linear span of the set $\{x_1, \ldots, x_n\}$ lies in B. Consider

$$E = \bigcap_{r_1,\ldots,r_n \in \mathbb{Q}} \Big(B - \sum_{i=1}^{n} r_i x_i^*\Big).$$

This is a residual subset of B, so it contains an element $x_{n+1} \in B$ which is linearly independent from the set $\{x_1, \ldots, x_n\}$. Indeed, if not then $Z := \mathrm{span}(x_1, \ldots, x_n)$ contains E and is hence residual; but Z is a nowhere dense set, being a closed and proper subspace of X, which is impossible by the Baire category theorem. According to the definition of E, the condition $x_{n+1} \in E$ implies that $x_{n+1} + \sum_{i=1}^{n} r_i x_n \in B$ for every $r_1, \ldots, r_n \in \mathbb{Q}$. Thus we get the required infinite sequence $\{x_n\}$ of linearly independent elements in B. $\square$

With the above result in mind, which can be applied to $\mathrm{NA}(X)$ for Banach spaces X with the Radon-Nikodým property or with an almost LUR norm, it would be nice to know if there is some Banach space X so that $\mathrm{NA}(X)$ is residual, but still $\mathrm{NA}(X)$ does not contain two-dimensional subspaces. Remark also that as a consequence of Proposition 5.5 in combination with Theorem 3.1, for a general Banach space X, if $\mathrm{span}(\mathrm{NA}(X))$ is weak-star modest then we again get that $\mathrm{NA}(X)$ is not residual.

Also, the following result of Fonf and Lindenstrauss [18, Theorem 4.3] is worth mentioning: for every non-reflexive Banach space X, $X^* \setminus \mathrm{NA}(X)$ is not a subset of a proper operator range, equivalently, $\mathrm{span}\big(X^* \setminus \mathrm{NA}(X)\big)$ is dense and barrelled (use [26, Theorem 1.1] for the equivalence).

On the other hand, for separable Banach spaces, if $\mathrm{span}(\mathrm{NA}(X))$ is modest (or weak-star modest), then $\mathrm{span}(\mathrm{NA}(X))$ has to be of the first Baire category, as we may prove using a theorem by Pettis.

Proposition 5.7. *Let X be a separable Banach space. If $\mathrm{span}(\mathrm{NA}(X))$ is of the second Baire category in X^*, then $\mathrm{span}(\mathrm{NA}(X)) = X^*$.*

Proof. The argument relies on notions and results from descriptive set theory that we'll recall in the course of the proof. A Polish space is a completely metrisable separable topological space, and an analytic set is a subset of a topological space which is a continuous image of some Polish space. Since X is separable, $\mathrm{NA}(X)$ is an analytic subset of X^* equipped with the weak-star topology; see [23, p. 221]. We shall argue that $\mathrm{span}(\mathrm{NA}(X))$ is analytic as well.

For $n \in \mathbb{N}$ define $f_n \colon \mathrm{NA}(X)^n \times \mathbb{R}^n \to X^*$ by

$$f_n(x_1^*, \dots, x_n^*, t_1, \dots, t_n) = \sum_{k=1}^{n} t_k x_k^*;$$

then

$$\mathrm{span}(\mathrm{NA}(X)) = \bigcup_{n\in\mathbb{N}} f_n(\mathrm{NA}(X)^n \times \mathbb{R}^n).$$

Now the class of analytic sets is closed under taking finite (even countable) products, continuous images, and countable unions; therefore $\mathrm{span}(\mathrm{NA}(X))$ is indeed analytic for the weak-star topology.

In a Hausdorff topological space, analytic sets are known to be F-Souslin [6, Theorem 6.6.8], that is, they can be represented as

$$\bigcup_{\sigma} \bigcap_{n=1}^{\infty} F_{\sigma_1,\dots,\sigma_n}$$

for closed sets $F_{\sigma_1,\dots,\sigma_n}$ where the union is taken over all sequences $\sigma = (\sigma_1, \sigma_2, \dots)$ of positive integers. Hence $\mathrm{span}(\mathrm{NA}(X))$ is F-Souslin for the weak-star topology and therefore also for the norm topology.

The next piece of information that we need concerns the Baire property. A subset of a topological space has the Baire property if it differs from an open set by a set of the first category; that is, if it can be written in the form $G \,\Delta\, M$ with G open and M of the first category where Δ denotes the symmetric difference. In a Hausdorff space, every F-Souslin set has the Baire property; see [31, Cor. 2.9.4]. Consequently, $\mathrm{span}(\mathrm{NA}(X))$ has the Baire property for the norm topology.

Finally we apply a theorem due to Pettis ([28, Theorem 1] or [24, page 92]) which, in particular, assures that a subspace of a normed linear space of the second Baire category which satisfies the Baire property is the whole space. □

We are grateful to W. Moors for indicating the above argument to us; in a preliminary version of this paper we had to make the far stronger assumption that X^* is separable.

We do not know whether separability can be dropped from Proposition 5.7.

As a consequence of Proposition 5.7, if X is separable and $\mathrm{span}(\mathrm{NA}(X))$ is weak-star modest, then $\mathrm{span}(\mathrm{NA}(X))$ has to be of the first Baire category. We do not know whether the converse is also true, but there is a partial answer.

Proposition 5.8. *Let X be a Banach space such that X^* is separable. If* $\mathrm{span}(\mathrm{NA}(X))$ *is not barrelled, then* $\mathrm{span}(\mathrm{NA}(X))$ *is modest (and so, X admits an equivalent Read norm).*

Proof. By [34, Theorem 15.2.1], it follows that $\operatorname{span}(\mathrm{NA}(X))$ is contained in a (dense) proper operator range. Now, [32, Theorem 1] shows that there is a dense operator range Y in X^* such that $Y \cap \operatorname{span}(\mathrm{NA}(X)) = \{0\}$, that is, $\operatorname{span}(\mathrm{NA}(X))$ is modest. □

We have to mention that this result does not produce new examples of spaces which admit Read norms, as the following result by Fonf shows: if $\operatorname{span}(\mathrm{NA}(X))$ is not barrelled, then X contains a copy of c_0 (see [26, Theorem 2.6] for a version of Fonf's result using this language).

Let us note that the task to find a Banach space X with X^* separable such that $\operatorname{span}(\mathrm{NA}(X))$ is weak-star modest and X does not contain c_0, requires to find a Banach space X such that $\operatorname{span}(\mathrm{NA}(X))$ is of the first Baire category and barrelled.

Finally, it would be interesting to find examples of Banach spaces X for which $\operatorname{span}(\mathrm{NA}(X))$ is modest (or weak-star modest) in their usual norm, as it happens with c_0. Another example of this kind is given in the papers [1, 2] by Acosta: let $w \in \ell_2 \setminus \ell_1$ with $0 < w_n < 1$ for all n and consider the space Z of sequences z of scalars for which

$$\|z\| := \|(1-w)z\|_\infty + \|wz\|_{\ell_1} < \infty$$

endowed with this function as norm. Then, the sequence $\{e_n\}$ of unit vectors is a 1-unconditional basic sequence of Z^* whose closed linear span $X(w)$ is an isometric predual of Z for which $\{e_n\}$ is a 1-unconditional basis whose biorthogonal basis $\{e_n^*\}$ is again the canonical unit vector basis [1, Lemma 2.1]. Then, $\operatorname{span}(\mathrm{NA}(X(w)))$ is modest in X^*. Indeed, it is shown in [2, Lemma 2.2] that if $x^* \in X(w)^*$ belongs to $\mathrm{NA}(X(w)^*)$, then $w\chi_{\operatorname{supp}(x^*)} \in \ell_1$; if we consider the bounded linear operator $T\colon \ell_1 \longrightarrow X(w)^*$ given by $T(e_n) = e_n^*$ for every $n \in \mathbb{N}$, it follows, as $w \notin \ell_1$, that $T(Y) \cap \operatorname{span}(\mathrm{NA}(X)) = \{0\}$ where Y is the operator range of ℓ_1 given in Lemma 2.5. Let us observe that $X(w)$ contains a copy of c_0 (since the basis is unconditional and shrinking and the space is not reflexive), so we already know from Section 4 that it admits an equivalent Read norm.

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