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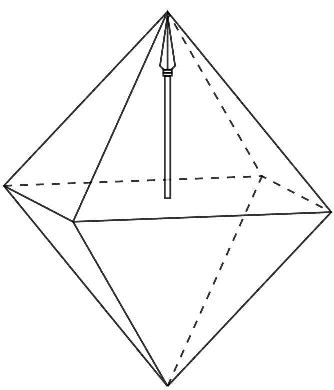
Spear operators between Banach spaces



$$J\colon \mathbb{A}(\mathbb{D}) \hookrightarrow C(\mathbb{T})$$

$$\mathscr{F} : L_1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$$

$$[\mathscr{F}(f)](\xi) = \int_{-\infty}^{+\infty} f(x)\, e^{-i\xi x}\, dx$$

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Preface

Let X and Y be Banach spaces. The main goal of this monograph is to study bounded linear operators $G\colon X \longrightarrow Y$ satisfying that for every other bounded linear operator $T\colon X \longrightarrow Y$ there is a modulus-one scalar ω such that the norm equality

$$\|G+\omega T\| = 1+\|T\| \tag{SPE}$$

holds. In this case, G is said to be a *spear operator*.

Our main motivation to develop this study is the case when $X=Y$ and G is the identity operator Id, for which the study goes back to the 1970's, when J. Duncan, C. McGregor, J. Pryce, and A. White proved that for $T\colon X \longrightarrow X$ bounded and linear, the existence of a modulus-one scalar ω such that the norm equality

$$\|\operatorname{Id}+\omega T\| = 1+\|T\| \tag{aDE}$$

holds, is equivalent to the equality between the numerical radius of T and its norm. The list of spaces for which the identity is a spear operator (they are called spaces with numerical index 1) contains all $C(K)$ spaces and $L_1(\mu)$ spaces, as well as some spaces of analytic functions and vector-valued functions, which motivated the intensive study of this class of spaces in the past decades. The equation (aDE) is named as the *alternative Daugavet equation*, as it is a variant of the *Daugavet equation*:

$$\|\operatorname{Id}+T\| = 1+\|T\|. \tag{DE}$$

This latter norm equality for operators takes its name from a 1963 result by I. Daugavet saying that every compact linear operator T on $C[0,1]$ satisfies (DE), a property which is shared by weakly compact operators on the space $C(K)$ when K is perfect, by weakly compact operators on the space $L_1(\mu)$ when μ is atomless, and by weakly compact operators on the disk algebra $A(\mathbb{D})$. The Daugavet equation has been deeply studied in several environments by many mathematicians in the last decades, and there are variants of it for polynomials and for Lipschitz operators between Banach spaces. The Daugavet equation is also related to the numerical range of operators: an operator T satisfies (DE) if and only if the norm of T coincides with the supremum of the real part of the values of the numerical range of T. Let us recall that the concepts of numerical range and numerical radius of operators played an important role in operator theory, particularly in the classification of operator algebras and in the study of the geometry of their unit balls.

An extension of the Daugavet equation to operators between different Banach spaces is another motivation for our study. An operator $G\colon X \longrightarrow Y$ is said to be a *Daugavet center* if the norm equality

$$\|G+T\| = 1 + \|T\|$$

holds for all rank-one bounded (and then for all weakly compact) linear operators $T\colon X \longrightarrow Y$.

The study of spear operators has been initiated recently by a paper by M. Ardalani from where the name is taken. For general operators G, a concept of numerical range of operators with respect to G is also introduced in this seminal work, and there is a relation between (SPE) and numerical ranges, analogous to the commented one for the case of (aDE). But, actually, such a new numerical range is deeply related to an intrinsic concept of numerical range of elements which takes its roots in the work by H. Bohnenblust and S. Karlin of the 1950's. This relation allows us to say that a spear operator is geometrically unitary in the strongest possible form. These concepts provide also a natural motivation for the study of spear operators.

Let us comment that, as we will see here, the extension to general operators produces other important examples of operators which are spear different from the identity. One of the most striking ones is the Fourier transform on the L_1 space on a locally compact Abelian group; another example is the inclusion of a unital uniform algebra into the space of bounded continuous functions on its Choquet boundary.

The property of G being a spear operator is formulated in terms of all bounded linear operators between two Banach spaces, which leads to many difficulties for its study in abstract spaces, and also in concrete ones. It would be much more convenient to have a geometric definition of this property (in terms of G). Unfortunately a description of this property in pure geometrical terms has not been discovered until now, even for the case when $G = \mathrm{Id}$. In order to manage this difficulty for this latter case, two other Banach space properties were introduced: the alternative Daugavet Property (aDP in short) and lushness. These two properties are of geometric nature, the aDP is weaker and lushness is stronger than the fact that the identity is a spear operator. On the other hand, in some classes of Banach spaces these properties are equivalent (say, in Asplund spaces, in spaces with the Radon-Nikodým property and, more generally, in the so called SCD spaces introduced in 2010). The study of these two properties has been crucial in the development of the theory in the case when $G = \mathrm{Id}$. So, now naturally appears the task of re-constructing the theory of the aDP and lushness in such a way that it could be applied to spear operators. The definition of the aDP is easy to explain: we just require the equation (SPE) to be satisfied by rank-one operators, and so it can be written in terms of G and the geometry of the domain and range spaces. On the other hand, the definition of a lush operator is more tricky: a norm-one operator $G\colon X \longrightarrow Y$ is lush if for every $\varepsilon > 0$, every $x_0 \in X$ with $\|x_0\| \leqslant 1$, and every $y \in Y$ with $\|y\| = 1$, there is a subset F of the unit ball of X such that the distance of x_0 to the absolutely closed convex hull of F is not bigger than ε and that $\|Gx+y\| > 2-\varepsilon$ for every x which belongs to the convex hull of F. Lush operators are spear operators and spear operators (trivially) have the aDP. We will study these three properties for general operators here. On this way, we not only transfer the known results to the new setting, but in fact do much more. Namely, we introduce a unified approach to a huge number of previously known results, substantially simplify the system of notations and, in many cases, present the general results for operators G in a more clear way than it was done earlier for the identity operator. To do so it has been crucial to study the concepts of spear set and target operator which are newly introduced in this book. Another very important concept for us is the one of SCD set. A bounded subset A of a Banach space is said to be *slicely countably determined* (*SCD* in short) if there is a sequence $\{S_n\colon n \in \mathbb{N}\}$ of slices of A such that if B intersects all the S_n's, then the closed convex hull of B contains A. Separable Radon-Nikodým bounded convex sets and separable bounded convex sets not containing ℓ_1-sequences, are the basic examples of SCD sets. One of the main utilities for us of SCD sets is that an aDP operator for which the unit ball of its domain is SCD is actually a lush operator and, in particular, a spear operator. Let us also mention that the aDP and lushness are separably determined properties, while we do not know whether the concept of spear operator is. This separable determination allows us to use the full power of the theory of SCD sets and operators, a task which is crucial in the development of the subject. In particular, it allows to show that an operator $G\colon X \longrightarrow Y$ which has the aDP is actually lush whenever X has the Radon-Nikodým property or X does not contain copies of ℓ_1.

Another motivation to study spear operators between different spaces was the potential applicability of the extended theory to the study of non-linear Lipschitz spears. Namely, the standard technique of Lipschitz-free spaces reduces equation (aDE) for a non-linear Lipschitz map $T: X \longrightarrow X$ to an analogous equation for the linearization of T, but this linearization acts from the Lipschitz-free space $\mathcal{F}(X)$ to X. Hence, in order to use this technique, we are in need of studying equation (SPE) instead of (aDE), that is, to study spear operators between two different spaces.

Outline of the book

The first chapter contains an overview of the known results for the identity, that is, about Banach spaces with numerical index 1, and it also contains the notation and terminology we will need along the book in section 1.1. The concepts and main results on numerical ranges of operators and numerical index of Banach spaces are compiled in section 1.2; moreover, we also present there the concepts of numerical ranges with respect to an operator (intrinsic and approximate spatial) and their relationship with spear operators, Daugavet centers, and the aDP. We next expose the properties which are stronger than the numerical index 1 and which are related to the extremal structure of the unit ball (section 1.3), the Daugavet property and the alternative Daugavet property (section 1.4), and lush spaces (section 1.5). The section which is most important for the rest of the book is the one devoted to slicely countably determined Banach spaces (section 1.6). Here, the concepts, the examples, the main results, and also the applications for Banach spaces with numerical index 1 are presented; the ideas presented here will be used profusely along the book. We finish the chapter with a pair of diagrams which present the relationship between the concepts presented in the chapter.

In chapter 2 we recall the concept of spear vector and introduce the new concept of spear set. These concepts are used here as "leitmotiv" to give a unified presentation of the concepts of spear operator, lush operator, alternative Daugavet property, and other notions that we will introduce here for operators. We collect some properties of spear sets and vectors, together with some (easy) examples of spear vectors.

Chapter 3 includes the main definitions of the manuscript for operators: spearness, the alternative Daugavet property, and lushness. We start presenting some preliminary results and easy examples of spear operators in section 3.1. Next, in section 3.2 we study bounded linear operators $G\colon X \longrightarrow Y$ with the alternative Daugavet Property (aDP). We give several characterizations of them (some of them in terms of spear sets) and prove that this is a separably determined property. Section 3.3 starts with the definition of target operator for G. This is a property for operators $T\colon X \longrightarrow Y$ ensuring that there is a modulus-one scalar ω such that $\|G+\omega T\| = 1+\|T\|$. Interestingly, if G has the aDP and the operator T is SCD, then T is a target for G, and this will be frequently used to deduce important results. Our new concept of target operator naturally plays an analogous role that the one played by strong Daugavet operators in the study of the Daugavet property and in the study of Daugavet centers. Let us say that even for the case $G = \mathrm{Id}_X$, this concept is new and provides with non-trivial new results. We characterize target operators for a given operator G, show that this property is separably determined, and prove that if G has the aDP, then every operator whose restriction to separable subspaces is SCD is a target for G. In section 3.4, we introduce the notion of lush operator, which generalizes the concept of lush space. This generalization is closely connected with target operators from the previous section, which, on the one hand, reduces some results about lush spaces to results from the previous section, and on the other hand, gives more motivation for the study of target operators. We give several characterizations of lush operators, prove that this property is separably determined, show that the aDP and lushness are equivalent when every separable subspace of the domain space is SCD (so, for instance, when the domain is Asplund, has the Radon-Nikodým Property, or does not contain copies of ℓ_1), and present some sufficient conditions for lushness which will be used in the chapter about examples and applications. Besides, we prove that lush operators with separable domain fulfill a stronger version of lushness which has to do with spear functionals.

Chapter 4 is devoted to present some examples in classical Banach spaces. Among other results, we show that the Fourier transform is lush, we characterize operators from $L_1(\mu)$ spaces which have the aDP, and we study lushness, spearness and the aDP for operators which arrive to spaces of continuous functions. In particular, we show that every uniform algebra isometrically embeds by a lush operator into the space of bounded continuous functions on a normal Hausdorff topological space (for unital algebras, this space is just its Choquet boundary). Also, we present here a family of spear operators which are not lush.

Next, we devote chapter 5 to provide further results on our properties. We characterize lush operators when the domain space has the Radon-Nikodým Property or the codomain space is Asplund, and we get better results when the domain or the codomain is finite-dimensional or when the operator has rank-one. Further, we study the behaviour of lushness, spearness and the aDP with respect to the operation of taking adjoint operators; in particular, we show that these properties pass from an operator to its adjoint if the domain has the Radon-Nikodým Property or the codomain is M-embedded; we also show that the aDP and spearness pass from an operator to its adjoint when the codomain is L-embedded.

In chapter 6 we provide with some isomorphic and isometric consequences of the properties as, among others, that the dual of the domain of an operator with the aDP and infinite rank contains ℓ_1 in the real case, and that lush operators always attain their norm, a property which is not shared by aDP operators. Besides, many results showing that the aDP, spearness and lushness do not combine well with rotundity or smoothness properties are also presented.

We study Lipschitz spear operators in chapter 7. These are just the spear vectors of the space of Lipschitz operators between two Banach spaces endowed with the Lipschitz norm. The main result here is that every (linear) lush operator is a Lipschitz spear operator, a result which can be applied, for instance, to the Fourier transform. We also provide with analogous results for aDP operators and for Daugavet centers.

A collection of stability results for our properties is given in chapter 8. We include results for various operations like absolute sums, vector-valued function spaces, and ultraproducts. The results we got are, in most cases, extensions of previously known results for the case of the identity, but some results are new even in this case.

Finally, we complete the book with a collection of open problems in chapter 9.

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Chapter 1

Historical introduction: a walk on the results for Banach spaces with numerical index 1

Maybe every Functional Analysis course contains the famous formula

$$\|T\| = \sup\{|\langle Tx,x\rangle| : x \in H,\ \|x\| = 1\} \tag{1.1}$$

for the norm of a selfadjoint operator T in a Hilbert space H, which is one of the cornerstones of selfadjoint operators theory. In general, for non-selfadjoint operators the above formula is no longer true, but the right hand side of (1.1) still makes sense and is called the *numerical radius* $v(T)$ of T. In complex Hilbert spaces the following inequality [51, Page 114] holds true for every $T \in \mathcal{L}(H)$

$$\frac{1}{2}\|T\| \leqslant v(T) \leqslant \|T\|,$$

which means, in particular, that the numerical radius is an equivalent norm on the space $\mathcal{L}(H)$ of operators.

There is a natural way to extend the concept of numerical radius to operators on an arbitrary Banach space X using the action of a functional on an element instead of the inner product: for a bounded linear operator $T\colon X \longrightarrow X$, its *numerical radius* is

$$v(T) := \sup\{|x^*(Tx)| : x \in X,\, x^* \in X^*,\ \|x\| = \|x^*\| = x^*(x) = 1\}. \tag{1.2}$$

Surprisingly, in some important Banach spaces X (like $C(K)$ or $L_1(\mu)$) one has $v(T) = \|T\|$ for ALL bounded linear operators $T\colon X \longrightarrow X$ so, in some sense, in these spaces every operator behaves like a selfadjoint one. Such spaces of *numerical index* 1 can be characterized equivalently [38] as those spaces X where for every bounded linear operator $T\colon X \longrightarrow X$ there is a modulus-one scalar ω such that

$$\|\operatorname{Id} + \omega T\| = 1 + \|T\|. \tag{1.3}$$

There is a rich and actively developing theory devoted to numerical index 1 spaces and their interplay with two other related classes of Banach spaces: spaces with the alternative Daugavet property and lush spaces.

In this book we extend and rebuild this theory on the basis of the recently introduced concept of *spear operator* [7]: a norm-one linear operator $G\colon X \longrightarrow Y$ satisfying that for every other bounded linear operator $T: X \longrightarrow Y$, there is a modulus-one scalar ω such that the norm equality

$$\|G+\omega T\| = 1+\|T\|$$

holds true.

The current introductory chapter is addressed to present the main results on Banach spaces with numerical index 1 using a somehow historical perspective. After giving the notation and terminology that we will use all along the book, we present the definitions and main results about numerical ranges, numerical radius, and numerical indices; the main geometrical properties implying numerical index 1 which have to do with the extremal estructure of the unit ball, the Daugavet property and the alternative Daugavet property, and lush spaces. Finally, we present the concepts and main examples of slicely countably determined (SCD) sets, spaces, and operators and their relationship with numerical index 1 spaces. We finish with two diagrams relating the notions presented in the chapter. Some proofs are included to give the reader the flavour of how to use the involved properties.

We think that this chapter will help to put the reader in the perspective of how is the situation for Banach spaces with numerical index 1 for a better understanding of the general theory of spear operators. Nevertheless, the results given here are not needed in the rest of the book with the only exceptions of the notation and terminology (section 1.1) and of the content of section 1.6 about SCD sets, spaces and operators, since these concepts will be crucial in the study of the relationship between spearness, lushness, and the aDP.

1.1 Notation and terminology

We first present the notation which is standard and next some new notation and terminology that is important for the main part of the book.

Standard notation and terminology

By $\mathbb{K}$ we denote the scalar field ($\mathbb{R}$ or $\mathbb{C}$), and we use the standard notation

$$\mathbb{T} := \{\omega \in \mathbb{K} \colon |\omega| = 1\}$$

for its unit sphere. We use the notation $\mathrm{Re}(\cdot)$ to denote the real part function, which is nothing more than the identity when we are dealing with real numbers.

We use the letters X, Y, Z for Banach spaces over $\mathbb{K}$ and by subspace we always mean closed subspace. In some cases, we have to distinguish between the real and the complex case, but for most results this difference is insignificant. The closed unit ball and unit sphere of X are denoted respectively by B_X and S_X. We denote the Banach space of all bounded linear operators from X to Y by $\mathcal{L}(X,Y)$, and write $\mathcal{L}(X)$ for $\mathcal{L}(X,X)$. The identity operator is denoted by Id, or Id_X if it is necessary to precise the space. The dual space of X is denoted by X^*, and $J_X : X \longrightarrow X^{**}$ denotes the natural isometric inclusion of X into its bidual X^{**}.

For a subset $A \subset X$ and for $x \in X$ we write

$$\mathbb{T}A := \{\omega a \colon \omega \in \mathbb{T},\ a \in A\} \quad \text{and} \quad \mathbb{T}x := \{\omega x \colon \omega \in \mathbb{T}\}.$$

A subset A of X is said to be *rounded* if $\mathbb{T}A = A$.

Given $A \subset X$, we denote by $\mathrm{conv}\, A$ or $\mathrm{conv}\,(A)$ the *convex hull* of A, by $\overline{\mathrm{conv}}\, A$ or $\overline{\mathrm{conv}}(A)$ the *closed convex hull*, by $\mathrm{aconv}\, A$ or $\mathrm{aconv}\,(A)$ its *absolutely convex hull*, i.e. $\mathrm{aconv}\,(A) = \mathrm{conv}\,(\mathbb{T}A)$, and by $\overline{\mathrm{aconv}}(A)$ the absolutely closed convex hull of A. We say that $A \subset B_X$ is *norming* for $Z \subset X^*$ if $\|f\| = \sup_{x\in A} |f(x)|$ for every $f \in Z$ or, equivalently, $B_X = \overline{\mathrm{aconv}}^{\sigma(X,Z)}(A)$ where $\sigma(X,Z)$ is the topology on X of the pointwise convergence for the elements of Z. For $x_0^* \in X^*$ and $y_0 \in Y$, we write $x_0^* \otimes y_0$ to denote the rank-one operator given by $[x_0^* \otimes y_0](x) = x_0^*(x) y_0$ for every $x \in X$.

We write $X = Y \oplus_1 Z$ and $X = Y \oplus_\infty Z$ to mean that X is, respectively, the ℓ_1-*sum* and the ℓ_∞-*sum* of Y and Z, i.e. every element $x \in X$ has unique representation $x = y + z$ with $y \in Y$, $z \in Z$, and $\|x\| = \|y\| + \|z\|$ if we speak about ℓ_1-sum, and $\|x\| = \max\{\|y\|, \|z\|\}$ if we speak about ℓ_∞-sum. In the first case, we say that Y is an *L-summand* of X; in the second case, we say that Y is an *M-summand* of X.

Let $A \subset X$ be a non-empty subset. A point $x \in A$ is said to be *extreme* if it does not belong to the relative interior of any non-void straight line segment whose endpoints are in A. In other words, for every $u, v \in A$ and $0 < \lambda < 1$, if $x = \lambda u + (1-\lambda)v$, then $u = v = x$. We denote by $\mathrm{ext}(A)$ (or $\mathrm{ext}A$) the set of extreme points of A.

A *slice* of A is a not empty part of A that is cut out by a hyperplane. Given $x^* \in X^*$ and $\alpha > 0$, denote the corresponding slice as

$$\mathrm{Slice}(A, x^*, \alpha) := \left\{x \in A \colon \mathrm{Re}\, x^*(x) > \sup_A \left(\mathrm{Re}\, x^*\right) - \alpha\right\}.$$

A *face* of A is a (non-empty) subset of the form

$$\mathrm{Face}(A, x^*) := \left\{x \in A \colon \mathrm{Re}\, x^*(x) = \sup_A \left(\mathrm{Re}\, x^*\right)\right\},$$

where $x^* \in X^*$ is such that its real part attains its supremum on A. In Figure 1.1, F is face and S is a slice.

If $A \subset X^*$ and the functional defining the slice or the face is taken in the predual, i.e. $x^* = x \in X \equiv J_X(X) \subset X^{**}$, then $\mathrm{Slice}(A, x, \alpha)$ is called a *w^*-slice* of A and $\mathrm{Face}(A, x)$ is called a *w^*-face* of A.

A point $x \in B_X$ is said to be *strongly extreme* if given a sequence $(y_n)_{n\in\mathbb{N}}$ in X such that $\|x \pm y_n\| \longrightarrow 1$, we have that $\lim y_n = 0$.

A point $x \in A$ is *denting* if it belongs to slices of A of arbitrarily small diameter. If X is a dual space and the corresponding small slices can be chosen to be w^*-slices, then the point is called *w^*-denting*. Observe that denting points are strongly extreme points, strongly extreme points are extreme points, and none of the implications reverses in general (see [77], for instance), although they are obviously the same notion if A is norm compact. A typical example of extreme point that is not denting is $\mathbb{1}_{[0,1]} \in B_{C[0,1]}$ (here and in the sequel $\mathbb{1}_A$ stays, as usual, for the characteristic function of a set A). In Figure 1.1, a is a denting point and b is not.

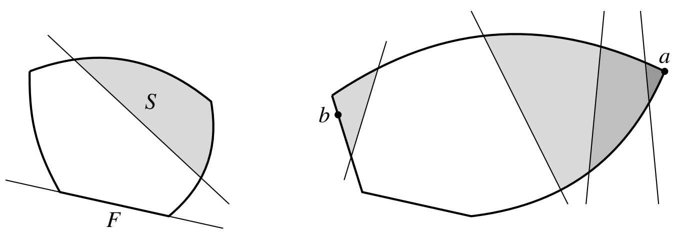


Fig. 1.1 Slice, face, denting, and not denting points

We write $\mathrm{dent}(A)$ to denote the set of denting points of A. We say that A is *dentable* (in the sense of Ghoussoub-Godefroy-Maurey-Schachermayer [45, §III]) if $A = \overline{\mathrm{conv}}(\mathrm{dent} A)$ [45, Proposition III.3]. The concept of *Radon-Nikodým set* was originally defined in terms of the validity of the Radon-Nikodým theorem for vector measures, but we will use the following equivalent definition (see [11, §5] or [20, §2]): a closed convex set $A \subset X$ possesses the *Radon-Nikodým property* (*RNP* in short), if all its closed convex bounded subsets are dentable. In particular, the whole space X also may have this property. Typical examples of spaces with the RNP are reflexive spaces and separable dual spaces (in particular, ℓ_1), and typical examples of spaces that do not enjoy this property are c_0 and $L_1[0,1]$.

We will mention also the so-called Asplund property, a concept related to differentiability of convex continuous functions, which can be equivalently reformulated in terms of separability and duality [20, §5]. A separable closed convex bounded subset A of a Banach space X has the *Asplund property* if and only if the seminormed space (X^*, ρ_A) is separable, where

$$\rho_A(x^*) = \sup\{|x^*(a)| \colon a \in A\} \qquad (x^* \in X^*).$$

A Banach space X is an *Asplund space* if for every separable subspace $Y \subset X$ the dual space Y^* is separable. X is an Asplund space if and only if X^* has the Radon-Nikodým property. Of course, separable closed convex bounded subsets of Asplund spaces have the Asplund property.

Recall that a Banach space X is said to be *strictly convex* if $\operatorname{ext} B_X = S_X$, and *smooth* if the mapping $x \longmapsto \|x\|$ is Gâteaux differentiable at every point of $X \setminus \{0\}$ (equivalently, for each $0 \neq x \in X$ there is a unique $x^* \in S_{X^*}$ with $x^*(x) = \|x\|$). If moreover, the mapping $x \longmapsto \|x\|$ is Fréchet differentiable at every point of $X \setminus \{0\}$, then X is said to be *Fréchet smooth*.

We will mention many times the expression *operator that do not fix copies of* some special Banach space E, say *operator that do not fix copies of* ℓ_1, or *of* c_0, etc. All this terminology comes from the following definition: $T \in \mathcal{L}(X,Y)$ *fixes a copy of a Banach space* E, if there is a subspace $Z \subset X$ such that Z is isomorphic to E and the restriction of T to Z is an isomorphism between Z and $T(Z)$.

Finally, we recall some common notation for (vector-valued) function spaces. Given a Hausdorff topological space Ω and a Banach space X, we write $C_b(\Omega,X)$ to denote the Banach space of all bounded continuous functions from Ω to X, endowed with the supremum norm. In the case when $X = \mathbb{K}$, we just write $C_b(\Omega) \equiv C_b(\Omega,\mathbb{K})$. If K is a compact Hausdorff topological space and X is a Banach space, we write $C(K,X) = C_b(K,X)$ which is the Banach space of all continuous functions from K into X endowed with the supremum norm, and we just write $C(K) = C(K,\mathbb{K})$. Given a positive measure space (Ω,Σ,μ) and a Banach space X, $L_\infty(\mu,X)$ is the Banach space of all (clases of) strongly measurable functions from Ω into X which are essentially bounded, endowed with the essential supremum norm

$$\|f\|_\infty = \inf\{c > 0 \colon \|f(t)\|_X \leqslant c \text{ for a.e. } t \in \Omega\}$$

and we just write $L_\infty(\mu) = L_\infty(\mu,\mathbb{K})$. The Banach space of all (clases of) Bochner-integrable functions from Ω into X, endowed with the integral norm

$$\|f\|_1 = \int_\Omega \|f(t)\|_X \, d\mu(t),$$

is denoted by $L_1(\mu,X)$. We simply write $L_1(\mu) = L_1(\mu,\mathbb{K})$. Recall that an element $A \in \Sigma$ is an *atom* if $\mu(A) \neq 0$ and for every $B \in \Sigma$ with $B \subset A$ it holds that $\mu(B) = 0$ or $\mu(A \setminus B) = 0$. We say that μ is *atomless* if it does not contain atoms, and it is called *purely atomic* whether every measurable set B with $\mu(B) \neq 0$ contains an atom. When μ is the counting measure on a set Γ, then we write $\ell_\infty(\Gamma) = L_\infty(\mu)$ and $\ell_1(\Gamma) = L_1(\mu)$. If moreover Γ is a finite set of n-elements, we simply denote these spaces as ℓ_∞^n and ℓ_1^n respectively.

Specific notation and terminology

For a subset $A \subset X$ we write $\|A\| := \sup\{\|x\| : x \in A\}$ if A is bounded and $\|A\| = \infty$ if it is unbounded. Observe that this function has the following properties:

$$\|\lambda A\| = |\lambda|\,\|A\|, \quad \|A+B\| \leqslant \|A-C\| + \|C+B\|, \quad \|A-B\| \geqslant \big|\|A-C\| - \|B-C\|\big|,$$

for every $\lambda \in \mathbb{K}$ and every bounded subsets A, B, C of X. The diameter of a (bounded) set $A \subset X$ can be calculated as $\operatorname{diam}(A) = \|A - A\|$. We will also denote

$$\|F \pm x\| := \max\{\|F + x\|, \|F - x\|\}.$$

Given $B \subset B_X$, $F \subset B_{X^*}$ and $\varepsilon > 0$, we define

$$\operatorname{gSlice}(B,F,\varepsilon) := \left\{x \in B\colon \sup_{x^* \in F} \operatorname{Re} x^*(x) > 1 - \varepsilon\right\}$$

and we call it a *generalized slice* of B (observe that it is a union of slices when non-empty). We also define

$$\operatorname{gFace}(B,F) := \left\{x \in B\colon \sup_{x^* \in F} \operatorname{Re} x^*(x) = 1\right\},$$

and call it a *generalized face* of B. If $F = \{z^*\}$, then we just write

$$\operatorname{gSlice}(B,F,\varepsilon) = \operatorname{gSlice}(B,z^*,\varepsilon)$$

(which is a slice of B when non-empty). The following easy results about generalized slices will be frequently used.

Remark. Let X be a Banach space, let $B \subset B_X$ be a rounded set, let $A \subset B_{X^*}$ be a set, and let $z^* \in S_{X^*}$. Then:

(a) $\operatorname{aconv}\big(\operatorname{gSlice}(B,A,\varepsilon)\big) = \operatorname{conv}\big(\operatorname{gSlice}(B,\mathbb{T}A,\varepsilon)\big)$;
(b) if $\operatorname{Re} z^*$ attains its supremum on B, then

$$\operatorname{gFace}(B,\mathbb{T}z^*) = \big\{x \in B\colon |z^*(x)| = 1\big\} = \mathbb{T}\operatorname{Face}(B,z^*).$$

We will introduce more notation on the way.

1.2 Numerical range, numerical radius, and numerical index

Our aim here is to give a short account on numerical ranges which will be very useful for the motivation and better understanding of the concepts of spear vector and spear operator.

The numerical range of a linear operator on a normed linear space is a subset of the scalar field, constructed in such a way that it is related both to the algebraic and the norm structures of the operator. For an operator on a Hilbert space, the numerical range has a very natural definition which was introduced, for finite-dimensional spaces, by O. Toeplitz in 1918 [118] as follows. Let H denote a Hilbert space with scalar product $\langle , \rangle$. The *numerical range* of $T \in \mathcal{L}(H)$ is the set $W(T)$ of scalars defined by

$$W(T) = \{\langle Tx, x\rangle : x \in S_H\}.$$

Observe that the numerical range of an operator is the image of S_H by the action of the quadratic form associated to the operator and, for this, it is sometimes called the *field of values* of the operator.

An excellent account of the Hilbert space numerical range and its properties can be found in a book of P. Halmos [51] about Hilbert spaces, and in a book of K. Gustafson and D. Rao [50] about numerical range.

An extension of the concept of numerical range to elements of unital Banach algebras was used in 1955 by H. Bohnenblust and S. Karlin [13] to relate geometrical and algebraic properties of the unit, and in the development of Vidav's characterization of C^*-algebras. Later on, in the 1960's, F. Bauer and G. Lumer gave independent but related extensions of Toeplitz's numerical range to bounded linear operators on arbitrary Banach spaces, which do not use the algebraic structure of the space of operators. All these notions are essential to define and study when an operator on a general Banach space is hermitian, skew-hermitian, dissipative, etc., and in the study and classification of operator algebras (see the fundamental F. Bonsall and J. Duncan books [15, 16], the survey paper [67], and the sections 2.1 and 2.9 of the recent book [23]).

In the 1985 paper [101], the following abstract notion of numerical range, which had already appeared implicitly in the aforementioned 1950's paper [13], was developed. We refer to sections 2.1 and 2.9 of the very recent book [23] for more information and background.

Given a Banach space Z and a distinguished element $u \in S_Z$, we define the *numerical range* and the *numerical radius* of $z \in Z$ *with respect to* (Z,u) as, respectively,

$$V(Z,u,z) := \big\{z^*(z) : z^* \in \operatorname{Face}(S_{Z^*},u)\big\}, \quad v(Z,u,z) := \max\big\{|\lambda| : \lambda \in V(Z,u,z)\big\}.$$

Then, the *numerical index* of (X,u) is

$$N(X,u) := \inf\big\{v(Z,u,z) : z \in S_Z\big\} = \max\big\{k \geqslant 0 : k\|z\| \leqslant v(Z,u,z)\ \forall z \in Z\big\}.$$

With this notation, $u \in S_Z$ is a *vertex* if $v(Z,u,z) \neq 0$ for every $z \in Z \setminus \{0\}$ (that is, $\operatorname{Face}(S_{Z^*},u)$ separates the points of Z), $u \in S_Z$ is a *geometrically unitary* element if the linear hull of $\operatorname{Face}(S_{Z^*},u)$ equals the whole space Z^* or, equivalently, if $N(X,u) > 0$ (see e.g. [23, Theorem 2.1.17]). By Definition 2.1, $u \in S_Z$ is a *spear vector* if $\|u + \mathbb{T}z\| = 1 + \|z\|$ for every $z \in Z$. By the Hahn-Banach Theorem, for $z \in Z$ we have that

$$\sup \operatorname{Re} V(Z,u,z) = \|z\| \quad \text{if and only if} \quad \|u+z\| = 1 + \|z\| \tag{1.4}$$

Therefore,

$$v(Z,u,z) = \|z\| \quad \text{if and only if} \quad \exists \omega \in \mathbb{T} \text{ with } \|u+\omega z\| = 1 + \|z\|. \tag{1.5}$$

As a consequence,

$$u \text{ is a spear vector} \qquad \Longleftrightarrow \qquad N(Z,u) = 1. \tag{1.6}$$

So, spear vectors are geometrically unitary elements (in the strongest possible way!), geometrically unitary elements are vertices, and vertices are extreme points. None of these implications reverses (see [23, Section 2.1]). Let us also comment that the celebrated Bohnenblust-Karlin Theorem [13] states that algebraic unitary elements of a unital complex Banach algebra (i.e. invertible elements u such that u and u^{-1} have norm one) are geometrically unitary, see [108] for a detailed account on this. Finally, let us mention that the concept of spear vector appeared, without name, in the paper [80] by Å. Lima about intersection properties of balls. It had also appeared tangentially in the monograph [81] by J. Lindenstrauss about extension of compact operators.

We then have an easy way to define the numerical range of an operator: given a Banach space X and $T \in \mathcal{L}(X)$, the *algebra numerical range* or *intrinsic numerical range* of T is just

$$V(T) := V(\mathcal{L}(X), \mathrm{Id}, T).$$

As in order to study this concept we have to deal with the (wild) dual of $\mathcal{L}(X)$, there are other concepts of numerical range which simplify such a task. The (Bauer) *spatial numerical range* of T is defined as

$$W(T) := \bigcup\nolimits_{x\in S_X} V(X,x,Tx) \;=\; \big\{x^*(Tx) \colon x \in S_X,\, x^* \in S_{X^*},\, x^*(x) = 1\big\}.$$

It is a classical result that the two numerical ranges are related as follows:

$$\overline{\operatorname{conv}}\big(W(T)\big) = V(\mathcal{L}(X), \mathrm{Id}, T) \tag{1.7}$$

(see e.g. [23, Proposition 2.1.31]), so they produce the same *numerical radius* of operators. Namely, the numerical radius of $T \in \mathcal{L}(X)$ can be computed as

$$v(T) := \sup\{|\lambda| \colon \lambda \in V(T)\} = \sup\{|\lambda| \colon \lambda \in W(T)\} = v(\mathcal{L}(X), \mathrm{Id}, T),$$

which in its turn equals to the quantity given in the basic definition (1.2). It is clear that v is a seminorm on $\mathcal{L}(X)$, and $v(T) \leqslant \|T\|$ for every $T \in \mathcal{L}(X)$. Quite often, v is actually a norm and it is equivalent to the operator norm $\|\cdot\|$. Thus it is natural to consider the so called *numerical index* of the space X, namely the constant $n(X)$ defined by

$$n(X) := \inf\{v(T) \colon T \in S_{\mathcal{L}(X)}\} = N(\mathcal{L}(X), \mathrm{Id}).$$

Equivalently, $n(X)$ is the greatest constant $k \geqslant 0$ such that $k\|T\| \leqslant v(T)$ for every $T \in \mathcal{L}(X)$. Note that $0 \leqslant n(X) \leqslant 1$, and $n(X) > 0$ if and only if v and $\|\cdot\|$ are equivalent norms (i.e. Id is geometrically unitary). Thanks to (1.6) and (1.7), the following result holds.

Proposition 1.1. *Let X be a Banach space. Then, $n(X) = 1$ if and only if* Id *is a spear element of $\mathcal{L}(X)$ (that is, $\|\mathrm{Id} + \mathbb{T}T\| = 1 + \|T\|$ for every $T \in \mathcal{L}(X)$).*

The concept of numerical index of a Banach space was first suggested by G. Lumer in a lecture to the North British Functional Analysis Seminar in 1968. At that time, it was known that in a complex Hilbert space (of dimension greater than 1) $\|T\| \leqslant 2v(T)$ for all $T \in \mathcal{L}(H)$. The real case is different. In a real Hilbert space H with dimension greater than 1 it is easy to build a norm-one operator T such that Tx is orthogonal to x for every $x \in S_H$, so $W(T) = \{0\}$. In other words, for a Hilbert space H of dimension greater than 1, $n(H) = 1/2$ if H is complex, and $n(H) = 0$ if it is real. G. Lumer [87] proved that $\|T\| \leqslant 4v(T)$ for every bounded linear operator T on a complex Banach space X, so $n(X) \geqslant 1/4$, and so, the numerical radius is an equivalent norm in the space of all bounded operators. In 1970, B. Glickfeld [46] improved this estimate by just writing in terms of the numerical radius an inequality due to F. Bohnenblust and S. Karlin [13].

Theorem 1.2 ([46, Theorem 1.4])**.** *Let X be a complex Banach space. Then*

$$\|T\| \leqslant \mathrm{e}\, v(T)$$

for all $T \in \mathcal{L}(X)$. Equivalently, $n(X) \geqslant \mathrm{e}^{-1}$.

Glickfeld also proves in [46] that e^{-1} is the best possible constant in a strong sense: there is a complex Banach space X and $T \in \mathcal{L}(X)$ such that $\|T\| = 1$ and $v(T) = \mathrm{e}^{-1}$. Therefore, $n(X) = e^{-1}$ and the infimum defining $n(X)$ is attained. Finally, J. Duncan, C. McGregor, J. Pryce, and A. White [38] determined (also in 1970) the range of variation of the numerical index.

Theorem 1.3 ([38, Theorems 3.5 and 3.6])**.** *For every $t \in [0,1]$ (resp. $t \in [\mathrm{e}^{-1}, 1]$), there is a real (resp. complex) Banach space X such that $n(X) = t$. Actually, X can be taken to be two-dimensional.*

The somehow surprising appearance of the number e in this world was due to the use of holomorphic techniques in the proof of the inequality by Bohnenblust and Karlin (see [13] for details). An elementary and direct proof of Theorem 1.2 can be found in [110, Proposition 1.3].

Computing the numerical index of concrete spaces may be hard. For instance, the numerical index of ℓ_p for $p \neq 1, 2, \infty$ is yet unknown, even though it is known that it cannot be zero [96]. However, there are some classical spaces whose numerical indexes have been calculated in the literature. In [38], the authors gave the first example of a Banach space such that the norm and the numerical radius coincide for all operators on it, that is, a space with numerical index 1: $C(K)$ for every compact Hausdorff topological space K [38, Theorem 2.1]. To find new examples, we can look at the relation between the numerical indexes of a Banach space and its dual.

It is clear that $W(T) \subset W(T^*)$ for every bounded linear operator T on a Banach space X, where T^* is the adjoint of T. There is also a result by G. Lumer [87, Lemma 12] showing that $\overline{\text{conv}}\, W(T) = \overline{\text{conv}}\, W(T^*)$. Therefore,

$$v(T) = v(T^*)$$

for every $T \in \mathcal{L}(X)$. Moreover, by using a refinement of the Bishop-Phelps theorem which enables to approximate pairs $(x, x^*) \in S_X \times S_{X^*}$ with $x^*(x)$ close to 1 by pairs $(y, y^*) \in S_X \times S_{X^*}$ with $y^*(y) = 1$ (this refinement is called now the Bishop-Phelps-Bollobás theorem), B. Bollobás [14] proved that, actually, one has that

$$W(T) \subset W(T^*) \subset \overline{W(T)}.$$

We can now state:

Proposition 1.4 ([38, Proposition 1.3])**.** *The inequality $n(X^*) \leqslant n(X)$ holds true for every Banach space X.*

Back to the examples, [38, Theorem 2.2] gives us two families of Banach spaces with numerical index 1: L-spaces and M-spaces. Indeed, the dual of an L-space and the bidual of an M-space are isometric to a space of continuous functions on some compact Hausdorff topological space, and the above proposition applies. In particular, every $L_1(\mu)$ space possesses numerical index 1.

It is natural to ask for the behavior of the numerical index under some operations. It is shown in [98] that the numerical index of a c_0-, ℓ_1-, or ℓ_∞-sum of Banach spaces can be computed in the expected way. Given an arbitrary family $\{X_\lambda : \lambda \in \Lambda\}$ of Banach spaces, let us denote by $[\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}$ (resp. $[\oplus_{\lambda\in\Lambda} X_\lambda]_{l_1}$, $[\oplus_{\lambda\in\Lambda} X_\lambda]_{l_\infty}$) the c_0-sum (resp. ℓ_1-sum, ℓ_∞-sum) of the family.

Proposition 1.5 ([98, Proposition 1])**.** *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of Banach spaces. Then*

$$n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}\Big) = n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{l_1}\Big) = n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{l_\infty}\Big) = \inf_\lambda n(X_\lambda).$$

An analogous result is known for spaces of vector-valued functions.

Theorem 1.6 ([98, Theorems 5 and 8] and [100, Theorem 2.3])**.** *Let K be a compact Hausdorff space, and let (Ω, Σ, μ) be a σ-finite positive measure space. Then*

$$n(C(K,X)) = n(L_1(\mu,X)) = n(L_\infty(\mu,X)) = n(X)$$

for every Banach space X.

Let us finally mention one further example of a Banach space with numerical index 1, namely the disk algebra (see [30, Theorem 3.3]). It actually follows from [124] that all uniform algebras have numerical index 1. We will speak about the extension of this result to the so-called C-rich subspaces later.

As an application of Proposition 1.5, one can exhibit an example of a real Banach space X such that the numerical radius is a norm in $\mathcal{L}(X)$, but it is not equivalent to the operator norm, i.e. $n(X) = 0$ (see [98, Example 2.b]); in other words, the identity is a vertex of the unit ball of $\mathcal{L}(X)$ but it is not geometrically unitary (see [23, Proposition 2.1.39]).

An useful application of the numerical range of operators is the following characterization of the generator of a uniformly continuous semigroup of isometries of a real Banach space. We refer to [110, Theorem 1.4] for an elementary proof.

Proposition 1.7. *Let X be a real Banach space and let $T \in \mathcal{L}(X)$. Then, the following are equivalent:*

(i) $v(T) = 0$,
(ii) $\exp(\rho T)$ *is a surjective isometry for every $\rho \in \mathbb{R}^+$, i.e. T is the generator of a uniformly continuous one-parameter semigroup of isometries.*

With this result in mind, the following terminology is understandable. Given a Banach space X, the group of all surjective isometries on X is called the *Lie group* of X and the subspace of all $T \in \mathcal{L}(X)$ with $v(T) = 0$ is called the *Lie algebra* of X and its elements are called *skew-hermitian operators*; the result above can be read as that the Lie algebra of X is the tangent space to the Lie group of X. See more details in the already cited paper [110]

1.2.1 Numerical ranges with respect to a given operator

Let now X, Y be Banach spaces and let us deal with numerical ranges with respect to a fixed operator $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$. First, the *intrinsic numerical range* of $T \in \mathcal{L}(X,Y)$ *with respect to* G is easy to define: just consider

$$V(\mathcal{L}(X,Y),G,T) = \left\{\Phi(T)\colon \Phi \in \mathcal{L}(X,Y)^*,\ \|\Phi\| = \Phi(G) = 1\right\},$$

and so we have the corresponding numerical radius $v(\mathcal{L}(X,Y),G,T)$ and numerical index $N(\mathcal{L}(X,Y),G)$. Observe that by (1.6), $N(\mathcal{L}(X,Y),G) = 1$ if and only if G is a spear element of $\mathcal{L}(X,Y)$ (i.e. G is a *spear operator*: for every $T \in \mathcal{L}(X,Y)$ there is $\omega \in \mathbb{T}$ such that $\|G + \omega T\| = 1 + \|T\|$). There are two notions which are also connected to numerical ranges: Daugavet centers [18, 17] and the aDP. An operator $G \in \mathcal{L}(X,Y)$ is a *Daugavet center* if

$$\|G + T\| = 1 + \|T\|$$

for every rank-one operator $T \in \mathcal{L}(X,Y)$. Observe that T satisfies the above equation if and only if $\|T\| = \sup \operatorname{Re} V(\mathcal{L}(X,Y),G,T)$, see (1.4). An operator $G \in \mathcal{L}(X,Y)$ has the *alternative Daugavet property* (*aDP* in short) if for every $T \in \mathcal{L}(X,Y)$ of rank-one, there is $\omega \in \mathbb{T}$ such that

$$\|G + \omega T\| = 1 + \|T\|.$$

Observe that T satisfies this equality if and only if $\|T\| = v(\mathcal{L}(X,Y),G,T)$, see (1.5).

Let us observe that the definition of intrinsic numerical range forces us to deal with the dual of $\mathcal{L}(X,Y)$, which is not a nice task. On the other hand, the possible extension of the definition of spatial numerical range to this setting has many problems as, for instance, it is empty if G does not attain its norm; moreover, even in the case when G is an isometric embedding, it does not always have a good behavior, see [95]. Very recently, a new notion has appeared [7]: the *approximated spatial numerical range* of $T \in \mathcal{L}(X,Y)$ *with respect to* G is defined by

$$\widetilde{W}_G(T) := \bigcap\nolimits_{\varepsilon>0} \overline{\big\{y^*(Tx) \colon y^* \in S_{Y^*},\, x \in S_X,\, \operatorname{Re} y^*(Gx) > 1 - \varepsilon\big\}}.$$

We then have the corresponding numerical radius and numerical index:

$$v_G(T) = \sup\big\{|\lambda| \colon \lambda \in \widetilde{W}_G(T)\big\}, \quad n_G(X,Y) = \inf\big\{v_G(T) \colon T \in \mathcal{L}(X,Y),\ \|T\| = 1\big\}.$$

The relationship between these two numerical ranges is analogous to the one for the identity operator [93, Theorem 2.1]:

$$\operatorname{conv}\big(\widetilde{W}_G(T)\big) = V(\mathcal{L}(X,Y),G,T)$$

for every norm-one $G \in \mathcal{L}(X,Y)$ and every $T \in \mathcal{L}(X,Y)$. Therefore, both concepts produce the same numerical radius of operators and so, the same numerical index of G, the same concepts of vertex and geometrically unitary elements. In particular, $G \in \mathcal{L}(X,Y)$ is a spear operator if and only if $N\big(\mathcal{L}(X,Y),G\big) = 1$ if and only if $n_G(X,Y) = 1$. We will give a direct proof of this fact in Proposition 3.2. Analogous results for Daugavet centers and the aDP also hold.

1.3 Banach spaces with numerical index 1 in relation to geometry of the extreme points and the faces of the unit ball

A Banach space X has numerical index 1 if and only if for every $T \in \mathcal{L}(X)$ the norm of T can be evaluated as

$$\|T\| = \sup\{|x^*(Tx)| \colon x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1\}.$$

The guiding open question on these spaces is the following.

Problem 1.8. Find necessary and sufficient conditions for a Banach space to have numerical index 1 which do not involve operators.

It is also natural to ask what are the consequences of this property on the geometry (or the topology) of a Banach space. For instance, is it possible to find an infinite-dimensional reflexive Banach space having numerical index 1? Or, which infinite-dimensional Banach spaces have (or can be re-normed to have) the property?

In 1971, C. McGregor [102, Theorem 3.1] gave a characterization of numerical index 1 spaces in the finite-dimensional case.

Theorem 1.9 ([102, Theorem 3.1]). *A finite-dimensional space X satisfies $n(X)=1$ if and only if*

$$|x^*(x)| = 1 \text{ for every } x \in \operatorname{ext}(B_X) \text{ and every } x^* \in \operatorname{ext}(B_{X^*}). \tag{1.8}$$

This implies easily that in the real case all such spaces must be *polyhedral*, that is, their unit balls are polyhedrons. Moreover, the only (up to isometry) real space of dimension 2 with numerical index 1 is $\ell_1^2 \equiv \ell_\infty^2$ whose unit ball is a square but, the bigger the dimension is, the greater the variety of examples of numerical index 1 spaces is.

It is not clear how to extend McGregor's result to arbitrary Banach spaces. If we use literally (1.8) in the infinite-dimensional context, we do not get a sufficient condition, since the set $\operatorname{ext}(B_X)$ may be empty and this does not imply numerical index 1 (e.g. $\operatorname{ext}\left(B_{c_0(\ell_2)}\right) = \emptyset$ but $n(c_0(\ell_2)) < 1$). One could reformulate McGregor's condition in a natural way: $|x^{**}(x^*)| = 1$ for every $x^* \in \operatorname{ext}(B_{X^*})$ and every $x^{**} \in \operatorname{ext}(B_{X^{**}})$. It is easy to show that this condition is sufficient to ensure $n(X) = 1$. Unfortunately, this condition is not necessary. Even more, there is a space X having $n(X) = 1$ and such that for every $x^* \in S_{X^*}$ there is an $x^{**} \in \operatorname{ext}(B_{X^{**}})$ with $|x^{**}(x^*)| < 1$ [64, Remarks 4.2.c], see Example 4.27.

Necessary conditions in the spirit of McGregor's result were given in 1999 by G. López, M. Martín, and R. Payá [84]. The key idea was considering denting points instead of general extreme points.

Proposition 1.10 ([84, Lemma 1]). *Let X be a Banach space with numerical index 1. Then:*

(i) $|x^{**}(x^*)| = 1$ *for every* $x^{**} \in \operatorname{ext}(B_{X^{**}})$ *and every* w^**-denting point* $x^* \in B_{X^*}$.
(ii) $|x^*(x)| = 1$ *for every* $x^* \in \operatorname{ext}(B_{X^*})$ *and every* $x \in \operatorname{dent}(B_X)$.

Let us comment that, like McGregor original result, the conditions in Proposition 1.10 are not sufficient in the infinite-dimensional context. Indeed, the space $X = C([0,1], \ell_2)$ does not have numerical index 1, while B_X has no denting points and there are no w^*-denting points in B_{X^*}. Actually, all the slices of B_X and the

w^*-slices of B_{X^*} have diameter 2 (see [71, Lemma 2.2 and Example on p. 858], for instance).

The above proposition can be combined with a useful sufficient condition for a *real* Banach space to contain a subspace isomorphic either to c_0 or to ℓ_1, which follows easily from Rosenthal's ℓ_1-Theorem [109] and Fonf's Theorem on containment of c_0 [41].

Proposition 1.11 ([84, Proposition 2]). *Let X be a real Banach space and assume that there is an infinite set $A \subset S_X$ such that $|x^*(a)| = 1$ for every $a \in A$ and all $x^* \in \operatorname{ext}(B_{X^*})$. Then X contains (an isomorphic copy of) c_0 or ℓ_1.*

A proof of this result will be given in Proposition 2.11.j.

The way to use Proposition 1.11 and Lemma 1.10 should be clear: take a real Banach space with numerical index 1 and infinitely many denting points (or w^*-denting points in its dual), and you obtain that the Banach space (or the dual) contains isomorphic copies of c_0 or ℓ_1. A natural (isomorphic) assumption on an infinite-dimensional Banach space providing a lot of denting points is the Radon-Nikodým property. On the other hand, if X is an Asplund space (equivalently X^* has the RNP), then B_{X^*} is the weak-star closed convex hull of its w^*-strongly exposed (hence w^*-denting) points (see [103]). Therefore, we get:

Theorem 1.12 ([84, Theorem 3]). *Let X be an infinite-dimensional real Banach space with $n(X) = 1$. If X has the RNP, then X contains ℓ_1. If X is an Asplund space, then X^* contains ℓ_1.*

Note that the second part of the above theorem does not follow directly from the first one, because we require only $n(X) = 1$ instead of more restrictive assumption $n(X^*) = 1$.

Some interesting consequences of the above theorem are obtained by using the relationship between the RNP, containment of c_0 or ℓ_1, reflexivity, etc. For instance, an Asplund space cannot contain ℓ_1, so if X is a real Asplund space satisfying the RNP, and $n(X) = 1$, then X is finite-dimensional. As a special case, a reflexive or quasi-reflexive real Banach space with numerical index 1 must be finite-dimensional. Actually, if the quotient X^{**}/X is separable, it is known (see [37, pp. 219]) that X has the RNP and it is an Asplund space. Therefore, if X is an infinite-dimensional real Banach space with $n(X) = 1$, then X^{**}/X is non-separable. All these results can be understood as necessary conditions for a Banach space to be re-normable with numerical index 1. We emphasize the following.

Corollary 1.13. *An infinite-dimensional real Asplund space with the RNP cannot be re-normed to have numerical index* 1.

Unfortunately, it is not known how to extend the above results to the complex case. There, the knowledge of Banach spaces with numerical index 1 is too poor.

It even remains to be an open question whether an infinite-dimensional reflexive complex space may have numerical index 1.

In the remaining part of this section we discuss various sufficient conditions for a Banach space to have numerical index 1.

The eldest of these properties was introduced in the fifties by O. Hanner [52]: a real Banach space has the *intersection property 3.2* (*3.2.I.P.* in short) if every collection of three mutually intersecting closed balls has nonempty intersection. The 3.2.I.P. was systematically studied by J. Lindenstrauss [81] and Å. Lima [79], and typical examples of spaces with this property are $L_1(\mu)$ and their isometric preduals. The fact that an infinite-dimensional real Banach space with the 3.2.I.P. cannot be reflexive was known to J. Lindenstrauss and R. Phelps in 1968 [82, Corollary 2.4].

Another isometric property, weaker than the 3.2.I.P. but still ensuring numerical index 1, was introduced by R. Fullerton in 1960 [42]. A real or complex Banach space is said to be a *CL-space* if its unit ball is the absolutely convex hull of every maximal convex subset of the unit sphere. If the unit ball is merely the closed absolutely convex hull of every maximal convex subset of the unit sphere, we say that the space is an *almost-CL-space* (J. Lindenstrauss [81] and Å. Lima [80]). Both definitions appeared only for real spaces, but they extend literally to the complex case. Let us remark that the complex space ℓ_1 is an almost-CL-space which is not a CL-space [99, Proposition 1], but we do not know if such an example exists in the real case.

Problem 1.14. Is there any real almost-CL-space which is not a CL-space?

In 1990, M. Acosta proved that real CL-spaces have numerical index 1 (see [2] and [3, Teorema 5.5]). The result was extended to both real and complex almost-CL-spaces in [88, Proposition 12]. A demonstration of a stronger result can be found in Proposition 1.40 of Section 1.5.

Proposition 1.15. *If X is an almost-CL-space, then $n(X) = 1$.*

In the converse direction, the basic examples of Banach spaces with numerical index 1 are known to be almost-CL-spaces (see [99] and [16, Theorem 32.9]). Moreover, all finite-dimensional spaces with numerical index 1 are CL-spaces [80, Corollary 3.7], and a Banach space with the Radon-Nikodým property and numerical index 1 is an almost-CL-space [89, Theorem 1]. Nevertheless, there are Banach spaces with numerical index 1 which are not almost-CL-spaces. Actually, this happens with the space given in Example 1.42 in Section 1.5 below (see [21, Example 3.4]).

Let us also comment that it is easy to show, using Proposition 1.11, that the dual of every infinite-dimensional real almost-CL-space contains a copy of ℓ_1, with no isomorphic assumption on the space [99, Theorem 5].

To finish this section, we cite a result obtained by S. Reisner in 1991 [107], which emphasizes the difference between spaces with the 3.2.I.P. and CL-spaces. In 1981,

A. Hannsen and Å. Lima had given a structure theorem for real finite-dimensional spaces with the 3.2.I.P. [53]: any such space is obtained from the real line by repeated ℓ_1- and ℓ_∞-sums. That is, it can be constructed in a finite sequence of steps, using only one type of "brick", which is the real line, and two "construction tools", ℓ_1- and ℓ_∞-sums. In [107], Reisner proved that nothing similar can be expected for CL-spaces. He showed that it does not exist a finite set of "bricks" which is sufficient to construct all finite-dimensional real CL-spaces by ℓ_1- and ℓ_∞-sums (see [107, Section 3] for details).

For more information and background on CL-spaces and 3.2.I.P. we refer the interested reader to [6], and to the already mentioned [42, 53, 79, 80, 81, 107]. In section 1.5 we are going to speak about a some more general sufficient condition – lushness – which is now the "mainstream" of the numerical index 1 spaces theory.

1.4 Daugavet Property and alternative Daugavet Property: slices come into play.

In frames of approximation theory, it is often significant whether for a given subspace Y of a Banach space X there is a norm-one linear projection $P \in \mathcal{L}(X)$. It is usually a good exercise for students to find an example of $Y \subset X$ where such a norm-one projection does not exist. An easy solution is the subspace Y of $X = C[0,1]$, consisting of functions satisfying the condition $f(0) = 0$. In 1963, I. Daugavet [31] discovered the following effect: for every compact operator $T \in \mathcal{L}(C[0,1])$ the identity

$$\|\operatorname{Id} + T\| = 1 + \|T\|, \tag{1.9}$$

called now *the Daugavet equation*, holds true. The proof can be easily generalized to perfect compact Hausdorff topological spaces K (K is called *perfect* if it does not have isolated points). An evident corollary of this is that every projection on a finite-codimensional subspace of $C[0,1]$ has at least norm 2. On the other hand, if K is not perfect, i.e. if K has an isolated point τ, then (1.9) is not true for the following very simple rank-one operator $T \in \mathcal{L}(C(K))$: $[Tf](t) = -f(t)\mathbb{1}_{\{\tau\}}$ for $t \in K$, $f \in C(K)$.

Remark that (1.9) for T implies the same equation for αT for all $\alpha > 0$ (see Remark 2.2). This observation enables us to consider only operators of norm 1 in the demonstration of results like Daugavet's theorem.

The reader noticed immediately the analogy between (1.9) and the characterization (1.3) of operators having numerical index 1. Nevertheless, the theories of the Daugavet equation and the one of numerical index 1 spaces, were being developed by different people and independently one from the other. Only at the beginning of the 21^{st} century, the exchange of ideas and methods between these two theories started, which enriched the theories enormously.

Before passing to the connections with numerical index 1 spaces, let us speak about the development of the Daugavet equation theory. We do not pretend to present here the complete picture, our goal is to give some historical comments to explain the relations to general Banach space theory, and to concentrate on those concepts and results that are of importance for this book.

On the initial stage of this study, a number of authors generalized the Daugavet's theorem mainly in two directions: to some other spaces and to wider classes of operators. For example, C. Foiaş and I. Singer [40] extended the Daugavet's theorem to almost diffuse operators in $C(K)$ on perfect compact K, and L. Weis and D. Werner [123] demonstrated the Daugavet equation for operators on $C(K)$ not fixing a copy of $C[0,1]$. The same equation (1.9) holds true for compact operators in $L_1[0,1]$, which was demonstrated first (in a more general setting) by G. Lozanovskii [85] in 1966. In fact, as remarked A. Pełczyński (published in [40] with his permission), the Daugavet's type theorem for $L_1(\mu)$ with atomless μ follows easily from the $C(K)$ case by a duality argument: the dual space of $L_1(\Omega,\Sigma,\mu)$ is isometric to $C(K)$ for a very abstract perfect compact K, so (1.9) holds true for adjoint compact operators on $L_1(\Omega,\Sigma,\mu)^*$, but since the norm of an operator and the one of its adjoint coincide, and adjoints of compact operators are again compact, the proof is over.

An important step was done by A. Plichko and M. Popov [105] in 1990 when they introduced narrow operators in spaces of measurable functions. In the case of operators defined on $L_1(\mu)$, the corresponding class of *PP-narrow operators* is defined as follows: an operator $T \in \mathcal{L}(L_1(\mu),X)$ is said to be PP-narrow, if for every $A \in \Sigma$ of positive measure and every $\varepsilon > 0$ there is a partition $A = B \cup C$ with $\mu(B) = \mu(C) = \frac{1}{2}\mu(A)$ such that $\|T(\mathbb{1}_B - \mathbb{1}_C)\| < \varepsilon$. They demonstrated that for every atomless measure μ, all PP-narrow operators $T \in \mathcal{L}(L_1(\mu))$ satisfy (1.9). This result is applicable, in particular, to operators not fixing copies of $L_1[0,1]$. Much more information and recent developments may be found in the monograph [106] by M. Popov and B. Randrianantoanina.

Motivated by the above results, V. Kadets and M. Popov [69] introduced the concept of narrow operator on $C[0,1]$. This concept is easily extendable to arbitrary sup-normed spaces $C_b(\Omega)$ [27]. Afterwards, this extended definition appeared to be of high importance for the numerical index theory, so we are going to speak about it in more detail.

Definition 1.16. Let Ω be a Hausdorff topological space and let X be a Banach space. An operator $T \in \mathcal{L}(C_b(\Omega),X)$ is said to be *C-narrow* whether for every $\varepsilon > 0$ and every non-void open subset $U \subset \Omega$ there is $f \in S_{C_b(\Omega)}$ with $f(\Omega) \subset [0,1]$, $\operatorname{supp}(f) \subset U$ and $\|Tf\| < \varepsilon$.

Remark that in the original definition, an operator $T \in \mathcal{L}(C[0,1],X)$ is said to be C-narrow if for every non-void open subset $U \subset [0,1]$ the restriction of T on the subspace of those functions with support inside U is not bounded from below, that is, for every $\varepsilon > 0$ and every non-void open subset $U \subset [0,1]$ there is a function $f \in S_{C[0,1]}$ with $\operatorname{supp}(f) \subset U$ and $\|Tf\| < \varepsilon$. This apparently weaker definition turns out to be equivalent to the one given above, as the next useful lemma demonstrates that the hypothesis $f(\Omega) \subset [0,1]$ in Definition 1.16 can be omitted when the topological space Ω is normal. It was proved originally in [69] for $C[0,1]$ and for real scalars. The demonstration below follows the same idea but, formally speaking, it is extracted from [12], where the result is proved in the setting of spaces $C(K,Y)$ of vector-valued continuous functions on a compact K.

Lemma 1.17. *Let Ω be a normal Hausdorff topological space. Then, an operator $T \in \mathcal{L}(C_b(\Omega),X)$ is C-narrow whether for every $\varepsilon > 0$ and every non-void open subset $U \subset \Omega$ there is $g \in S_{C_b(\Omega)}$ with $\operatorname{supp}(g) \subset U$ and $\|Tg\| < \varepsilon$.*

Proof. Without loss of generality, we may and will assume that $\|T\| = 1$. For every $r > 0$ write $V(r) = \{t \in \mathbb{K}\colon |t-1| < r\}$ to denote the r-neighborhood of 1 in the scalar field $\mathbb{K}$. Let us fix $\varepsilon > 0$ and an open set U in Ω. Using the hypothesis, we find a function $f_1 \in S_{C_b(\Omega)}$ with $\operatorname{supp}(f_1) \subset U$ and $\|Tf_1\| < \frac{\varepsilon}{2}$. Multiplying, if necessary, f_1 by a modulus-one scalar, we may assume that there is a point $t_1 \in U$ where $f_1(t_1)$ is real and $f(t_1) > \frac{1}{2}$. Put $U_1 = U$ and define $U_2 = f_1^{-1}(V(\frac{1}{2}))$. Then, $t_1 \in U_2$, so $U_2 \neq \emptyset$ and we may apply the hypothesis to U_2 and $\frac{\varepsilon}{2}$ and obtain, as above, a $t_2 \in U_2$ and a function $f_2 \in S_{C_b(\Omega)}$ with $\operatorname{supp}(f_2) \subset U_2$, $f_2(t_2) > \frac{3}{4}$, and $\|Tf_2\| < \frac{\varepsilon}{2}$. We denote $U_3 = f_2^{-1}(V(\frac{1}{4}))$ and continue the process. In the j^{th} step, we get a non-empty open set $U_j = f_{j-1}^{-1}(V(\frac{1}{2^{j-1}})) \subset U_{j-1}$ and apply the hypothesis to obtain a function f_j corresponding to U_j and $\frac{\varepsilon}{2}$.

Choose $n \in \mathbb{N}$ such that $\frac{4}{n} < \frac{\varepsilon}{2}$ and put $f = \frac{1}{n}(f_1 + f_2 + \cdots + f_n)$. Now, using Urysohn's Lemma, we may find a continuous function $g_j\colon \Omega \longrightarrow [0,1]$ which equals 1 on U_{j+1} and equals 0 outside of U_j. Writing $g = \frac{1}{n}(g_1 + \cdots + g_n)$, we obtain a positive function $g \in S_{C_b(\Omega)}$ with $\operatorname{supp}(g) \subset U$. We claim that $\|f - g\| < \frac{\varepsilon}{2}$. Indeed, by our construction, if $t \in \Omega \setminus U_1$, then $|[f-g](t)| = 0$. For $t \in U_{n+1}$, we have

$$\begin{aligned} |[f-g](t)| &= \left|\frac{1}{n}\big[(f_1 - g_1) + \cdots + (f_n - g_n)\big](t)\right| \\ &= \left|\frac{1}{n}\big((f_1(t) - 1) + \cdots + (f_n(t) - 1)\big)\right| \\ &\leqslant \frac{1}{n}\left(\frac{1}{2} + \cdots + \frac{1}{2^n}\right) < \frac{1}{n} < \frac{\varepsilon}{2}. \end{aligned}$$

Finally, if $t \in U_k \setminus U_{k+1}$ for $k \in \{1, \ldots, n\}$, then

$$\begin{aligned} |[f-g](t)| &= \left|\frac{1}{n}\big[(f_1 - g_1) + \cdots + (f_n - g_n)\big](t)\right| \\ &= \left|\frac{1}{n}\big((f_1(t) - 1) + \cdots + (f_{k-1}(t) - 1) + (f_k(t) - g_k(t)) + g_{k+1}(t)\big)\right| \\ &\leqslant \frac{1}{n}\left(\frac{1}{2} + \cdots + \frac{1}{2^{k-1}} + 2 + 1\right) < \frac{4}{n} < \frac{\varepsilon}{2}, \end{aligned}$$

which demonstrates our claim. Moreover,

$$\|Tf\| \leqslant \frac{1}{n}(\|Tf_1\| + \|Tf_2\| + \cdots + \|Tf_n\|) < \frac{\varepsilon}{2}.$$

Thus $\|T(g)\| = \|T(g-f)\| + \|T(g)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ and we are done. □

A typical example of C-narrow operator is an operator on $C[0,1]$ which does not fix copies of $C[0,1]$, but the class of C-narrow operators on $C[0,1]$ contains also some operators fixing copies of $C[0,1]$.

We state now the expected result:

Theorem 1.18 ([69]). *Let K be a perfect compact, then every C-narrow operator $T \in \mathcal{L}(C(K))$ satisfies the Daugavet equation.*

The proof of this result will be given in Proposition 1.30.

The concept of *C-rich subspace* of $C[0,1]$ was introduced in the same paper [69]. In the sequel we will need the analogous definition for $C(K)$ spaces and for $C_b(\Omega)$ spaces.

Definition 1.19. Let Ω be a Hausdorff topological space. A subspace X of $C_b(\Omega)$ is said to be C-rich, if the quotient map $q\colon C_b(\Omega) \longrightarrow C_b(\Omega)/X$ is C-narrow.

Observe that if Ω is completely regular, then $C_b(\Omega)$ is C-rich in itself.

It is known [69] that in C-rich subspaces of $C[0,1]$ the Daugavet equation is valid for operators not fixing copies of $C[0,1]$.

Using the definition of C-narrow operator (and Lemma 1.17) one can reformulate C-richness of a subspace $X \subset C_b(\Omega)$ without using the notion of quotient space:

Proposition 1.20. *Let Ω be a Hausdorff topological space. A subspace X of $C_b(\Omega)$ is C-rich if and only if for every non-void open subset $U \subset \Omega$ and for every $\varepsilon > 0$ there are $f \in S_X$ and $g \in S_{C_b(\Omega)}$ with $g(\Omega) \subset [0,1]$, $\operatorname{supp}(g) \subset U$ and $\|f - g\| < \varepsilon$. Moreover, the hypothesis $g(\Omega) \subset [0,1]$ can be omitted if Ω is normal (in particular, if it is compact).*

It is easy to get examples from the above proposition. We need some notation: given a Hausdorff compact topological space K and a closed subset L of K, we write $C_0(K\|L)$ for the subspace of $C(K)$ consisting of those continuous functions on K which are zero on L. If $K \setminus L$ is dense, then Urysohn's Lemma gives that $C_0(K\|L)$ and every subspace of $C(K)$ containing it are C-rich.

Corollary 1.21. *Let K be a compact Hausdorff topological space and let L be a closed subspace of K such that $K \setminus L$ is dense. Then, every subspace X of $C(K)$ containing $C_0(K\|L)$ is C-rich.*

The following definition will be useful in the next section. Given a compact Hausdorff topological space K, a nowhere-dense closed subset L of K, and a subspace $E \subset C(L)$, we write

$$C_E(K\|L) := \{f \in C(K)\colon f|_L \in E\}.$$

By the corollary above, all these spaces are C-rich in $C(K)$.

The spaces $C[0,1]$ and $L_1[0,1]$ appear in the Banach space theory in many instances when an example of a space with some "bad" property is needed. For instance, both of them are not isomorphic to a dual space. In 1992, P. Wojtaszczyk discovered that the validity of the Daugavet equation for rank-one operators is the common reason for $C[0,1]$ and $L_1[0,1]$ to be non-dual (or, to be more precise, to be spaces without the Radon-Nikodým property).

It was discovered in 1993 [59] that another common feature of $C[0,1]$ and $L_1[0,1]$ – the absence of an unconditional basis – also follows from the Daugavet equation for compact operators. Let us show this.

Assume that in a Banach space X the Daugavet equation takes place for compact operators, assume also that X has an unconditional basis $(e_n)_{n\in\mathbb{N}}$ and denote $(e_n^*)_{n\in\mathbb{N}}$ the corresponding biorthogonal functionals. Then the identity operator on X can be represented as the point-wise unconditionally convergent sum of rank-one operators $T_n = e_n^* \otimes e_n$:

$$\mathrm{Id} = \sum_{n\in\mathbb{N}} T_n.$$

Denote by $\mathrm{Fin}(\mathbb{N})$ the set of finite subsets of $\mathbb{N}$. By the well-known properties of unconditional bases (deduced from properties of unconditionally convergent series and the Banach-Steinhaus uniform boundedness principle, see, for example, the beginning of the proof of [83, Proposition 1.c.6]), the quantity

$$\alpha = \sup\left\{\left\|\sum_{n\in A} T_n\right\| : A \in \mathrm{Fin}(\mathbb{N})\right\}$$

is finite. Also, whenever $B \subset \mathbb{N}$, then

$$\left\|\sum_{n\in B} T_n\right\| \leqslant \sup\left\{\left\|\sum_{n\in A} T_n\right\| : A \in \mathrm{Fin}(\mathbb{N}),\ A \subset B\right\} \leqslant \alpha.$$

Let $\varepsilon > 0$ and pick $A_0 \in \mathrm{Fin}(\mathbb{N})$ such that $\|\sum_{n\in A_0} T_n\| \geqslant \alpha - \varepsilon$. Then, from our assumption, we obtain that

$$\begin{aligned} 0 = \left\|\mathrm{Id} - \sum_{n\in\mathbb{N}} T_n\right\| &\geqslant \left\|\mathrm{Id} - \sum_{n\in A_0} T_n\right\| - \left\|\sum_{n\notin A_0} T_n\right\| \\ &\overset{(1.9)}{=} 1 + \left\|\sum_{n\in A_0} T_n\right\| - \left\|\sum_{n\notin A_0} T_n\right\| \geqslant 1 + \left\|\sum_{n\in A_0} T_n\right\| - \alpha \geqslant 1 - \varepsilon. \end{aligned}$$

This contradiction completes the proof.

The isomorphic consequences of the validity of the Daugavet equation for some "good" class of operators in a given space motivated V. Kadets, R. Shvidkoy, G. Sirotkin, and D. Werner to introduce in 2000 [71] the following Banach space property and to launch its study.

Definition 1.22. A Banach space X has the *Daugavet property* if the Daugavet equation (1.9) holds true for every rank-one operator $T \in \mathcal{L}(X)$.

The selection of the class of rank-one operators (instead of, say, compact ones) in the above definition allows to reformulate it in a purely geometrical language.

Theorem 1.23 **([71, Lemma 2.2]).** *Let X be a Banach space. The following assertions are equivalent:*

(i) X has the Daugavet property,
(ii) for every $x \in S_X$, for every $\varepsilon > 0$, and for every slice S of the unit ball B_X there is some $y \in S$ such that

$$\|x+y\| > 2-\varepsilon. \tag{1.10}$$

After this reformulation, the Daugavet property starts to be one more property of the unit ball, which is a quite traditional type of properties considered in the Banach space theory. Taking a $z \in S$ and applying Theorem 1.23 to $x = -z$, we get that for a Banach space X with the Daugavet property, every slice of B_X has diameter 2. In particular, X fails the Radon-Nikodým property, a fact originally due to Wojtaszczyk [126].

Item (ii) of Theorem 1.23 easily implies the following stronger version:

(iii) For every $x \in S_X$, for every $\varepsilon > 0$, and for every slice S_1 of the unit ball B_X there is another slice $S_2 \subset S_1$ such that (1.10) holds true for all $y \in S_2$.

Applying this property (iii) step-by-step to the element x_1 and the slice S_1, then to x_2 and S_2, etc., one gets the following.

Proposition 1.24. *Let X be a Banach space with the Daugavet property. Then for every finite subset of $A \subset S_X$, every slice S of the unit ball B_X, and every $\varepsilon > 0$, there is $y \in S$ such that (1.10) holds true for all $x \in A$.*

After some play with epsilons and ε-nets, one even gets the following.

Lemma 1.25 **([71, Lemma 2.8]).** *If X is a Banach space with the Daugavet property, then for every finite-dimensional subspace Y of X, every $\varepsilon > 0$ and every slice S of B_X there is another slice $S_1 \subset S$ of B_X such that*

$$\|y+tx\| \geqslant (1-\varepsilon_0)(\|y\|+|t|) \qquad \forall y \in Y,\ \forall x \in S_1,\ \forall t \in \mathbb{R}. \tag{1.11}$$

This leads to the presence of a number of subspaces isomorphic to ℓ_1 in every space with the Daugavet property.

Proposition 1.26. *Let X be a Banach space with the Daugavet property. Then for every sequence S_n, $n = 0, 1, \dots$ of slices of the unit ball B_X and every $\varepsilon > 0$, there are elements $x_n \in S_n$ such that the sequence $(x_n)_{n=0}^{\infty}$ is ε-equivalent to the canonical basis of ℓ_1; namely, for every $a = (a_n) \in \ell_1$*

$$(1-\varepsilon)\sum_{n=0}^{\infty}|a_n| \leqslant \left\|\sum_{n=0}^{\infty}a_n x_n\right\| \leqslant \sum_{n=0}^{\infty}|a_n|.$$

Surprisingly, although Definition 1.22 deals only with rank-1 operators, it implies the validity of the Daugavet equation for much wider classes of operators. Recall, that $T \in \mathcal{L}(X,Y)$ is called a *strong Radon-Nikodým operator*, if $\overline{T(B_X)}$ is a Radon-Nikodým set. We present the corresponding result with the proof, in order to enable the reader to feel the flavor of the geometry of slices technique, which will appear very often in this book.

Theorem 1.27 ([71, Theorem 2.3])**.** *If a Banach space X possesses the Daugavet property, then the Daugavet equation remains valid for all strong Radon-Nikodým operators in X, in particular, for all compact and all weakly compact operators.*

Proof. Let $T \in \mathcal{L}(X)$ be a strong Radon-Nikodým operator with $\|T\| = 1$. Then $K = \overline{T(B_X)}$ has the RNP and, therefore, coincides with the closed convex hull of its denting points. So, for every $\varepsilon > 0$ there is a denting point x_0 of K with

$$\|x_0\| > \sup\{\|y\| \colon y \in K\} - \varepsilon = 1 - \varepsilon,$$

and for some $0 < \delta < \varepsilon$ there is a slice $S = \{y \in K \colon \operatorname{Re} y^*(y) > 1 - \delta\}$ of K containing x_0 and having diameter $< \varepsilon$; here $y^* \in X^*$ and $\sup_{y\in K} \operatorname{Re} y^*(y) = 1$. Consider $x^* = T^* y^*$. By construction, $\|x^*\| = 1$ and

$$\begin{aligned} T(\operatorname{Slice}(B_X, x^*, \delta)) &= \{Tx \colon x \in B_X,\ \operatorname{Re} x^*(x) > 1 - \delta\} \\ &= \{Tx \colon x \in B_X,\ \operatorname{Re} y^*(Tx) > 1 - \delta\} \subset S. \end{aligned}$$

Now, by Theorem 1.23, we may select an element $y_0 \in \operatorname{Slice}(B_X, x^*, \delta)$ such that $\big\|y_0 + x_0/\|x_0\|\big\| > 2 - \varepsilon$ and hence $\|x_0 + y_0\| > 2 - 2\varepsilon$. But $Ty_0 \in S$, so $\|Ty_0 - x_0\| < \varepsilon$, and we have

$$\|\operatorname{Id} + T\| \geqslant \|[\operatorname{Id} + T](y_0)\| \geqslant \|y_0 + Ty_0\| \geqslant \|y_0 + x_0\| - \varepsilon \geqslant 2 - 3\varepsilon,$$

as desired. □

An analogous result for operators not fixing copies of ℓ_1 was demonstrated by R. Shvidkoy [115]; the proof needs an extension of Theorem 1.23 in which weak open sets instead of slices are considered.

In [72] two more classes of operators related to the Daugavet property were introduced. The definition is a bit technical, but very useful in frames of this theory.

Definition 1.28. Let X, E be Banach spaces.

(a) An operator $T \in \mathcal{L}(X,E)$ is said to be a *strong Daugavet operator* if for every two elements $x,y \in S_X$ and for every $\varepsilon > 0$ there is an element $z \in S_X$ such that $\|z+x\| > 2-\varepsilon$ and $\|Tz-Ty\| < \varepsilon$.
(b) An operator $T \in \mathcal{L}(X,E)$ is said to be *narrow* if for every $x,y \in S_X$, $\varepsilon > 0$ and every slice S of the unit ball of X containing y, there is an element $z \in S$ such that $\|x+z\| > 2-\varepsilon$ and $\|Tz-Ty\| < \varepsilon$.

Every narrow operator is strong Daugavet, and there is an easy connection between strong Daugavet operators and the Daugavet equation.

Theorem 1.29 ([72, Lemma 3.2]). *Let X be a Banach space. If $T \in \mathcal{L}(X)$ is a strong Daugavet operator, then T satisfies the Daugavet equation.*

Proof. We assume without loss of generality that $\|T\| = 1$. Given $\varepsilon \in (0,1/2)$, pick $y \in S_X$ such that $\|Ty\| \geqslant 1-\varepsilon$. If $x = Ty/\|Ty\|$ and z is chosen according to Definition 1.28, then

$$2-\varepsilon < \|z+x\| \leqslant \|z+Ty\| + \varepsilon \leqslant \|z+Tz\| + 2\varepsilon,$$

hence

$$\|\operatorname{Id}+T\| \geqslant \|[\operatorname{Id}+T](z)\| = \|z+Tz\| \geqslant 2-3\varepsilon,$$

which proves the result. □

One can guess that narrow operators are related to C-narrow and PP-narrow introduced earlier. As an easy illustration, let us demonstrate the following.

Proposition 1.30. *Let E be a Banach space and let Ω be a Hausdorff topological space. Then, every C-narrow operator $T \in \mathcal{L}(C_b(\Omega),E)$ is a strong Daugavet operator.*

Observe that this result, together with Theorem 1.29, provides a proof of Theorem 1.18.

Proof (of Proposition 1.30). Consider two arbitrary functions $x,y \in C_b(\Omega)$ with $\|x\| = \|y\| = 1$ and fix $\varepsilon > 0$. Our goal is to find $z \in S_{C_b(\Omega)}$ such that

$$\|z+x\| > 2-\varepsilon \quad \text{and} \quad \|Tz-Ty\| < \varepsilon.$$

Remark that the condition $z \in S_{C_b(\Omega)}$ can be relaxed to $\|z\| \leqslant 1+\varepsilon$: if we divide such a z by its norm, we get what we need with a little bit spoiled ε, which is not significant.

Denote $\delta = \varepsilon/2$. Without loss of generality, we assume that $\sup_{t\in\Omega} \operatorname{Re} x(t) = 1$ (this can be achieved by multiplying x and y by the same modulus-one constant).

Denote $U = (\operatorname{Re} x)^{-1}((1-\delta, +\infty))$. By continuity of y, there is an open subset $V \subset U$ and $c \in \mathbb{C}$ such that $|c| \leqslant 1$ and $|y(t) - c| < \delta$ for all $t \in V$. From the definition of C-narrow operator, there is a non-negative function $g \in S_{C_b(\Omega)}$ with $\sup(g) \subset V$ and $\|Tg\| < \delta$. Consider $z = y + (1-c)g$. For $t \in \Omega \setminus V$ we have $|z(t)| = |y(t)| \leqslant 1$; for $t \in V$ we have

$$\begin{aligned}|z(t)| &= |y(t) - c + c + (1-c)g(t)| \leqslant \delta + |c + (1-c)g(t)| \\ &= \delta + |g(t) + c(1-g(t))| \leqslant \delta + g(t) + (1-g(t)) \leqslant 1 + \delta,\end{aligned}$$

consequently $\|z\| < 1 + \varepsilon$. Further, $\|Tz - Ty\| = |1-c|\,\|Tg\| < 2\delta \leqslant \varepsilon$, and finally

$$\begin{aligned}\|z + x\| &\geqslant \sup_{t \in V} \operatorname{Re}\big(x(t) + z(t)\big) \geqslant 1 - \delta + \sup_{t \in V} \operatorname{Re}\big(y(t) + (1-c)g(t)\big) \\ &\geqslant 1 - 2\delta + \sup_{t \in V} \operatorname{Re}\big(c + (1-c)g(t)\big) \geqslant 2 - 2\delta = 2 - \varepsilon,\end{aligned}$$

finishing thus the proof. □

Note that if Ω is a normal space and has no isolated points, then for every non-void open subset $U \subset \Omega$ we have that the subspace of all $f \in C_b(\Omega)$ with $\operatorname{supp}(f) \subset U$ is infinite-dimensional. Thus every rank-1 operator on $C_b(\Omega)$ is evidently C-narrow, and Proposition 1.30 together with Theorem 1.29 gives a demonstration of the Daugavet property of $C_b(\Omega)$.

One can say more [72]: for operators on $C(K)$ with perfect K, the classes of C-narrow and narrow operators coincide, and although for operators on $L_1(\mu)$ with atomless measure μ the classes of PP-narrow and narrow operators are not the same, every PP-narrow operator is narrow in the new sense. Also, all operators not fixing copies of ℓ_1 and strong Radon-Nikodým operators acting from a space with the Daugavet property are narrow. Finally, let us mention that although the class of narrow operators is not closed under ordinary sums, the sum of operators not fixing copies of ℓ_1 or of strong Radon-Nikodým operators is again narrow.

Analogously to Definition 1.19, the general concept of rich subspace of a Banach space X with the Daugavet property comes in the following natural way:

Definition 1.31. A subspace Y of a Banach space X with the Daugavet property is said to be *rich* if the quotient map $q\colon X \longrightarrow X/Y$ is narrow.

It is demonstrated in [72] that every rich subspace $Y \subset X$ shares the Daugavet property of X. In particular, every subspace $Y \subset X$ of finite codimension in a Banach space with the Daugavet property X also has the Daugavet property. The following result demonstrates that the concept of richness is very natural.

Theorem 1.32 ([72, Theorem 5.12])**.** *For subspace Y of a Banach space X with the Daugavet property, the following conditions are equivalent:*

(i) Y is rich.

(i) Every subspace $E \subset X$ containing Y possesses the Daugavet property.

A detailed survey of the development of the Daugavet equation theory until 2001 can be found in [125]. Besides, the Daugavet equation has been deeply studied in several environments by many mathematicians in the last decades, see the recent papers [5, 19, 65, 112] and references therein, for instance. We finally would like to mention that the Daugavet equation (1.9) is, in some sense, the only possible norm-equality that can be satisfied by all rank-one operators in a Banach space, see [62] for details.

Let us go to the relationship with Banach spaces with numerical index 1. As we already mentioned, $v(T) = \|T\|$ if and only the following equality holds

$$\|\mathrm{Id} + \mathbb{T}T\| = 1 + \|T\| \tag{1.12}$$

(see [97, Lemma 2.3] for an explicit proof). Therefore, it was known since 1970 that every bounded linear operator on $C(K)$ or $L_1(\mu)$ satisfies (1.12), a fact that was rediscovered and reproved in some papers from the eighties and nineties as the ones by Y. Abramovich [1], J. Holub [56], and K. Schmidt [114].

This latest equation was named as the *alternative Daugavet equation* by M. Martín and T. Oikhberg in [97], where the following property was introduced.

Definition 1.33. A Banach space X is said to have the *alternative Daugavet property* (*aDP* for short) if every rank-one operator on X satisfies (1.12).

As before, in the above definition it is sufficient to consider operators T of norm 1 (see Remark 2.2).

Let us comment that, contrary to the Daugavet property, the aDP depends upon the base field (e.g. $\mathbb{C}$ has aDP as a complex space but not as a real space). For more information on the alternative Daugavet property we refer to the already cited paper [97] and also to [91]. From the former one we take the following list of geometric characterizations.

Proposition 1.34 ([97, Propositions 2.1 and 2.6]). *Let X be a Banach space. Then, the following assertions are equivalent.*

(i) *X has the alternative Daugavet property.*

(ii) *For all $x_0 \in S_X$, $x_0^* \in S_{X^*}$ and $\varepsilon > 0$, there is some $x \in S_X$ such that*

$$|x_0^*(x)| \geqslant 1 - \varepsilon \qquad \text{and} \qquad \|x + x_0\| \geqslant 2 - \varepsilon.$$

(ii^*) *For all $x_0 \in S_X$, $x_0^* \in S_{X^*}$ and $\varepsilon > 0$, there is some $x^* \in S_{X^*}$ such that*

$$|x^*(x_0)| \geqslant 1 - \varepsilon \qquad \text{and} \qquad \|x^* + x_0^*\| \geqslant 2 - \varepsilon.$$

(iii) *$B_X = \overline{\mathrm{conv}}\Big(\mathbb{T}\left[B_X \setminus \big(x + (2-\varepsilon)B_X\big)\right]\Big)$ for every $x \in S_X$ and every $\varepsilon > 0$ (see Figure 1.2 below).*

(*iii**) $B_{X^*} = \overline{\operatorname{conv}}^{w^*}\left(\mathbb{T}\left[B_{X^*} \setminus \left(x^* + (2-\varepsilon)B_{X^*}\right)\right]\right)$ *for every* $x^* \in S_{X^*}$ *and every* $\varepsilon > 0$.
(*iv*) $B_{X^* \oplus_\infty X^{**}} = \overline{\operatorname{conv}}^{w^*}\left(\{(x^*, x^{**}) \colon x^* \in \operatorname{ext}(B_{X^*}),\, x^{**} \in \operatorname{ext}(B_{X^{**}}),\, |x^{**}(x^*)| = 1\}\right)$.

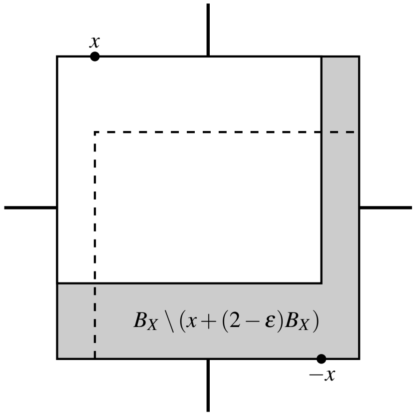


Fig. 1.2 ℓ_∞^2 has the alternative Daugavet property

Analogously to Theorem 1.27, if X possesses the aDP, then (1.12) remains true for all strong Radon-Nikodým operators in X, in particular for all compact and all weakly compact operators [97, Theorem 2.2].

It is clear that both spaces with the Daugavet property and spaces with numerical index 1 have the alternative Daugavet property. Both converses are false: the space $c_0 \oplus_1 C([0,1], \ell_2)$ has the alternative Daugavet property but fails the Daugavet property and its numerical index is not 1 [97, Example 3.2]. Nevertheless, under certain isomorphic conditions, the alternative Daugavet property forces the numerical index to be 1.

Proposition 1.35 ([84, Remark 6])**.** *Let X be a Banach space with the alternative Daugavet property. If X has the Radon-Nikodým property or X is an Asplund space, then* $n(X) = 1$.

With this result in mind, one realizes that the necessary conditions for a real Banach space to be renormed with numerical index 1 given in Section 1.3 (namely Theorem 1.12 and Corollary 1.13), can be written in terms of the alternative Daugavet property. Even more, in the proof of Proposition 1.10 given in [84], only rank-one operators are used and, therefore, it can be also written in terms of the alternative Daugavet property.

Proposition 1.36 ([84, Lemma 1 and Remark 6])**.** *Let X be a Banach space with the alternative Daugavet property. Then,*

(a) $|x^{**}(x^*)| = 1$ *for every* $x^{**} \in \operatorname{ext}(B_{X^{**}})$ *and every* w^**-denting point* $x^* \in B_{X^*}$.
(b) $|x^*(x)| = 1$ *for every* $x^* \in \operatorname{ext}(B_{X^*})$ *and every denting point* $x \in B_X$.

Proposition 1.37 ([97, Remark 2.8]). *Let* X *be an infinite-dimensional real Banach space with the alternative Daugavet property. If* X *has the Radon-Nikodým property, then* X *contains* ℓ_1. *If* X *is an Asplund space, then* X^* *contains* ℓ_1. *In particular,* X^{**}/X *is not separable.*

There is a natural reason why it is difficult to find characterizations of Banach spaces with numerical index 1 that do no involve operators, namely, it is not easy to construct noncompact operators on an abstract Banach space. Thus, when one uses the assumption that a Banach space has numerical index 1, only the alternative Daugavet property can be easily exploited. Of course, things are easier if one is working in a context where the alternative Daugavet property ensures numerical index 1, as it happens with Asplund spaces and spaces with the Radon-Nikodým property. Therefore, it would be desirable to find more isomorphic properties ensuring that the alternative Daugavet property implies numerical index 1. Such a very general property called SCD will appear a few pages later in Section 1.6.

Let us remark that, on the other hand, it is not possible to find isomorphic properties ensuring that the alternative Daugavet property and the Daugavet property are equivalent.

Proposition 1.38 ([97, Corollary 3.3]). *Let* X *be a Banach space with the aDP. Then there exists a Banach space* Y, *isomorphic to* X, *which has the aDP but fails the Daugavet property.*

Indeed, if X fails the Daugavet property then we can take $Y = X$. In the opposite case, we take a one-codimensional subspace $Z \subset X$ and consider $Y = Z \oplus_1 \mathbb{K}$.

1.5 Lush spaces and the duality problem for the numerical index

As we already mentioned in Proposition 1.4

$$n(X^*) \leqslant n(X) \tag{1.13}$$

for every Banach space X. The question if this is actually an equality had been around from the beginning of the subject (see [73, pp. 386], for instance). Let us comment some partial results which led to think that the answer could be positive. Namely, it is clear that $n(X) = n(X^*)$ for every reflexive space X, and this equality also holds whenever $n(X^*) = 1$, in particular when X is an L- or an M-space. It is also true that $n(X) = n(X^*)$ when X is a C^*-algebra or a von Neumann algebra predual [73]. Moreover, if X is L-embedded in its bidual, then $n(X) = n(X^*)$; if X is an M-ideal of its bidual and $n(X) = 1$, then $n(X^*) = n(X^{**}) = 1$ [92].

Nevertheless, in 2007 K. Boyko, V. Kadets, M. Martín, and D. Werner [22] answered the question in the negative by giving an example of a Banach space whose numerical index equals 1 and it is strictly greater than the numerical index of its dual. The answer was given in three steps: at first, a new sufficient condition for $n(X)=1$ (called *lushness*) was introduced, at second, it was demonstrated that all C-rich subspaces of $C(K)$ are lush, and finally, the example in question (which is a C-rich subspace of a $C(K)$ space) was constructed. In this section we will review all these steps, and afterwards present some results and applications of lush spaces that were obtained in the last decade.

Definition 1.39. We say that a Banach space X is *lush* if for every $x,y\in S_X$ and every $\varepsilon>0$, there exists $y^*\in S_{Y^*}$ such that $y\in \operatorname{Slice}(B_X,y^*,\varepsilon)$ and

$$\operatorname{dist}\big(x,\operatorname{conv}\big(\mathbb{T}\operatorname{Slice}(B_X,y^*,\varepsilon)\big)\big)<\varepsilon.$$

Evidently, every almost-CL-space is lush. The proof of the fact that almost-CL-spaces have numerical index 1 can be straightforwardly extended to lush spaces.

Proposition 1.40. *Let X be a lush Banach space. Then, $n(X)=1$.*

Proof. For $T\in\mathcal{L}(X)$ with $\|T\|=1$, and $0<\varepsilon<1/2$ fixed, we take $x_0\in S_X$ such that $\|Tx_0\|>1-\varepsilon$, and we apply the definition of lushness to x_0 and $y_0=\dfrac{Tx_0}{\|Tx_0\|}$ to get $y^*\in S_{Y^*}$ with $y_0\in\operatorname{Slice}(B_X,y^*,\varepsilon)$ and $x_1,\dots,x_n\in\operatorname{Slice}(B_X,y^*,\varepsilon)$, $\theta_1,\dots,\theta_n\in\mathbb{T}$ such that a convex combination $z=\sum\lambda_k\theta_kx_k$ of the elements $\theta_1x_1,\dots,\theta_nx_n$ approximates x_0 up to ε. Then

$$|y^*(Tz)|=\left|y^*(y_0)-y^*\left(T\left(\frac{x_0}{\|Tx_0\|}-z\right)\right)\right|>1-4\varepsilon.$$

On the other hand, $y^*(Tz)$ is a convex combination of $y^*(\theta_1Tx_1),\dots,y^*(\theta_nTx_n)$, so there is an index j such that

$$|y^*(Tx_j)|=|y^*(\theta_jTx_j)|>1-4\varepsilon.$$

Now, we have

$$\begin{aligned}\|\operatorname{Id}+\mathbb{T}T\|&\geqslant\max_{\omega\in\mathbb{T}}\big|y^*\big([\operatorname{Id}+\omega T](x_j)\big)\big|\geqslant\max_{\omega\in\mathbb{T}}\big|y^*(x_j)+\omega y^*(Tx_j)\big|\\&=|y^*(x_j)|+|y^*(Tx_j)|>2-5\varepsilon.\end{aligned}$$

This shows that $n(X)=1$ by Proposition 1.1. □

The demonstration of the following theorem (in a generalized form) will be given in Section 4, Theorem 4.6.

Theorem 1.41. *Let Ω be a Hausdorff topological space and let X be a C-rich subspace of $C_b(\Omega)$. Then, X is lush and, therefore, $n(X) = 1$.*

We are now able to present the promised example of a Banach space with numerical index 1 and whose dual does not share the property.

Example 1.42. Let us consider the countable compact subset of $\mathbb{R}$ given by

$$K = \left\{1 - \tfrac{1}{n+1} : n \in \mathbb{N}\right\} \cup \left\{2 - \tfrac{1}{n+1} : n \in \mathbb{N}\right\} \cup \left\{3 - \tfrac{1}{n+1} : n \in \mathbb{N}\right\} \cup \{1,2,3\}$$

and define the Banach space

$$X = \left\{f \in C(K) \colon f(1) + f(2) + f(3) = 0\right\}.$$

Then, X is C-rich in $C(K)$ (so $n(X) = 1$) and $n(X^*) < 1$.

Proof. Consider $L = \{1,2,3\} \subset K$ and observe that $U = K \setminus L$ is open and dense in K. Then, as $C_0(K\|L) \subset X \subset C(K)$, it follows from Corollary 1.21 that X is C-rich in $C(K)$ and so Theorem 1.41 gives us that $n(X) = 1$. Let us show that $n(X^*) < 1$. Write $\nu = \delta_1 + \delta_2 + \delta_3 \in C(K)^*$ and observe that $X = \ker \nu$. Since K is countable, every measure μ on K is purely atomic and can be written as $\mu = \sum_{t\in K} a_t \delta_t$ with $\|\mu\| = \sum_{t\in K} |a_t|$. Consequently, $C(K)^*$ can be written as $C(K)^* = Y \oplus_1 Z$, where Y consists of measures concentrated on isolated points of K, and $Z = \operatorname{span}\{\delta_1, \delta_2, \delta_3\}$. Now,

$$X^* = \left[C(K)^*\right]/\operatorname{span}\{\nu\} = Y \oplus_1 (Z/\operatorname{span}\{\nu\}),$$

where in the last equality we have used that $\nu \in Z$. By Proposition 1.5,

$$n(X^*) \leqslant n(Z/\operatorname{span}\{\nu\}).$$

But $Z/\operatorname{span}\{\nu\}$ is isometric to the two-dimensional space $W = \ell_1^3/\operatorname{span}\{(1,1,1)\}$ and it routinely follows from Theorem 1.9 that $n(W) < 1$, so the proof is done. Actually, in the real case the unit ball of W is a hexagon (its unit ball has 6 extreme points, see Figure 1.3) and $n(W) = 1/2$ by [94, Theorem 1]. □

Remark that the same space gives an example of an Asplund space that has numerical index 1, but is not an almost-CL-space [22, Example 3.4].

With just a little bit of work, Example 1.42 can be pushed out to produce even better counterexamples.

Proposition 1.43 **([22, Examples 3.3]).**

(a) There exists a real Banach space X such that $n(X) = 1$ and $n(X^*) = 0$.
(b) There exists a complex Banach space X such that $n(X) = 1$ and $n(X^*) = 1/\mathrm{e}$.

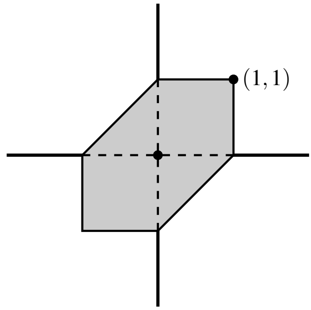


Fig. 1.3 The unit ball of W

Let us comment that the example X given in [22, Examples 3.3] which is shown in item (a) above satisfies that $n(X)=1$, $n(X^*)=0$ but the only operator $T\in\mathcal{L}(X^*)$ with $v(T)=0$ is $T=0$. Therefore, as shown in Proposition 1.7, both X and X^* fail to have one-parameter uniformly continuous semigroup of isometries. It is not too complicated to get a better result with the ideas of this section.

Example 1.44 ([90]). Let $K=[0,1]\times[0,1]$, let $L=[0,1]\times\{0\}$ (which is closed and nowhere dense in K), and consider $E=\ell_2$ viewed as a subspace of $C(L)$. Then the space $X=C_E(K\|L)$ has numerical index 1 but there are infinitely many linearly independent operators $T\in\mathcal{L}(X^*)$ with $v(T)=0$. In particular, X fails to have one-parameter uniformly continuous semigroups of isometries, while X^* contains infinitely many of them; equivalently, the Lie algebra of X is trivial while the one of X^* is infinite-dimensional.

The idea of the proof is the following. On the one hand, X is C-rich in $C(K)$ by Corollary 1.21, so $n(X)=1$ by Theorem 1.41. On the other hand,

$$X^*\equiv C_0(K\|L)^*\oplus_1 E^*$$

(see the proof of [90, Theorem 3.3] for the details), so every operator in $\mathcal{L}(E^*)$ with numerical radius 0 naturally extends to an operator on X^* with numerical radius 0 [90, Proposition 2.4]. But now, we just recall that there are infinitely many linearly independent operators with numerical radius zero in $E^*\equiv\ell_2$.

Squeezing the above construction and using hard topological constructions, P. Koszmider, M. Martín, and J. Merí [76] provided in 2011 with the following surprising example.

Example 1.45 ([76, Example 6.3]). There exists a real Banach space X whose only surjective isometries are $\pm\mathrm{Id}$ but such that X^* admits infinitely many one-parameter uniformly continuous semigroups of isometries.

Actually, the space X above is constructed as a space $C_E(K\|L)$ where K is a compact Hausdorff topological space constructed in the same paper with "exotic" topological properties, see [76, Theorem 6.1] for the details.

Once we know that the numerical index of a Banach space and the one of its dual do not coincide, another natural question could be if two isometric preduals of a given Banach space should have the same numerical index. The answer is again negative.

Proposition 1.46 ([22, Examples 3.6]). *There is a Banach space Z (in fact $Z \simeq \ell_1$) with two isometric preduals X_1 and X_2 such that $n(X_1)$ and $n(X_2)$ are not equal.*

A very useful example of C-rich subspace comes from the inclusion $c_0 \subset \ell_\infty$ and some topological argument. Namely, $\ell_\infty = C(\beta\mathbb{N})$, where $\beta\mathbb{N}$ is the Stone-Čech compactification of $\mathbb{N}$. Then, $\mathbb{N}$ is a dense open subset of $\beta\mathbb{N}$, $c_0 = C_0(\beta\mathbb{N}\|[\beta\mathbb{N}\setminus\mathbb{N}])$ and Corollary 1.21 applies. This example leads to the following renorming theorem which, to the best of our knowledge, is the only general renorming theorem for numerical index 1 spaces that is known by now. The statement below is a little bit generalized version of the original [21, Corollary 3.6.].

Recall that a *countable norming system of functionals* of a Banach space X is a bounded subset $\{x_n^*\colon n \in \mathbb{N}\}$ of X^* for which there is a constant $K \geqslant 0$ such that

$$\|x\| \leqslant K \sup_{n\in\mathbb{N}} \left|x_n^*(x)\right| \qquad (x \in X).$$

Banach spaces with a countable norming system of functionals are those for which there is a bounded subset of the dual with non-empty interior which is weak-star separable or, equivalently, those which are isomorphic to closed subspaces of ℓ_∞, see [33, p. 254] for instance.

Proposition 1.47 (extended [21, Corollary 3.6]). *Every Banach space containing an isomorphic copy of c_0 and possessing a countable norming system of functionals (in particular, every separable space containing a copy of c_0) can be equivalently renormed to be lush and, in particular, to have numerical index* 1.

Proof. Let X be a Banach space containing an isomorphic copy of c_0 and possessing a countable norming system of functionals. Then, the Lindenstrauss-Rosenthal theorem [83, Theorem 2.f.12(i)] implies that X is isomorphic to a closed subspace X_1 of ℓ_∞ containing the canonical copy of c_0 inside ℓ_∞ (see [61, Lemma 4.2] for details). This X_1 is a C-rich subspace of ℓ_∞ by Corollary 1.21, so it is lush by Theorem 1.41. □

The multiple applications of lushness motivated a detailed study of this property in the last decade. In the following we present some of the most important results obtained. We start with one of the main features of lushness: it is a separably determined property.

Proposition 1.48 ([21, Theorem 4.2]). *A Banach space X is lush if and only if every separable subspace $E \subset X$ is contained in a separable lush subspace $Y \subset X$.*

This result becomes very useful when it is combined with the following fact about separable lush spaces.

Theorem 1.49 ([63, Theorem 4.3] and [78, Proposition 2.1]). *Let X be a separable lush space. Then, there exists a G_δ subset $\widetilde{K}$ of S_{X^*} which is norming for X and satisfies that*

$$B_X = \overline{\text{aconv}}\big(\text{Face}(B_X, x^*)\big)$$

for every $x^* \in \widetilde{K}$.

The above two results combined provide with the following result.

Corollary 1.50 ([63, Corollary 4.9]). *The dual of every infinite-dimensional real lush space contains a copy of ℓ_1.*

Indeed, let X be an infinite-dimensional real lush space. By Proposition 1.48 and the lifting property of ℓ_1, we may suppose that the space is separable. Once in this case, it is an easy consequence of Theorem 1.49 that

$$|x^{**}(x^*)| = 1$$

for every $x^* \in \widetilde{K}$ and every $x^{**} \in \text{ext}(B_{X^{**}})$ (use Lemma 2.5.b). As X is infinite-dimensional and $\widetilde{K}$ is norming for X, it has infinite cardinal, and then Proposition 1.11 gives us that X^* contains c_0 or ℓ_1. But a dual space contains ℓ_∞ (hence ℓ_1) at the moment it contains c_0. See Theorem 6.1 for a detailed proof of a more general result.

As it happens for Banach spaces with numerical index 1 and for the aDP, lushness has good stability properties under some usual Banach space operations. The next proposition summarizes the known results, which can be found in [21, Corollary 4.4, Proposition 5.1, Theorem 5.2, and Proposition 5.3] and [104, Proposition 1, Theorem 2, Theorem 3].

Proposition 1.51. *Let X be a Banach space, let $\{X_n : n \in \mathbb{N}\}$ be a countable family of Banach spaces, let K be a compact Hausdorff topological space, $\mathscr{U}$ be a free ultrafilter on $\mathbb{N}$, and let $E = (\mathbb{R}^n, \|\cdot\|)$ be a Banach space with an absolute norm.*

(a) E is lush if and only if for every collection $X_1, X_2, \ldots, X_n$ of lush spaces, their E-direct sum $X = \big[X_1 \oplus X_2 \oplus \ldots \oplus X_n\big]_E$ is lush.
(b) Lushness is inherited by L-summands and M-ideals (in particular, by M-summands).
(c) If X_n is lush for every $n \in \mathbb{N}$, so are $(X_n)_{\mathscr{U}}$, $[\oplus_{n\in\mathbb{N}} X_n]_{c_0}$, $[\oplus_{n\in\mathbb{N}} X_n]_{\ell_1}$, and $[\oplus_{n\in\mathbb{N}} X_n]_{\ell_\infty}$.
(d) If X is lush, so is $C(K,X)$.

C-rich subspaces of $C(K)$ spaces form the most important class of lush spaces. The next result shows that, in the real case, C-richness can be characterized in terms of lushness when K is perfect.

Proposition 1.52 ([64, Theorem 6.2]). *Let K be a perfect compact space and let Y be a subspace of the real space $C(K)$. Then, Y is C-rich if and only if every subspace $Z \subset X$ containing Y is lush.*

It is also remarked in [64] that the situation for $L_1[0,1]$ is completely different to that of $C(K)$, as no one-codimensional subspace of $L_1[0,1]$ is lush.

To conclude our review about lush spaces, we recall two kinds of obstructive results concerning them. The first one shows that, in the real case, lush spaces cannot be strictly convex nor smooth, unless they are one-dimensional [63, Corollary 4.6]. The second one asserts that the only lush separable rearrangement invariant space on $[0,1]$ is $L_1[0,1]$ and that the only lush separable rearrangement invariant spaces on $\mathbb{N}$ are c_0 and ℓ_1 [66, Theorem 3.3 and Theorem 4.2].

Most of the results presented above will be demonstrated in a more general form and mainly with simplified proofs in the main part of our book. Let us only remark that, although all classical spaces possessing numerical index 1 are lush, with some effort one can construct an example of non-lush space X with $n(X) = 1$ [64, Theorem 4.1], a fact that will be proved and generalized in subsection 4.3.1 of this book.

Lushness property has been also used to study polynomial numerical indices of Banach spaces [44, 75, 78]. Let us finally say that lushness is surprisingly related to the study of Tingley's problem about extensions of surjective isometries between unit spheres of Banach spaces [117] and to the study of norm attaining operators [28, 74].

1.6 Slicely countably determined sets, spaces, and operators

One of the milestones of the theory was reached in the 2010 paper "Slicely countably determined Banach spaces" by A. Avilés, V. Kadets, M. Martín, J. Merí, and V. Shepelska [9], where a very general additional condition was found, ensuring, in particular, that the alternative Daugavet property implies lushness (and hence implies numerical index 1).

Definition 1.53 ([9]). Let X, Y be Banach spaces, $A \subset X$ be a bounded subset. A countable family $\{U_n \colon n \in \mathbb{N}\}$ of non-empty subsets of A is called *determining* for A if for each $B \subset X$ that intersects all the U_n with $n \in \mathbb{N}$, it holds that $A \subset \overline{\mathrm{conv}}(B)$. The set A is said to be *slicely countably determined* (*SCD* in short) if there exists a countable family of slices which is determining for A. The space X is said to be *slicely countably determined* (or *SCD* in short) if every convex bounded subset of X is SCD. Finally, a bounded linear operator $T \in \mathcal{L}(X,Y)$ is an *SCD operator* if $T(B_X)$ is an SCD subset of Y.

Note that every SCD set is clearly separable. Also, it follows routinely from the definition that a bounded set is SCD if and only if its closure is, see [9, Remark 2.7]. Let us further remark that the Hahn-Banach separation theorem leads to the following reformulation of the definition of determining sequence.

Lemma 1.54 **(**[9, Proposition 2.2] and [68, Lemma 1.2]**).** *Let X be a Banach space and let $A \subset X$ be a bounded set. A sequence $\{V_n : n \in \mathbb{N}\}$ of non-empty subsets of A is determining for A if every slice of A contains one of the V_n.*

It is routine to show that we may replace in the definition of an SCD set the sequence of slices by a sequence of convex combinations of slices. Now, a well-known result of J. Bourgain (see [119, Lemma 7.3], for instance) shows that every relatively weakly open subset of a *convex* bounded subset contains a convex combination of slices. Therefore, *for convex sets*, we may replace in the definition of an SCD set the sequence of slices by a sequence of relatively weakly open subsets.

Proposition 1.55 **(**[9, Proposition 2.18]**).** *Let X be a Banach space and let $A \subset X$ be a bounded and* convex *set. If there exits a determining sequence of relatively weakly open subsets of A, then A is SCD.*

This result is not true for non-convex sets, see [68, Proposition 2.6]. On the other hand, we will show in Proposition 7.17 that a bounded set A is SCD if and only if $\operatorname{conv}(A)$ is SCD.

The next result contains the main examples of SCD and non-SCD convex sets, spaces, and operators.

Examples 1.56 **(**[9]**).** Let X, Y be Banach spaces.

(a) A separable convex bounded subset A of X is SCD provided:
  (a.1) A has the convex point of continuity property; in particular, A has the Radon-Nikodým property.
  (a.2) A does not contain ℓ_1-sequences; in particular, A is Asplund.
(b) The following conditions on X imply that every separable subspace of X is SCD:
  (b.1) X has the convex point of continuity property; in particular, X has the Radon-Nikodým Property.
  (b.2) X does not contain copies of ℓ_1; in particular, X is Asplund.
(c) If a Banach space has the Daugavet property, then its unit ball is not an SCD set. Indeed, it follows from Proposition 1.26 that given $x_0 \in S_X$ and a sequence of slices $(S_n)_{n\in\mathbb{N}}$ of S_X we can find $x_n \in S_n$ for each $n \in \mathbb{N}$ so that the sequence $(x_n)_{n=0}^{\infty}$ is equivalent to the canonical basis of ℓ_1, and consequently x_0 does not

belong to the closed linear hull of $\{x_n : n \in \mathbb{N}\}$. In particular, the unit balls of $C[0,1]$ and $L_1[0,1]$ are not SCD sets.

(d) The following conditions on an operator $T \in \mathcal{L}(X,Y)$ guarantee that its restriction to every separable subspace of X is SCD:
   (d.1) T does not fix any copy of ℓ_1.
   (d.2) $T(B_X)$ has the convex point of continuity property; in particular, T is a strong Radon-Nikodým operator.

Let us comment that the proofs of the fact that the main examples given in (a) are SCD sets are of different nature. For (a.1) and for separable Asplund sets, the proofs are elementary, see [9, Proposition 2.8, Example 2.9, Example 2.12]. On the other hand, for subsets which do not contain copies of ℓ_1 the proof is involved and needs a highly non-trivial result by S. Todorčević: if a separable bounded convex set A does not contain ℓ_1 sequences, then A admits a countable π-base of the relative weak topology (i.e. a family of relatively weakly open subsets of A such that any other relatively weakly open subset of A contains one elements of the family). Now, this countable family of weakly open subsets is determining by Lemma 1.54 and then, Proposition 1.55 shows that A is SCD. See the proof of [9, Theorem 2.22] for the details.

Another class of examples: the unit ball of every space with a 1-unconditional basis is SCD [66, Theorem 3.1] and the unit ball of a locally uniformly rotund separable Banach space is also SCD [9, Example 2.10]. It is an open question whether every Banach space with an unconditional basis is an SCD space.

The main applications of the SCD property in our context are the following.

Proposition 1.57 ([9, §4]). *Let X be a Banach space. If X is separable, has the aDP and B_X is SCD, then X is lush. If X is non-separable, has the aDP, and B_Y is SCD for every separable subspace Y of X, then X is lush. In particular, if X has the aDP and it has the convex point of continuity property, the Radon-Nikodým Property, or it does not contain copies of ℓ_1, then X is lush.*

Proposition 1.58 ([9, §5]). *Let X be a Banach space with the aDP. Then, for every $T \in \mathcal{L}(X)$ such that $T(B_Y)$ is SCD for every separable subspace Y of X, one has $\|\mathrm{Id} + \mathbb{T}T\| = 1 + \|T\|$. In particular, this happens if $T(B_X)$ has the convex point of continuity property, the Radon-Nikodým Property, or if T does not fix copies of ℓ_1.*

Observe that SCD sets are separable, and we do not know whether numerical index 1 is separably determined. However, the aDP and lushness are (see Proposition 3.7 and Theorem 3.14 for generalizations of these facts), facts which were crucial in the way the results above were proved.

From these results we may provide with the more general necessary condition for a real Banach space to be renormed with numerical index 1 that we know.

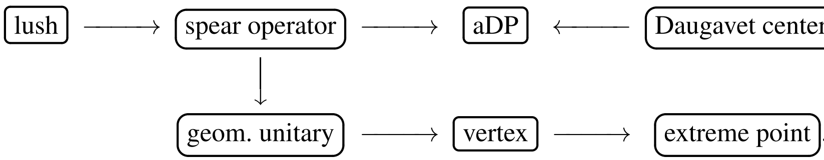
Corollary 1.59 ([9, Corollary 4.9])**.** *Let X be a real infinite-dimensional Banach space. If X admits an equivalent norm with the aDP (in particular, with numerical index* 1*), then X^* contains ℓ_1.*

Indeed, if X itself contains ℓ_1, then X^* contains ℓ_1. Otherwise, Proposition 1.57 gives that X can be renormed to be lush and, once in this case, Corollary 1.50 gives that X^* contains ℓ_1. Let us comment that this result will be generalized in Theorem 6.1.

The applications of SCD sets to the theory of Daugavet equation are of the same nature. For example, the following results are shown in [9, Proposition 5.8 and Theorem 5.11]: if X possesses the Daugavet property, $T \in \mathcal{L}(X,Y)$ and $T(B_X)$ is SCD, then the operator T is a strong Daugavet operator; if, moreover, all convex closed subsets of $T(B_X)$ are SCD, then T is narrow . More applications in this vein can be found in [17, Section 3] and [65, Theorems 3.4 and 3.7].

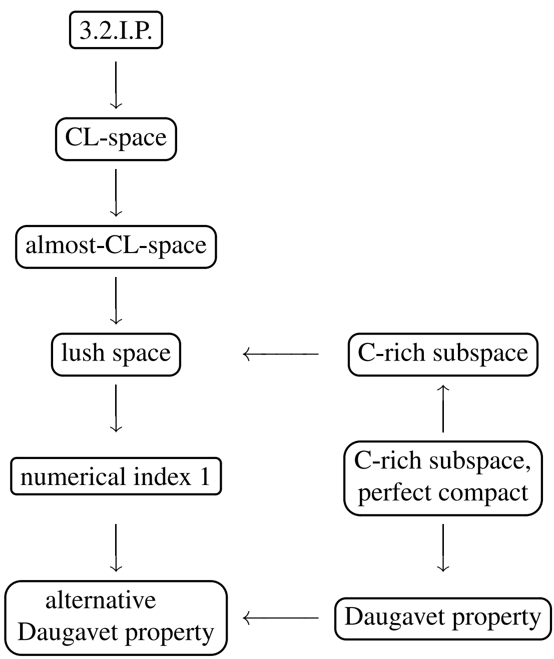
1.7 A pair of diagrams

We finish the introduction with a pair of diagrams. The first one shows the relationships between the properties of a norm-one operator $G \in \mathcal{L}(X,Y)$ that we have presented so far.



None of the implications above reverses, and Daugavet centers and spear operators do not imply each other.

When the above diagram is particularized to the case when $G = \mathrm{Id}$, we have introduced many more properties. The relationship between all of them is summarized in the diagram below.



Again, none of the implications above can be reversed, and the Daugavet property and numerical index 1 do not imply each other. On the other hand, lush spaces and spaces with the alternative Daugavet property are equivalent for Banach spaces with the Radon-Nikodým property and for Banach spaces not containing copies of ℓ_1.

Chapter 2

Spear vectors and spear sets

The following definition will be crucial in our further discussion.

Definition 2.1 **(**[7, Definition 4.1]**).** Let X be a Banach space. An element $z \in S_X$ is a *spear vector* (or *spear*) if $\|z + \mathbb{T}x\| = 1 + \|x\|$ for every $x \in X$. We write $\mathrm{Spear}(X)$ to denote the set of all elements of a Banach space X which are spear.

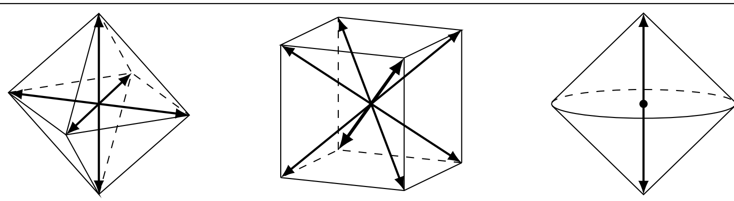


Fig. 2.1 Spear vectors in the real spaces ℓ_1^3, ℓ_∞^3 and $\ell_2^2 \oplus_1 \mathbb{R}$, respectively.

As we commented in section 1.2, $z \in \mathrm{Spear}(X)$ if and only if $N(X,z) = 1$. In particular, the definition was motivated in [7] by the fact that Id_X is a spear element of $\mathcal{L}(X)$ if and only if X has numerical index 1. As we already mentioned in the introduction, the concept of spear vector appeared, without name, in the paper [80] by Å. Lima about intersection properties of balls. It had also appeared tangentially in the monograph [81] by J. Lindenstrauss about extension of compact operators.

Observe that we are dealing with the norm-equation

$$\|x + y\| = \|x\| + \|y\|$$

which is obviously true for collinear vectors looking in the same direction, being the converse result true in strictly convex spaces, but not in general Banach spaces. Let us emphasize here that this equality for a pair of vectors is equivalent to the same statements for all positive multiples of the vectors, which allows us to use only norm-one elements in the definition of spear vector.

Remark 2.2. *Let X be a Banach space.*

(a) *If $x,y \in X$ satisfy that $\|x+y\| = \|x\| + \|y\|$, then*

$$\|ax+by\| = a\|x\| + b\|y\|$$

for every $a,b>0$.

(b) *$z \in S_X$ is a spear vector if and only if $\|z+\mathbb{T}x\| = 2$ for every $x \in S_X$.*

Indeed, by symmetry we may assume $a \geqslant b$. Then,

$$\|ax+by\| = \|a(x+y)-(a-b)y\| \geqslant a\|(x+y)\| - (a-b)\|y\| = a\|x\| + b\|y\|.$$

Observe that geometrically speaking, $x,y \in S_X$ satisfy $\|x+y\| = 2$ if and only if the whole segment

$$[x,y] := \{tx+(1-t)y : t \in [0,1]\}$$

connecting x and y belongs to S_X.

The next notion extends the definition of spear from vectors to sets.

Definition 2.3. Let X be a Banach space. $F \subset B_X$ is called a *spear set* if

$$\|F+\mathbb{T}x\| = 1 + \|x\|$$

for every $x \in X$.

Observe that if $F \subset B_X$ is a spear set, then every subset of B_X containing F is also a spear set. In particular, if a subset F of B_X contains a spear vector, then F is a spear set. On the other hand, it is not true that every spear set contains a spear vector: in every Banach space X, $F = S_X$ is obviously a spear set, but there are Banach spaces containing no spear vectors at all (for instance, a two-dimensional Hilbert space, see Example 2.12.f).

We start the exposition with the following fundamental result.

Theorem 2.4. *Let X be a Banach space, let F be a subset of B_X and let $\mathscr{A} \subset B_{X^*}$ with $B_{X^*} = \overline{\operatorname{conv}}^{w^*}(\mathscr{A})$. The following statements are equivalent:*

(i) $\|F+x\| = 1 + \|x\|$ *for every* $x \in X$.

(ii) $B_{X^*} = \overline{\operatorname{conv}}^{w^*}(\operatorname{gSlice}(\mathscr{A},F,\varepsilon))$ *for every* $\varepsilon > 0$.

(iii) $B_{X^*} = \overline{\operatorname{conv}}^{w^*}(\operatorname{gSlice}(\operatorname{ext} B_{X^*},F,\varepsilon))$ *for every* $\varepsilon > 0$.

(iv) $\operatorname{gFace}(\operatorname{ext} B_{X^*},F)$ *is a dense subset of* $(\operatorname{ext} B_{X^*},w^*)$.

If $X = Y^*$ is a dual Banach space, this is also equivalent to

(v) $B_Y = \overline{\operatorname{conv}}(\operatorname{gSlice}(S_Y,F,\varepsilon))$ *for every* $\varepsilon > 0$.

Moreover, remark that the set $\operatorname{gFace}(\operatorname{ext} B_{X^*},F)$ is G_δ, a fact which makes item (iv) more applicable.

Previously to give the proof of the theorem, we recall some properties of the set of extreme points of compact convex sets which will be needed here and all along the book. When $A \subset X$ is convex and it is compact in a locally convex Hausdorff topology τ, then the set $\operatorname{ext} A$ of its extreme points, outside of being not empty and generating the whole A as the τ-closure of its convex hull (Krein-Milman theorem), has many good topological properties which are less known as the following ones which we will profusely use all along the book.

Lemma 2.5. *Let X be a Banach space and let $A \subset X^*$ convex and weak-star compact.*

(a) (Choquet's Lemma) If $x^* \in \operatorname{ext} A$, then for every weak-star neighborhood U of x^* in A there is a w^*-slice S of A such that $x^* \in S \subset U$. In other words, the w^*-slices of A containing x^* form a base of the relative weak-star neighborhoods of x^* in A.
(b) (Milman's Theorem) If $D \subset A$ satisfies that $\overline{\operatorname{conv}}^{w^*}(D) \supset A$, then $\overline{D}^{w^*} \supset \operatorname{ext} A$.
(c) $(\operatorname{ext} A, w^*)$ is a Baire space, so the intersection of every sequence of G_δ dense subsets of $\operatorname{ext} A$ is again (G_δ) dense.

These are well-known results which can be found, for instance, in the volume 2 of the *Lectures on Analysis* by G. Choquet [29]. Concretely, assertion (a) can be found in [29, p. 107]; (b) is an immediate consequence of (a) and Hahn-Banach separation Theorem; (c) appears in [29, p. 146, Theorem 27.9].

Proof (of Theorem 2.4). (i) $\Rightarrow$ (ii): Given $\varepsilon > 0$, we just have to check that every w^*-slice S of B_{X^*} intersects $\operatorname{gSlice}(\mathscr{A}, F, \varepsilon)$. We can assume that $S = \operatorname{Slice}(B_{X^*}, x_0, \delta)$ for $x_0 \in S_X$ and $\varepsilon > \delta > 0$. Using (i) and the condition on $\mathscr{A}$, we can find $x_0^* \in \mathscr{A}$ and $z_0 \in F$ such that

$$\operatorname{Re} x_0^*(z_0) + \operatorname{Re} x_0^*(x_0) > 2 - \delta.$$

In particular, $\operatorname{Re} x_0^*(z_0) > 1 - \delta$ and $\operatorname{Re} x_0^*(x_0) > 1 - \delta$, so $x_0^* \in S \cap \operatorname{gSlice}(\mathscr{A}, F, \varepsilon)$.

(ii) $\Rightarrow$ (i): Given $x \in S_X$ and $\varepsilon > 0$, the hypothesis allows us to find $x^* \in \operatorname{gSlice}(\mathscr{A}, F, \varepsilon)$ such that $\operatorname{Re} x^*(x) > 1 - \varepsilon$. Also, by definition of $\operatorname{gSlice}(\mathscr{A}, F, \varepsilon)$, there is $z \in F$ such that $\operatorname{Re} x^*(z) > 1 - \varepsilon$. Now,

$$\|F + x\| \geqslant \|z + x\| \geqslant \operatorname{Re} x^*(z) + \operatorname{Re} x^*(x) > 2 - 2\varepsilon,$$

and the arbitrariness of ε gives the result.

The equivalence between (i) and (iii) is just a particular case of the already proved equivalence between (i) and (ii) since $\mathscr{A} = \operatorname{ext} B_{X^*}$ satisfies the condition above by the Krein-Milman Theorem.

(iii) $\Rightarrow$ (iv): For each $\varepsilon > 0$, $\operatorname{gSlice}(\operatorname{ext} B_{X^*}, F, \varepsilon)$ is a relatively weak-star open subset of $\operatorname{ext} B_{X^*}$, as it can be written as union of w^*-slices. Moreover, condition (iii) together with Milman's Theorem (see Lemma 2.5.b) yields that the set $\operatorname{gSlice}(\operatorname{ext} B_{X^*}, F, \varepsilon)$ is weak-star dense in $\operatorname{ext} B_{X^*}$. Using that the set $(\operatorname{ext} B_{X^*}, w^*)$ is a Baire space (see Lemma 2.5.c), we conclude that

$$\mathrm{gFace}(\mathrm{ext}\, B_{X^*}, F) = \bigcap_{n\in\mathbb{N}} \mathrm{gSlice}(\mathrm{ext}\, B_{X^*}, F, 1/n) \tag{2.1}$$

satisfies the properties above.

(iv) $\Rightarrow$ (iii): Given $\varepsilon > 0$, since $\mathrm{gFace}(\mathrm{ext}\, B_{X^*}, F) \subset \mathrm{gSlice}(\mathrm{ext}\, B_{X^*}, F, \varepsilon)$ we deduce that this last set is also dense in $(\mathrm{ext}\, B_{X^*}, w^*)$, and so using the Krein-Milman Theorem we conclude that

$$B_{X^*} = \overline{\mathrm{conv}}^{w^*}(\mathrm{ext}\, B_{X^*}) \subset \overline{\mathrm{conv}}^{w^*}(\mathrm{gSlice}(\mathrm{ext}\, B_{X^*}, F, \varepsilon)).$$

Finally, if $X = Y^*$, (v) is a particular case of (ii) with $\mathscr{A} = S_Y$ by Goldstine's Theorem.

The "moreover" part follows from equation (2.1). □

Now we may present a characterization of spear sets which is an easy consequence of the above theorem and the fact that $\|F + \mathbb{T}x\| = \|\mathbb{T}F + x\|$ for every set F and every vector x.

Corollary 2.6. *Let X be a Banach space and let $\mathscr{A} \subset B_{X^*}$ with $\overline{\mathrm{conv}}^{w^*}(\mathscr{A}) = B_{X^*}$. For $F \subset B_X$, the following assertions are equivalent:*

(i) F is a spear set, i.e. $\|F + \mathbb{T}x\| = 1 + \|x\|$ for each $x \in X$.
(ii) $B_{X^*} = \overline{\mathrm{aconv}}^{w^*}(\mathrm{gSlice}(\mathscr{A}, F, \varepsilon))$ for every $\varepsilon > 0$.
(iii) $\mathrm{gFace}(\mathrm{ext}\, B_{X^*}, \mathbb{T}F)$ is a dense G_δ subset of $(\mathrm{ext}\, B_{X^*}, w^*)$.

If $X = Y^*$ is a dual Banach space, this is also equivalent to

(iv) $B_Y = \overline{\mathrm{aconv}}(\mathrm{gSlice}(S_Y, F, \varepsilon))$ for every $\varepsilon > 0$.

The following result is of interest in the complex case.

Proposition 2.7. *Let X be a Banach space. If $F \subset B_X$ is a spear set, then*

$$\|F \pm x\|^2 \geqslant 1 + \|x\|^2$$

for every $x \in X$.

Proof. Let $x \in X$ and $\varepsilon > 0$. Using Corollary 2.6.ii, we get that

$$\begin{aligned}
\|F \pm x\|^2 &\geqslant \sup\{|x^*(z) \pm x^*(x)|^2 : z \in F, x^* \in \mathrm{gSlice}(S_{X^*}, F, \varepsilon)\} \\
&\geqslant \sup\{|x^*(z)|^2 + |x^*(x)|^2 : z \in F, x^* \in \mathrm{gSlice}(S_{X^*}, F, \varepsilon)\} \\
&\geqslant \sup\{(1-\varepsilon)^2 + |x^*(x)|^2 : x^* \in \mathrm{gSlice}(S_{X^*}, F, \varepsilon)\} \\
&= (1-\varepsilon)^2 + \|x\|^2,
\end{aligned}$$

giving the result. □

The case in which a spear set is a singleton coincides, of course, with the concept of spear vector of Definition 2.1. Most of the assertions of the next corollary follow from Corollary 2.6 and the fact that

$$\operatorname{gFace}(\operatorname{ext} B_{X^*}, \mathbb{T}z) = \{x^* \in \operatorname{ext} B_{X^*} : |x^*(z)| = 1\}$$

is weak-star closed. The other ones are consequences of the general results of the theory of numerical range spaces (see section 1.2) and we have taken them from [23, Section 2.1].

Corollary 2.8. *Let X be a Banach space and let $\mathscr{A} \subset B_{X^*}$ with $B_{X^*} = \overline{\operatorname{conv}}^{w^*}(\mathscr{A})$. The following assertions are equivalent for $z \in S_X$:*

(i) $z \in \operatorname{Spear}(X)$ *(that is,* $\|z + \mathbb{T}x\| = 1 + \|x\|$ *for every* $x \in X$*).*
(ii) $B_{X^*} = \overline{\operatorname{aconv}}^{w^*}(\operatorname{Slice}(\mathscr{A}, z, \varepsilon))$ *for each* $\varepsilon > 0$.
(iii)$_{\mathbb{R}}$ *If X is a real space,* $B_{X^*} = \operatorname{conv}\big(\operatorname{Face}(S_{X^*}, z) \cup -\operatorname{Face}(S_{X^*}, z)\big)$.
(iii)$_{\mathbb{C}}$ *If X is a complex space,* $\operatorname{int}(B_{X^*}) \subset \operatorname{aconv}\big(\operatorname{Face}(S_{X^*}, z)\big)$*;*
in particular, $B_{X^*} = \overline{\operatorname{aconv}}\big(\operatorname{Face}(S_{X^*}, z)\big)$.
(iv) $|x^*(z)| = 1$ *for every* $x^* \in \operatorname{ext} B_{X^*}$*, i.e.* $\operatorname{ext} B_{X^*} \subset \mathbb{T}\operatorname{Face}(S_{X^*}, z)$.

If $X = Y^*$ is a dual Banach space and $z = y^* \in S_{Y^*}$, this is also equivalent to:

(v) $B_Y = \overline{\operatorname{aconv}}\big(\operatorname{Slice}(S_Y, y^*, \varepsilon)\big)$ *for every* $\varepsilon > 0$.

Proof. The equivalence between (i), (ii) and (iv) is just a particular case of Corollary 2.6, as it is the equivalence with (v) when X is a dual space.

(iv) $\Rightarrow$ (iii) is contained in [23, Theorem 2.1.17] (both in the real and in the complex case), but we give the easy argument here. By (iv) and the Krein-Milman theorem,

$$B_{X^*} = \overline{\operatorname{aconv}}^{w^*}(\operatorname{Face}(S_{X^*}, z)). \tag{2.2}$$

In the real case, we have that the set

$$\operatorname{aconv}(\operatorname{Face}(S_{X^*}, z)) = \operatorname{conv}\big(\operatorname{Face}(S_{X^*}, z) \cup -\operatorname{Face}(S_{X^*}, z)\big)$$

is weak-star compact as so is $\operatorname{Face}(S_{X^*}, z)$, and the result follows from (2.2). In the complex case, for $0 < \rho < 1$ we take $n \in \mathbb{N}$ such that $(1-\rho)B_{\mathbb{C}} \subset \operatorname{conv}\{z_1, \ldots, z_n\}$, where $\{z_1, \ldots, z_n\}$ are the nth roots of 1 in $\mathbb{C}$. Then we have

$$\begin{aligned}(1-\rho)\operatorname{aconv}(\operatorname{Face}(S_{X^*}, z)) &= (1-\rho)\operatorname{conv}(B_{\mathbb{C}}\operatorname{Face}(S_{X^*}, z)) \\ &\subset \operatorname{conv}\left(\textstyle\bigcup_{k=1}^n z_k \operatorname{Face}(S_{X^*}, z)\right).\end{aligned}$$

Since $\operatorname{conv}\left(\bigcup_{k=1}^n z_k \operatorname{Face}(S_{X^*}, z)\right)$ is weak-star compact and is contained in the set $\operatorname{aconv}(\operatorname{Face}(S_{X^*}, z))$, it follows from (2.2) that $(1-\rho)B_{X^*} \subset \operatorname{aconv}(\operatorname{Face}(S_{X^*}, z))$, and this gives the result moving $\rho \downarrow 0$.

The implication (iii) $\Rightarrow$ (ii) is immediate taking $\mathscr{A} = S_{X^*}$. $\square$

We would like to comment that the complex case of (iii) in the Corollary above can not be improved to get that the whole unit ball is inside the absolutely convex hull, see [23, Example 2.1.18] for an example.

The next surprising result about spear vectors of a dual space appeared literally in [4, Corollary 3.5] and it is also consequence of the earlier [48, Theorem 2.3] using Corollary 2.8.iii. In both cases, the main tool is the use of norm-to-weak upper semicontinuity of the duality and pre-duality mappings. We include here an adaptation of the proof of [48, Theorem 2.3] to our particular situation which avoids the use of semicontinuities (which, on the other hand, are automatic in our context, see [23, Fact 2.9.3 and Theorem 2.9.18]).

Theorem 2.9 ([48, Theorem 2.3], [4, Corollary 3.5]). *Let X be a Banach space and let $z^* \in S_{X^*}$. Then, $z^* \in \operatorname{Spear}(X^*)$ if and only if $B_X = \overline{\operatorname{aconv}}\big(\operatorname{Face}(S_X, z^*)\big)$.*

Proof. The "if" part follows immediately from Corollary 2.8.v, so we just have to prove the "only if" part. To simplify, we will denote $F = \operatorname{gFace}(S_X, \{z^*\})$ (which we do not know a priori if it is non-empty) and $F^{**} = \operatorname{Face}(S_{X^{**}}, z^*) \neq \emptyset$.

Claim 1. If $A \subset B_{X^{**}}$ satisfies $\sup_A \operatorname{Re} z^* = 1$, then $\operatorname{dist}(A, F^{**}) = 0$. Indeed, let $\varepsilon > 0$ and $a^{**} \in A$ with $\operatorname{Re} z^*(a^{**}) > 1-\varepsilon$. Since $B_{X^{**}} = \overline{\operatorname{aconv}}\big(F^{**}\big)$ by Corollary 2.8.iii, we can find $x_1^{**}, \ldots, x_m^{**} \in F^{**}$ and $\lambda_1, \ldots, \lambda_m \in [0,1]$ with $\sum_{k=1}^m \lambda_k = 1$, and $\theta_1, \ldots, \theta_m \in \mathbb{T}$ such that

$$\left\| a^{**} - \sum_{n=1}^m \lambda_n \theta_n x_n^{**} \right\| < \varepsilon, \qquad \text{so} \qquad \sum_{n=1}^m \lambda_n \operatorname{Re} \theta_n = \operatorname{Re} z^* \left(\sum_{n=1}^m \lambda_n \theta_n x_n^{**} \right) > 1 - 2\varepsilon.$$

Therefore,

$$\begin{aligned}
\operatorname{dist}(A, F^{**}) &\leqslant \left\| a^{**} - \sum_{n=1}^m \lambda_n x_n^{**} \right\| \leqslant \varepsilon + \left\| \sum_{n=1}^m \lambda_n \theta_n x_n^{**} - \sum_{n=1}^m \lambda_n x_n^{**} \right\| \\
&\leqslant \varepsilon + \sum_{n=1}^m \lambda_n |1-\theta_n| \leqslant \varepsilon + \sum_{n=1}^m \lambda_n \sqrt{2(1-\operatorname{Re}\theta_n)} \\
&\leqslant \varepsilon + \sqrt{\sum_{n=1}^m 2\lambda_n (1-\operatorname{Re}\theta_n)} \leqslant \varepsilon + \sqrt{4\varepsilon}.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the claim is proved.

Claim 2. Given $x_0 \in B_X$ with $\operatorname{dist}(x_0, F^{**}) < \varepsilon$, the set $A = B_X \cap (x_0 + \varepsilon B_X)$ satisfies that $\sup_A \operatorname{Re} z^* = 1$. Indeed, let $x_0^{**} \in F^{**}$ with $\|x_0^{**} - x_0\| < \varepsilon$. Using the Principle of Local Reflexivity [39, Theorem 6.3], we have that for each $\delta > 0$ we can find an element $x_\delta \in X$ such that $\|x_\delta\| \leqslant 1$, $\operatorname{Re} z^*(x_\delta) > 1 - \delta$, and $\|x_\delta - x_0\| < \varepsilon$.

Claim 3. $F \neq \emptyset$ (so, $F = \operatorname{Face}(S_X, z^*)$) and if $A \subset B_X$ satisfies $\sup_A \operatorname{Re} z^* = 1$, then $\operatorname{dist}(A, F) = 0$. Indeed, let $A_0 = A$ and fix $\varepsilon > 0$. By Claim 1, we have that $\operatorname{dist}(A_0, F^{**}) = 0$, so taking $x_0 \in A_0$ with $\operatorname{dist}(x_0, F^{**}) < \frac{\varepsilon}{2}$, $A_1 = B_X \cap (x_0 + \frac{\varepsilon}{2} B_X)$

satisfies that $\sup_{A_1} \operatorname{Re} z^* = 1$. Repeating the process with A_1, we can find $x_1 \in A_1$ such that $A_2 := B_X \cap (x_1 + \frac{\varepsilon}{4} B_X)$ satisfies $\sup_{A_2} \operatorname{Re} z^* = 1$. Iterating this process, we will have a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ whose limit $z \in B_X$ satisfies that $\operatorname{dist}(z, A) \leqslant \varepsilon$ and that $z^*(z) = 1$, so $z \in F$.

Claim 4. $B_X = \overline{\operatorname{aconv}}(F)$. Indeed, given a slice S of B_X, Corollary 2.8.v shows that $\sup_{\mathbb{T}S} \operatorname{Re} z^* = 1$, so Claim 3 provides that $\operatorname{dist}(\mathbb{T}S, F) = 0$. It is now routine to show that $S \cap \mathbb{T}F \neq \emptyset$ for every slice S of B_X. □

Figure 2.2 illustrates the characterization of a spear vector $z^* \in S_{X^*}$ in terms of $\operatorname{Face}(S_X, z^*)$ for the dual pairs $(\ell_\infty^3, \ell_1^3)$, $(\ell_1^3, \ell_\infty^3)$, and $(\ell_2^2 \oplus_\infty \mathbb{R}, \ell_2^2 \oplus_1 \mathbb{R})$.

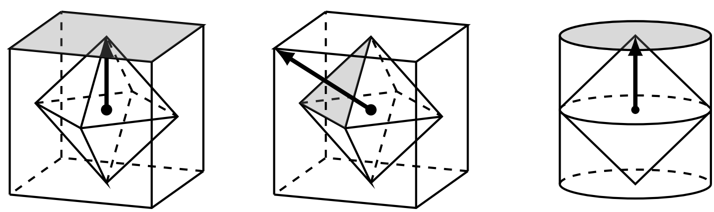


Fig. 2.2 Spear functionals and faces of the unit ball

The following proposition collects all the properties of spear vectors we know. In order to prove it, we need the following technical lemma which will be also used later.

Lemma 2.10. *Let X be a Banach space and let $(F_n)_{n\in\mathbb{N}}$ be a decreasing sequence of spear sets of X such that $\operatorname{diam}(F_n)$ tends to zero. If $z \in \bigcap_{n\in\mathbb{N}} F_n$, then z is a spear element.*

Proof. For every $x \in X$ we can write

$$\|z + \mathbb{T}x\| \geqslant \|F_n + \mathbb{T}x\| - \|F_n - z\| = 1 + \|x\| - \|F_n - z\|.$$

But the hypothesis implies that $\lim_n \|F_n - z\| = 0$. □

Proposition 2.11. *Let X be a Banach space. Then:*

(a) $\|z \pm x\|^2 \geqslant 1 + \|x\|^2$ *for each* $z \in \operatorname{Spear}(X)$ *and every* $x \in X$.
(b) Every $z \in \operatorname{Spear}(X)$ *is a strongly extreme point of* B_X.
In particular, $\operatorname{Spear}(X) \subset \operatorname{ext} B_X$.
(c) $J_X\big(\operatorname{Spear}(X)\big) \subset \operatorname{Spear}(X^{**})$. *In particular,* $J_X\big(\operatorname{Spear}(X)\big) \subset \operatorname{ext} B_{X^{**}}$.
(d) $\operatorname{Spear}(X)$ *is norm-closed and rounded.*
(e) If $\dim(X) \geqslant 2$, *then* $\operatorname{Spear}(X)$ *is nowhere-dense in* $(S_X, \|\cdot\|)$.
(f) If $B_X = \overline{\operatorname{conv}}(\operatorname{Spear}(X))$, *then* $\operatorname{Spear}(X^*) = \operatorname{ext} B_{X^*}$.

(g) If $B_{X^*} = \overline{\operatorname{conv}}^{w^*}(\operatorname{Spear}(X^*))$*, then* $x \in S_X$ *is a spear element if and only if* x *is a strongly extreme point of* B_X *if and only if* $x \in \operatorname{ext} B_{X^{**}}$.

(h) If X *is strictly convex and* $\dim(X) \geqslant 2$*, then* $\operatorname{Spear}(X) = \emptyset = \operatorname{Spear}(X^*)$.

(i) If X *is smooth and* $\dim(X) \geqslant 2$*, then* $\operatorname{Spear}(X) = \emptyset$.

If X *is a* ***real*** *space, we can add:*

(j) If $\operatorname{Spear}(X)$ *is infinite, then* X *contains a copy of* c_0 *or* ℓ_1.

(k) If $z^* \in \operatorname{Spear}(X^*)$ *and* $x^* \in X^*$ *is norm-attaining with* $\|z^* - x^*\| < 1 + \|x^*\|$*, then* $z^* + x^*$ *is norm-attaining and* $\|z^* + x^*\| = 1 + \|x^*\|$.

(l) If X *is smooth and* $\dim(X) \geqslant 2$*, then* $\operatorname{Spear}(X^*) = \emptyset$.

Proof. (a) is given by Proposition 2.7, and (b) is an obvious consequence of it.

(c). Fixed $z \in \operatorname{Spear}(X)$, we have that $B_{X^*} = \overline{\operatorname{aconv}}^{\|\cdot\|}\big(\operatorname{Face}(S_{X^*}, z)\big)$ by Corollary 2.8.iii so, using Goldstine's Theorem for X^*, we obtain that

$$B_{X^{***}} = \overline{\operatorname{aconv}}^{\sigma(X^{***},X^{**})} J_{X^*}\big(\operatorname{Face}(S_{X^*}, z)\big)$$

and, a fortiori,

$$B_{X^{***}} = \overline{\operatorname{aconv}}^{\sigma(X^{***},X^{**})}\big(\operatorname{Face}(S_{X^{***}}, J_X(z))\big).$$

Now, Milman's Theorem (see Lemma 2.5.b) gives that

$$\operatorname{ext}(B_{X^{***}}) \subset \overline{\mathbb{T}\operatorname{Face}(S_{X^{***}}, J_X(z))}^{\sigma(X^{***},X^{**})}.$$

Then, Corollary 2.8.iv gives that $J_X(z) \in \operatorname{Spear}(X^{**})$. An alternative proof is the following: for $z \in \operatorname{Spear}(X)$ we have that $|x^*(z)| = 1$ for every $x^* \in \operatorname{ext}(B_{X^*})$ by Corollary 2.8.iv, and so $|x^{***}(J_X(z))| = 1$ for every $x^{***} \in \operatorname{ext}(B_{X^{***}})$ by [113, Proposition 3.5], so $J_X(x) \in \operatorname{Spear}(X^{**})$ by using again Corollary 2.8.iv.

(d). Given a norm-convergent sequence $(x_n)_{n\in\mathbb{N}}$ in $\operatorname{Spear}(X)$, apply Lemma 2.10 to the family of sets $F_n := \overline{\{x_m \colon m \geqslant n\}}$.

(e). Fixed $e_0^* \in \operatorname{ext} B_{X^*}$, take an element $x_0 \in \ker e_0^* \cap S_X$. Given $z \in \operatorname{Spear}(X)$, there exists $\theta_0 \in \mathbb{T}$ such that $\|z + \theta_0 x_0\| = 2$ and so $\|z + \delta\theta_0 x_0\| = 1 + \delta$ for every $\delta > 0$ by Remark 2.2. Then, for each $\delta > 0$, $v = (z + \delta\theta_0 x_0)/(1+\delta)$ belongs to S_X, satisfies $\|v - z\| \leqslant 2\delta$, and $|e_0^*(v)| = (1+\delta)^{-1} \neq 1$, so v is not a spear by Corollary 2.8.iv.

(f). Fix $e^* \in \operatorname{ext} B_{X^*}$ and $x^* \in X^*$. The hypothesis implies that for every $\varepsilon > 0$ we can find $z \in \operatorname{Spear}(X)$ with $|x^*(z)| > \|x^*\| - \varepsilon$. Since $|e^*(z)| = 1$ (as z is a spear) we conclude that $\|\mathbb{T}e^* + x^*\| \geqslant |e^*(z)| + |x^*(z)| \geqslant 1 + \|x^*\| - \varepsilon$. This gives that $\operatorname{ext} B_{X^*} \subset \operatorname{Spear}(X^*)$ and the equality follows from (b).

(g). If we assume that $x \in \operatorname{ext} B_{X^{**}}$, then $|x^*(x)| = 1$ for each $x^* \in \operatorname{Spear}(X^*)$ by Corollary 2.8.iv. But Milman's theorem (see Lemma 2.5.c) applied to our assumption gives that $\operatorname{Spear}(X^*)$ is weak-star dense in $\operatorname{ext} B_{X^*}$, so we conclude that $|x^*(x)| = 1$ for every $x^* \in \operatorname{ext} B_{X^*}$, which implies that x is a spear by Corollary 2.8.iv.

(h). If $x_0 \in \operatorname{Spear}(X)$, we have that $\|x_0 + \mathbb{T}x\| = 2$ for every $x \in S_X$. If X is strictly convex, this implies that $S_X \subset \mathbb{T}x_0$ and so $\dim(X) = 1$. Next, suppose that there exists $z^* \in \operatorname{Spear}(X^*)$, so $B_X = \overline{\operatorname{aconv}}(\operatorname{Face}(S_X, z^*))$ by Theorem 2.9. If $\dim(X) \geqslant 2$ then $\operatorname{Face}(S_X, z^*)$ contains at least two points, so X is not strictly convex.

(i). If X is smooth, the set $\operatorname{Face}(S_{X^*}, z)$ is a singleton for every $z \in S_X$. If there is $z \in \operatorname{Spear}(X)$, then the above observation and Corollary 2.8.iii imply that X^* is one-dimensional, so X is one-dimensional as well.

(j) is just a reformulation of [84, Proposition 2], but we include the short argument for completeness. Suppose that X does not contain ℓ_1. Then, by Rosenthal's ℓ_1-Theorem [36, Chapter XI] there is a weakly Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ of distinct members of $\operatorname{Spear}(X)$. Write Y for the closed linear span of $\{x_n \colon n \in \mathbb{N}\}$ and observe that, obviously, $x_n \in \operatorname{Spear}(Y)$ for every $n \in \mathbb{N}$. Therefore, by Corollary 2.8.iv, the fact that the sequence $(x_n)_{n\in\mathbb{N}}$ is weakly Cauchy and that we are in the real case, we have that $\operatorname{ext}(B_{Y^*}) = \bigcup_{n\in\mathbb{N}} \big(E_n \cup -E_n\big)$ where

$$E_n := \big\{y^* \in \operatorname{ext}(B_{Y^*}) \colon y^*(x_k) = 1 \text{ for } k \geqslant n\big\} \qquad (n \in \mathbb{N}).$$

As $\{x_n \colon n \in \mathbb{N}\}$ separates the points of Y^*, each E_n is finite, so $\operatorname{ext}(B_{Y^*})$ must be countable. Fonf's Theorem [41] gives then that $X \supset c_0$, finishing the proof.

The proof of (k) is based on ideas of [80]. Let $F = \operatorname{Face}(S_{X^{**}}, z^*)$. By Corollary 2.8.iii, we have that $B_{X^{**}} = \operatorname{conv}(F \cup -F)$. Let $x^* \in X^*$ attain its norm at $x \in S_X$, i.e. $x^*(x) = \|x^*\|$, and suppose that $\|z^* - x^*\| < 1 + \|x^*\|$. We can write

$$x = (1-\lambda)x_1^{**} - \lambda x_2^{**}$$

for some $0 \leqslant \lambda \leqslant 1$ and $x_1^{**}, x_2^{**} \in F$. If $0 < \lambda \leqslant 1$, then $x_2^{**}(x^*) = -\|x^*\|$ necessarily, which is not possible as $|x_2^{**}(x^*) - x_2^{**}(z^*)| \leqslant \|x^* - z^*\| < 1 + \|x^*\|$. Therefore, $\lambda = 0$ and we get that $x = x_1^{**} \in F \cap S_X$, so $[z^* + x^*](x) = 1 + \|x^*\|$.

(l). Suppose that there exists $z^* \in \operatorname{Spear}(X^*)$. By (j), we can take a norm-attaining functional $x_0^* \in S_{X^*}$ with $0 < \|z^* - x_0^*\| < 2$, such that $z^* + x_0^*$ is norm-attaining and $\|z^* + x_0^*\| = 2$. Hence there is $x_0 \in S_X$ with $x_0^*(x_0) = z^*(x_0) = 1$, which means that X is not smooth. □

Below we list the known examples of spear vectors from [7] and some easy-to-check generalizations of those examples.

Example 2.12.

(a) Let X_1, X_2 be Banach spaces and let $X = X_1 \oplus_1 X_2$. Then, $(z_1, z_2) \in \operatorname{Spear}(X)$ if, and only if, either $z_1 \in \operatorname{Spear}(X_1)$ and $z_2 = 0$ or $z_1 = 0$ and $z_2 \in \operatorname{Spear}(X_2)$. The proof is straightforward. As a consequence, if the linear span of $z \in S_X$ is an L-summand of a Banach space X, then $z \in \operatorname{Spear}(X)$. Observe that the converse result does not hold; indeed, just note that $C(\Delta)$, where Δ is the Cantor set, contains a lot of spear vectors (see item (c) below), but contains no proper L-summand by Behrends L-M Theorem (see [54, Theorem I.1.8] for instance).

(b) Let (Ω,Σ,μ) be a positive measure space and let X be a Banach space. Then

$$\operatorname{Spear}(L_1(\mu,X)) = \left\{x\,\frac{\mathbb{1}_A}{\mu(A)} : x\in \operatorname{Spear}(X),\, A\in\Sigma \text{ atom},\, \mu(A)<\infty\right\}.$$

Indeed, every function of the given form belongs to $\{e\mathbb{1}_A \colon e\in X\}$ which is an L-summand isometric to X, and this function is a spear in that L-summand, so it is a spear vector by (a). To see the converse, let $f\in S_{L_1(\mu,X)}$ be a spear vector and let $A := \{t\in\Omega\colon \|f(t)\|>0\}\in\Sigma$ (this is an abuse of notations, as f formally is an equivalence class of functions). If A is not an atom, then we can find $B\subset A$ with $\mu(B),\mu(A\setminus B)\neq 0$ and so f can be written as a convex combination of norm-one functions, which contradicts that f is an extreme point of the unit ball. Hence A is an atom, and moreover $\mu(A)<\infty$ since f is nonzero on A. Thus f must be then of the form $f = x\,\mathbb{1}_A/\mu(A)$ where $x\in S_X$. If x is not spear, then there is $y\in S_X$ with $\|x+\mathbb{T}y\|<2$, so $g = y\,\mathbb{1}_A/\mu(A)$ satisfies that $\|f+\mathbb{T}g\| = \|x+\mathbb{T}y\|<2$ leading to a contradiction.
As a particular case, we deduce the already known result [7, p. 170] that if Γ is an arbitrary set, then

$$\operatorname{Spear}\big(\ell_1(\Gamma)\big) = \big\{\theta e_\gamma \colon \theta\in\mathbb{T},\, \gamma\in\Gamma\big\},$$

where e_γ is the function on Γ with value one at γ and zero on the rest.

(c) Let K be a compact Hausdorff space and X is a Banach space. Then,

$$\operatorname{Spear}(C(K,X)) = \{f\in C(K,X)\colon\, f(t)\in\operatorname{Spear}(X) \text{ for all } t\in K\}.$$

To see the equality, notice first that if f satisfies the condition that $f(t)$ is spear for each $t\in K$, then for each $g\in C(K,X)$ we have that

$$\|f+\mathbb{T}g\|_\infty = \max_{t\in K}\|f(t)+\mathbb{T}g(t)\| = \max_{t\in K}\big(1+\|g(t)\|\big) = 1+\|g\|_\infty.$$

For the converse, assume that f is spear but there is $t_0\in K$, $y_0\in S_X$ and $\delta>0$ such that $\|f(t_0)+\mathbb{T}y_0\|<2(1-\delta)$. Since f is continuous, there exists an open subset $U\subset K$ such that $\|f(t)+\mathbb{T}y_0\|<2(1-\delta)$ for each $t\in U$. By Urysohn's lemma there is $g\in S_{C(K)}$ with $g(K)\subset[0,1]$ and $\operatorname{supp}(g)\subset U$. Then $h := g\otimes x\in C(K,X)$ has norm-one and moreover

$$\|f+\mathbb{T}h\|_\infty = \max_{t\in K}\|f(t)+\mathbb{T}h(t)\| = \max_{t\in U}\|f(t)+\mathbb{T}g(t)y_0\| < 2-\delta.$$

In particular, we deduce the following result from [7, p. 170]

$$\operatorname{Spear}\big(C(K)\big) = \big\{f\in C(K)\colon\, |f(t)|=1 \text{ for every } t\in K\big\}.$$

That is, again spear vectors coincide with extreme points of the unit ball.

(d) Let (Ω,Σ,μ) be a positive measure space and let $g\in L_\infty(\mu)$. Then,

$$\operatorname{Spear}(L_\infty(\mu)) = \{g \in L_\infty(\mu)\colon\ |g(t)| = 1 \text{ for } \mu\text{-almost every } t\}.$$

Indeed, suppose that $g \in \operatorname{Spear}(L_\infty(\mu))$ and there exists a measurable subset A with $\mu(A) > 0$ such that $|g(t)| < 1$ for every $t \in A$. Then we have that

$$A = \bigcup_{n\in\mathbb{N}} \{t \in A\colon\ |g(t)| \leqslant 1 - 1/n\},$$

so there exists $n \in \mathbb{N}$ such that the measurable set $B := \{t \in A\colon\ |g(t)| \leqslant 1 - 1/n\}$ has positive measure. Now,

$$\|g + \mathbb{T}\,\mathbb{1}_B\|_\infty \leqslant 2 - 1/n < 2 = 1 + \|\mathbb{1}_B\|_\infty$$

and thus g is not a spear vector, a contradiction. Conversely, suppose that $|g(t)| = 1$ for almost every $t \in \Omega$. For $f \in L_\infty(\mu, Y)$ and $\varepsilon > 0$, there is $A \in \Sigma$ with $\mu(A) > 0$ such that $|f(t)| \geqslant \|f\|_\infty - \varepsilon$ for every $t \in A$. By the hypothesis, there is $A' \in \Sigma$, $A' \subset A$, $\mu(A') > 0$ such that $|g(t)| = 1$ for every $t \in A'$. Now, using the compactness of $\mathbb{T}$ we can give a lower bound for $\|g + \mathbb{T} f\|_\infty$. Indeed, fixed an ε-net $\mathbb{T}_\varepsilon$ of $\mathbb{T}$ we can find an element $\theta_1 \in \mathbb{T}_\varepsilon$ and a subset A'' of A' with positive measure such that $|g(t) + \theta_1 f(t)| \geqslant 1 + |f(t)|(1-\varepsilon)$ for every $t \in A''$. Therefore, we can write

$$\begin{aligned}\|g + \mathbb{T} f\|_\infty &\geqslant \inf_{t\in A''} |g(t) + \theta_1 f(t)| \\ &\geqslant \inf_{t\in A''} 1 + |f(t)|(1-\varepsilon) \geqslant 1 + (\|f\| - \varepsilon)(1-\varepsilon)\end{aligned}$$

and the arbitrariness of ε gives $\|g + \mathbb{T} f\|_\infty \geqslant 1 + \|f\|$.

(e) It will be proved in Corollary 4.22 that the vector-valued version of (d) is also valid. Let (Ω, Σ, μ) be a measure space, let X be a Banach space and let $g \in L_\infty(\mu, X)$. Then $g \in \operatorname{Spear}(L_\infty(\mu, X))$ if and only if $g(t) \in \operatorname{Spear}(X)$ for μ-almost every t. Actually, the proof of the "if" part is just a straightforward adaptation of the corresponding one for (d).

(f) The space $L_p(\mu)$ contains no spear vector if $1 < p < \infty$ and $\dim(L_p(\mu)) \geqslant 2$ (use Proposition 2.11.h, for instance).

(g) Let X_1, X_2 be Banach spaces and let $X = X_1 \oplus_\infty X_2$. Then, $(z_1, z_2) \in \operatorname{Spear}(X)$ if, and only if, $z_1 \in \operatorname{Spear}(X_1)$ and $z_2 \in \operatorname{Spear}(X_2)$. The proof is straightforward.

As an application of examples (b) and (e) above and Theorem 2.9, we get easily the following well-known old result (see [99] for an exposition which also covers the complex case).

Corollary 2.13. *Let X be a Banach space such that either X or X^* is isometrically isomorphic to an $L_1(\mu)$ space. Then, $B_X = \overline{\operatorname{aconv}}\big(\operatorname{Face}(S_X, x^*)\big)$ for every $x^* \in \operatorname{ext}(B_{X^*})$.*

This result can be read as that X is an almost-CL-space when X or X^* is isometrically isomorphic to an $L_1(\mu)$ space.

Chapter 3

Three definitions for operators: spearness, the alternative Daugavet property, and lushness

This is the main chapter of our manuscript, as we introduce and deeply study the main definitions: the one of spear operator, the weaker of operator with the alternative Daugavet property, and the stronger lush operator.

3.1 A first contact with spear operators

Even though it has been already given, we formally state the definition of spear operator as it is the main concept of the manuscript.

Definition 3.1. Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. We say that G is a *spear operator* if the norm equality

$$\|G + \mathbb{T}T\| = 1 + \|T\|$$

holds for every $T \in \mathcal{L}(X,Y)$, that is, if $G \in \operatorname{Spear}\big(\mathcal{L}(X,Y)\big)$.

We would like now to list some of the equivalent reformulations of the concept of spear operator which one can get particularizing the results of the previous chapter. We will also include a characterization in terms of numerical radius that comes from section 1.2 and follows from [93]. We include anyway a direct proof using the results of the previous chapter.

Proposition 3.2. *Let X, Y be Banach spaces, let $G \in \mathcal{L}(X,Y)$ be a norm-one operator and let $\mathscr{A} \subset B_{\mathcal{L}(X,Y)^*}$ such that $B_{\mathcal{L}(X,Y)^*} = \overline{\operatorname{conv}}^{w^*}(\mathscr{A})$. The following assertions are equivalent:*

(i) G is a spear operator.
(ii) $|\zeta(G)| = 1$ for every $\zeta \in \operatorname{ext}\big(B_{\mathcal{L}(X,Y)^*}\big)$, i.e. $\operatorname{Face}(S_{\mathcal{L}(X,Y)^*}, G)$ *is norming for $\mathcal{L}(X,Y)$.*

(iii) For every $\varepsilon > 0$*, the set* $\operatorname{Slice}(\mathscr{A}, G, \varepsilon)$ *is norming for* $\mathcal{L}(X,Y)$*.*
(iv) For every $\varepsilon > 0$*,*

$$\|T\| = \sup\big\{|y^*(Tx)|\colon\, y^* \in S_{Y^*}, x \in S_X, \operatorname{Re} y^*(Gx) > 1-\varepsilon\big\}$$

for every $T \in \mathcal{L}(X,Y)$*.*
(v) $n_G(X,Y) = 1$*, that is,* $v_G(T) = \|T\|$ *for every* $T \in \mathcal{L}(X,Y)$*.*

Proof. (i), (ii), and (iii) are equivalent by Corollary 2.8. To get the equivalence with (iv), we consider $\mathscr{A} = \{y^* \otimes x\colon\, y^* \in S_{Y^*},\ x \in S_X\}$ as a subset of the unit ball of $\mathcal{L}(X,Y)^*$ (indeed, $[y^* \otimes x](T) = y^*(Tx)$ for every $T \in \mathcal{L}(X,Y)$) and observe that $\mathscr{A}$ is rounded and norming for $\mathcal{L}(X,Y)$, so $B_{\mathcal{L}(X,Y)^*} = \overline{\operatorname{conv}}^{w^*}(\mathscr{A})$. Then, (iv) is just a reformulation of (iii) for this set $\mathscr{A}$; conversely, it is shown in Corollary 2.8 that the property (iii) for just one particular set $\mathscr{A}$ is sufficient to get that G is a spear operator.

That (iv) is equivalent to (v) follows routinely from the definition of numerical radius with respect to G. □

We next present some examples of spear operators which may help to better understand the definition and see how far from the Identity a spear operator can be. The first family appeared in [7, Theorem 4.2], but we will include an easy proof using the results of chapter 2.

Proposition 3.3. *Let* Γ *be an arbitrary set, let* X*,* Y *be Banach spaces, and let* $(e_\gamma)_{\gamma\in\Gamma}$ *be the canonical basis of* $\ell_1(\Gamma)$ *(as defined in Example 2.12.b).*
(a) $G \in \mathcal{L}(\ell_1(\Gamma),Y)$ *is a spear operator if and only if* $G(e_\gamma) \in \operatorname{Spear}(Y)\ \forall\,\gamma \in \Gamma$*.*
(b) $G \in \mathcal{L}(X, c_0(\Gamma))$ *is a spear operator if and only if* $G^*(e_\gamma) \in \operatorname{Spear}(X^*)\ \forall\,\gamma \in \Gamma$*.*

Proof. (a). We fix $T \in \mathcal{L}(\ell_1(\Gamma),Y)$ and $\varepsilon > 0$. As $B_{\ell_1(\Gamma)} = \overline{\operatorname{aconv}}\big(\{e_\gamma\colon\, \gamma \in \Gamma\}\big)$, it follows that

$$\|T\| = \sup_{\gamma\in\Gamma} \|Te_\gamma\|$$

for every $T \in \mathcal{L}(\ell_1(\Gamma),Y)$. If $Ge_\gamma \in \operatorname{Spear}(Y)$, it follows from Corollary 2.8 that $\mathscr{A}_\gamma := \operatorname{Slice}(S_{Y^*}, Ge_\gamma, \varepsilon)$ is norming for Y so, together with the previous equality, we get that

$$\|T\| = \sup_{\gamma\in\Gamma}\ \sup_{y^*\in\mathscr{A}_\gamma} |y^*(Te_\gamma)| = \sup\big\{|y^*(Te_\gamma)|\colon\, y^* \in S_{Y^*},\ \gamma\in\Gamma,\ \operatorname{Re} y^*(Ge_\gamma) > 1-\varepsilon\big\}.$$

It now follows from Proposition 3.2 that G is a spear operator. For the necessity, suppose that there is $\xi \in \Gamma$ such that $G(e_\xi) \notin \operatorname{Spear}(Y)$ and find $y_0 \in S_Y$ such that $\|G(e_\xi) + \mathbb{T} y_0\| < 2$. We then consider the norm-one operator $T \in \mathcal{L}(\ell_1(\Gamma),Y)$ given by $T(e_\xi) = y_0$ and $T(e_\gamma) = 0$ if $\gamma \neq \xi$. Therefore,

$$\|G + \mathbb{T}T\| = \sup_{\gamma\in\Gamma} \|G(e_\gamma) + \mathbb{T}T(e_\gamma)\| < 2,$$

so G is not a spear operator.

Let us prove (b). The sufficiency of the condition is given by the obvious fact that G is a spear operator when G^* is (as taking adjoint preserves the norm) and the result in (a). For the necessity, we may give an argument dual to the one of the case (a). Suppose that there is $\xi \in \Gamma$ such that $G^*(e_\xi) \notin \operatorname{Spear}(X^*)$ and find $x_0^* \in S_{X^*}$ such that $\|G^*(e_\xi) + \mathbb{T}x_0^*\| < 2$. We then consider the norm-one operator $T \in \mathcal{L}(X, c_0(\Gamma))$ given by $[Tx](\xi) = x_0^*(x)$ and $[Tx](\gamma) = 0$ if $\gamma \neq \xi$ for every $x \in X$, and observe that $T^*(e_\xi) = x_0^*$ and $T^*(e_\gamma) = 0$ for $\gamma \neq \xi$. Therefore,

$$\|G + \mathbb{T}T\| = \|G^* + \mathbb{T}T^*\| = \sup_{\gamma \in \Gamma} \|G^*(e_\gamma) + \mathbb{T}T^*(e_\gamma)\| < 2,$$

so G is not a spear operator. □

This result will be improved in Example 5.5. More involved examples of spear operators, lush operators, operators with the aDP... will appear in chapters 4, 5, 7.

The following observations follow straightforwardly from the definition of spear operator.

Remark 3.4. Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$.

(i) *Composing with isometric isomorphisms preserves spearness: Let X_1, Y_1 be Banach spaces, and let $\Phi_1 \in \mathcal{L}(X_1, X)$ and $\Phi_2 \in \mathcal{L}(Y, Y_2)$ be isometric isomorphisms. Then $G \in \operatorname{Spear}\big(\mathcal{L}(X,Y)\big)$ if and only if $\Phi_2 G \Phi_1 \in \operatorname{Spear}\big(\mathcal{L}(X_1, Y_1)\big)$.*
(ii) *We may restrict the codomain of a spear operator keeping the property of being spear operator: If G is a spear operator and Z is a subspace of Y containing $G(X)$, then $G\colon X \longrightarrow Z$ is a spear operator. On the other hand, the extension of the codomain does not always preserve spears: the map $j\colon \mathbb{K} \longrightarrow \mathbb{K} \oplus_\infty \mathbb{K}$ given by $j(x) = (x, 0)$ for every $x \in \mathbb{K}$, is not a spear operator.*
(iii) *As an easy consequence of (i) and (ii), we get that the following assertions are equivalent: (a) X has numerical index 1 (i.e. Id_X is a spear), (b) there exists a Banach space Z and an isometric isomorphism which is a spear in $\mathcal{L}(X,Z)$ or $\mathcal{L}(Z,X)$, (c) there exists a Banach space W and an isometric embedding of X into W which is a spear operator.*

3.2 Alternative Daugavet Property

We start presenting the definition of the alternative Daugavet property for an operator, which extends the analogous definition for a Banach space (through the Identity).

Definition 3.5. Let X, Y be Banach spaces. We say that $G \in \mathcal{L}(X,Y)$ has the *alternative Daugavet property* (*aDP* in short), if the norm equality

$$\|G+\mathbb{T}T\| = 1+\|T\| \tag{aDE}$$

holds for every rank-one operator $T \in \mathcal{L}(X,Y)$.

Substituting $T=0$ in (aDE) we deduce that if G has the aDP then $\|G\|=1$.

The following fundamental result characterizes the aDP of an operator in terms of the behaviour of the operator with respect to slices, spear sets...

Theorem 3.6. *Let $G \in \mathcal{L}(X,Y)$ be a norm-one operator between two Banach spaces X, Y, let $\mathscr{B} \subset B_X$ with $B_X = \overline{\operatorname{conv}}(\mathscr{B})$ and let $\mathscr{A} \subset B_{Y^*}$ with $B_{Y^*} = \overline{\operatorname{conv}}^{w^*}(\mathscr{A})$. The following assertions are equivalent:*

(i) G has the aDP.
(ii) $G(S)$ is a spear set for every slice S of $\mathscr{B}$.
(ii) $G^*(S^*)$ is a spear set for every w^*-slice S^* of $\mathscr{A}$.*
(iii) For every $y_0 \in S_Y$ and $\varepsilon > 0$

$$B_X = \overline{\operatorname{conv}}\big(\{x\in\mathscr{B}\colon \|Gx+\mathbb{T}y_0\| > 2-\varepsilon\}\big).$$

(iv) For every $x_0^* \in X^*$, the set

$$\big\{y^* \in \operatorname{ext} B_{Y^*}\colon \|G^*y^* + \mathbb{T}x_0^*\| = 1+\|x_0^*\|\big\}$$

is a dense G_δ set in $(\operatorname{ext} B_{Y^*}, w^*)$.

Proof. (i) $\Rightarrow$ (ii): Let $S = \operatorname{Slice}(\mathscr{B}, x_0^*, \varepsilon)$ be a slice of $\mathscr{B}$ with $x_0^* \in S_{X^*}$ and $0<\varepsilon<1$. Given any $0 \neq y_0 \in Y$ consider the rank-one operator $T = x_0^* \otimes y_0 \in \mathcal{L}(X,Y)$ which satisfies that $\|T\| = \|y_0\|$. Since $\|G+\mathbb{T}T\| = \|G^*+\mathbb{T}T^*\|$, for every $0<\delta<1$ there exists $y_0^* \in S_{Y^*}$ such that

$$\|G^*y_0^* + \mathbb{T}y_0^*(y_0)x_0^*\| > 1+\|y_0\|(1-\varepsilon\delta). \tag{3.1}$$

Using a rotation on y_0^* if necessary, we can assume that $0 \leqslant y_0^*(y_0) \leqslant \|y_0\|$. Using the hypothesis on $\mathscr{B}$, we deduce from (3.1) the existence of some $x_0 \in \mathscr{B}$ satisfying

$$|y_0^*(Gx_0)| + y_0^*(y_0)\operatorname{Re} x_0^*(x_0) > 1+\|y_0\|(1-\varepsilon\delta)$$

and, in particular,

$$\|Gx_0 + \mathbb{T}x_0^*(x_0)y_0\| > 1+\|y_0\|(1-\varepsilon\delta).$$

Since $\|Gx_0\| \leqslant 1$, we deduce from the first inequality that

$$\operatorname{Re} x_0^*(x_0) > 1-\varepsilon\delta > 1-\varepsilon.$$

Hence $x_0 \in S$ and so

$$\begin{aligned}\|G(S)+\mathbb{T}y_0\| \geqslant \|Gx_0+\mathbb{T}y_0\| &> \|Gx_0+\mathbb{T}x_0^*(x_0)y_0\| - \varepsilon\delta\|y_0\| \\ &> 1+\|y_0\|(1-\varepsilon\delta) - \|y_0\|\varepsilon\delta > 1+\|y_0\|(1-2\varepsilon\delta).\end{aligned}$$

(ii) $\Rightarrow$ (iii): Given $y_0 \in S_Y$, $\varepsilon > 0$ and a slice S of $\mathscr{B}$, since $G(S)$ is a spear set, we can find $x \in \mathbb{T}S$ with $\|Gx+y_0\| > 2-\varepsilon$, which means that every slice S of $\mathscr{B}$ intersects the set

$$\mathbb{T}\{x \in \mathscr{B}\colon \|Gx+y_0\| > 2-\varepsilon\} = \{x \in \mathscr{B}\colon \|Gx+\mathbb{T}y_0\| > 2-\varepsilon\}.$$

Therefore

$$B_X = \overline{\operatorname{conv}}(\mathscr{B}) \subset \overline{\operatorname{conv}}\big(\{x \in \mathscr{B}\colon \|Gx+\mathbb{T}y_0\| > 2-\varepsilon\}\big).$$

(iii) $\Rightarrow$ (i): Let $T \in \mathcal{L}(X,Y)$ be a rank-one operator. By Remark 2.2, we may and do suppose that $\|T\| = 1$. Then, it is of the form $T = x_0^* \otimes y_0$ for some $y_0 \in S_Y$ and $x_0^* \in S_{X^*}$. Given $\varepsilon > 0$, the hypothesis implies that $\operatorname{Slice}(\mathscr{B}, x_0^*, \varepsilon)$ intersects $\{x \in \mathscr{B}\colon \|Gx+\mathbb{T}y_0\| > 2-\varepsilon\}$, so there exists $x_0 \in \operatorname{Slice}(\mathscr{B}, x_0^*, \varepsilon)$ and $\theta_0 \in \mathbb{T}$ such that $\|G(x_0)+\theta_0 y_0\| > 2-\varepsilon$. Hence

$$\begin{aligned}\|G+\mathbb{T}T\| &\geqslant \|Gx_0+\theta_0 x_0^*(x_0)y_0\| \\ &\geqslant \|Gx_0+\theta_0 y_0\| - |x_0^*(x_0)-1| > 2-2\varepsilon.\end{aligned}$$

(ii) $\Rightarrow$ (iv): We may and do suppose that $\|x_0^*\| = 1$. Given $\varepsilon > 0$, the set $G\big(\operatorname{Slice}(\mathscr{B}, x_0^*, \varepsilon)\big)$ is a spear of B_Y by hypothesis, which by Corollary 2.6.iii means that the set $\operatorname{gFace}\big(\operatorname{ext} B_{Y^*}, \mathbb{T}G\big(\operatorname{Slice}(\mathscr{B}, x_0^*, \varepsilon)\big)\big)$ is a dense G_δ set in the Baire space $(\operatorname{ext} B_{Y^*}, w^*)$. Hence,

$$\bigcap_{m\in\mathbb{N}} \operatorname{gFace}\big(\operatorname{ext} B_{Y^*}, \mathbb{T}G\big(\operatorname{Slice}(\mathscr{B}, x_0^*, 1/m)\big)\big)$$

is also dense in $\operatorname{ext} B_{Y^*}$, and it is easy to check that

$$\bigcap_{m\in\mathbb{N}} \operatorname{gFace}\big(\operatorname{ext} B_{Y^*}, \mathbb{T}G\big(\operatorname{Slice}(\mathscr{B}, x_0^*, 1/m)\big)\big) = \big\{y^* \in \operatorname{ext} B_{Y^*}\colon \|G^*y^*+\mathbb{T}x_0^*\| = 2\big\}.$$

(iv) $\Rightarrow$ (ii*): Fix $x_0^* \in X^*$. If S^* is any w^*-slice of B_{Y^*}, then $S^* \cap \operatorname{ext} B_{Y^*}$ is a non-empty open subset of $(\operatorname{ext} B_{Y^*}, w^*)$. By hypothesis, S^* contains an element of $\{y^* \in \operatorname{ext} B_{Y^*}\colon \|G^*y^*+\mathbb{T}x_0^*\| = 1+\|x_0^*\|\}$, so $\|G^*(S^*)+\mathbb{T}x_0^*\| = 1+\|x_0^*\|$.

(ii*) $\Rightarrow$ (i): Let $T = x_0^* \otimes y_0$, where $x_0^* \in X^*$ and $y_0 \in S_Y$, be an arbitrary rank-one operator. Given any $\varepsilon > 0$ put $S^* = \operatorname{Slice}(\mathscr{A}, y_0, \varepsilon)$. Notice that for every $y^* \in S^*$,

$$\|T^*y^* - x_0^*\| = \|y^*(y_0)x_0^* - x_0^*\| < \|x_0^*\|\varepsilon,$$

so using that $G^*(S^*)$ is a spear, we deduce that

$$\|G+\mathbb{T}T\| = \|G^* + \mathbb{T}T^*\| \geqslant \|G^*(S^*) + \mathbb{T}x_0^*\| - \|x_0^*\|\varepsilon = 1 + (1-\varepsilon)\|x_0^*\|,$$

so $\|G+\mathbb{T}T\| = 1 + \|T\|$, giving (i). □

The next result shows that the aDP is separably determined, and will be very useful in the next section where we deal with SCD operators.

Proposition 3.7. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$. Then, G has the aDP if and only if for every separable subspaces $X_0 \subset X$ and $Y_0 \subset Y$, there exist separable subspaces X_∞, Y_∞ satisfying $X_0 \subset X_\infty \subset X$ and $Y_0 \subset Y_\infty \subset Y$ and such that $G(X_\infty) \subset Y_\infty$ and $G|_{X_\infty} : X_\infty \longrightarrow Y_\infty$ has the aDP.*

Proof. Suppose first that G has the aDP. Pick a sequence $(x_n)_{n\in\mathbb{N}}$ of S_X with $\sup_n \|Gx_n\| = 1$ and consider $X_1 = \overline{\operatorname{span}}(X_0 \cup \{x_n : n \in \mathbb{N}\})$ and $Y_1 = \overline{Y_0 + G(X_1)}$, both separable subspaces. By Theorem 3.6.iii, we have that

$$B_{X_1} \subset \overline{\operatorname{conv}}(\{x \in S_X : \|Gx + \mathbb{T}y_1\| > 2-\varepsilon\}) \quad \text{for every } y_1 \in S_{Y_1} \text{ and } \varepsilon > 0.$$

But since B_{X_1} and S_{Y_1} are separable, it is easy to deduce the existence of a countable set $A_1 \subset S_X$ such that

$$B_{X_1} \subset \overline{\operatorname{conv}}(\{x \in A_1 : \|Gx + \mathbb{T}y_1\| > 2-\varepsilon\}) \quad \text{for every } y_1 \in S_{Y_1} \text{ and } \varepsilon > 0.$$

Define then $X_2 = \overline{\operatorname{span}}(X_1 \cup A_1)$ and $Y_2 = \overline{Y_1 + G(X_2)}$, which are again separable. Repeating the same process as above, we can construct an increasing sequence of closed separable subspaces $X_n \subset X$ and $\overline{G(X_n)} \subset Y_n \subset Y$ such that

$$B_{X_n} \subset \overline{\operatorname{conv}}\big(\{x \in S_{X_{n+1}} : \|Gx + \mathbb{T}y_n\| > 2-\varepsilon\}\big) \quad \text{for every } y_n \in S_{Y_n} \text{ and } \varepsilon > 0.$$

This implies that $X_\infty := \overline{\bigcup_{n\in\mathbb{N}} X_n}$ and $Y_\infty := \overline{\bigcup_{n\in\mathbb{N}} Y_n}$, satisfy that

$$B_{X_\infty} \subset \overline{\operatorname{conv}}(\{x \in S_{X_\infty} : \|Gx + \mathbb{T}y\| > 2-\varepsilon\}) \quad \text{for every } y \in S_{Y_\infty} \text{ and } \varepsilon > 0,$$

which means that $G : X_\infty \longrightarrow Y_\infty$ has the aDP by using again Theorem 3.6.iii.

Conversely, take a non-null rank-one operator $T \in \mathcal{L}(X,Y)$, consider a separable subspace $X_0 \subset X$ such that $\|G|_{X_0}\| = \|G\| = 1$ and $\|T|_{X_0}\| = \|T\|$, and write

$$Y_0 = \overline{G(X_0) + T(X_0)}.$$

By hypothesis, there are separable subspaces $X_0 \subset X_\infty \subset X$ and $Y_0 \subset Y_\infty \subset Y$ such that $G|_{X_\infty} : X_\infty \longrightarrow Y_\infty$ has norm one and has the aDP. As T is rank-one and $T|_{X_0} \neq 0$, it follows that $T(X) \subset T(X_0) \subset Y_0 \subset Y_\infty$ and $\|T|_{X_\infty}\| = \|T\|$. Then we may apply that $G|_{X_\infty}$ has the aDP to get that $\|G|_{X_\infty} + \mathbb{T}T|_{X_\infty}\| = 1 + \|T\|$. But, clearly,

$$\|G+\mathbb{T}T\| \geqslant \|G|_{X_\infty} + \mathbb{T}T|_{X_\infty}\| = 1 + \|T\|,$$

and the reverse inequality is always true, so G has the aDP. □

As we did for spear operators, we may directly deduce from the definition the following three elementary results about operators with the aDP.

Remark 3.8. Let X,Y be Banach spaces and $G\in\mathcal{L}(X,Y)$.

(i) *The composition with isometric isomorphisms preserves the aDP: If X_1, Y_1 are Banach spaces and $\Phi_1\in\mathcal{L}(X_1,X)$, $\Phi_2\in\mathcal{L}(Y,Y_2)$ are isometric isomorphisms, then, $G\in\mathcal{L}(X,Y)$ has the aDP if and only if $\Phi_2G\Phi_1\in\mathcal{L}(X_1,Y_1)$ has the aDP.*

(ii) *If G has the aDP and Z is a subspace of Y containing $G(X)$, then $G\colon X\longrightarrow Z$ has the aDP. However, the property of aDP is not preserved by extending the codomain of the operator, as the same example of Remark 3.4 shows.*

(iii) *As an easy consequence of (i) and (ii), we have that the following statements are equivalent: (a) X has the aDP, (b) there exist a Banach space Z and an isometric isomorphism in $\mathcal{L}(X,Z)$ or in $\mathcal{L}(Z,X)$ which has the aDP, (d) there exist a Banach space W and an isometric embedding $G\in\mathcal{L}(X,W)$ which has the aDP.*

3.3 Target operators

Our goal in this section is to present and study the concept of target operator, which will be the key in the next section to relate the aDP and lushness so, in particular, to relate the aDP and spear operators. As far as we know, this is a new concept even in the particular case in which G is the identity operator of a Banach space.

Definition 3.9. Let X, Y, Z be Banach spaces and let $G\in\mathcal{L}(X,Y)$ be a norm-one operator. We say that $T\in\mathcal{L}(X,Z)$ is a *target* for G if each $x_0\in B_X$ has the following property:

$$\begin{aligned}&\text{For every } \varepsilon>0 \text{ and every } y\in S_Y, \text{ there is } F\subset B_X \text{ such that}\\ &\operatorname{conv}(F)\subset\{x\in B_X\colon \|Gx+y\|>2-\varepsilon\} \text{ and } \operatorname{dist}\big(Tx_0,T\big(\operatorname{aconv}(F)\big)\big)<\varepsilon.\end{aligned}\tag{$\diamond$}$$

Remark that if F satisfies ($\diamond$), then there is a finite subset of F satisfying the same condition.

At the end of the section we will include a result characterizing spear vectors in terms of target operators which will allow to better understand this definition, see Proposition 3.24. Next, we provide with several characterizations of this kind of operators which will be very useful in the sequel.

Proposition 3.10. *Let X, Y, Z be Banach spaces, let $G\in\mathcal{L}(X,Y)$ with $\|G\|=1$, let $T\in\mathcal{L}(X,Z)$, and let $\mathscr{A}\subset B_{Y^*}$ with $\overline{\operatorname{conv}}^{w^*}(\mathscr{A})=B_{Y^*}$. Given $x_0\in B_X$, the following assertions are equivalent:*

(i) *x_0 satisfies ($\diamond$).*

(ii) *For every $\varepsilon>0$ and $y\in S_Y$ there is $y^*\in\operatorname{Slice}(\mathscr{A},y,\varepsilon)$ such that*

$$\operatorname{dist}\big(Tx_0, T\left(\operatorname{aconv} \operatorname{gSlice}(S_X, G^*y^*, \varepsilon)\right)\big) < \varepsilon.$$

(iii) For every $\varepsilon > 0$, the set

$$\mathscr{D}_T^{\varepsilon}(\mathscr{A}, x_0) = \big\{y^* \in \mathscr{A} \colon \operatorname{dist}\big(Tx_0, T(\operatorname{aconv} \operatorname{gSlice}(S_X, G^*y^*, \varepsilon))\big) < \varepsilon\big\}$$

intersects every w^*-slice of $\mathscr{A}$.

Proof. (i) $\Rightarrow$ (ii): Let $\varepsilon > 0$ and $y \in S_Y$. Fixed $0 < \delta < 1$ such that $\delta^2 + \delta + \frac{\delta}{1-\delta} < \varepsilon$, by ($\diamond$) in Definition 3.9, we can find $F = \{x_1, \ldots, x_n\} \subset B_X$, $\lambda_1, \ldots, \lambda_n \geqslant 0$ with $\sum \lambda_k = 1$, and $\theta_1, \ldots, \theta_n \in \mathbb{T}$ such that

$$\left\|\sum_{k=1}^{n} \lambda_k G(x_k) + y\right\| > 2 - \delta^2 \qquad \text{and} \qquad \left\|Tx_0 - \sum_{k=1}^{n} \lambda_k \theta_k T(x_k)\right\| < \delta^2. \tag{3.2}$$

Let $a^* \in \mathscr{A}$ be such that

$$\operatorname{Re} a^* \left(\sum_{k=1}^{n} \lambda_k G(x_k) + y\right) > 2 - \delta^2.$$

Then $\operatorname{Re} a^*(y) > 1 - \delta^2 > 1 - \varepsilon$ and, moreover, $J = \{k \colon \operatorname{Re} a^*(Gx_k) > 1 - \delta\}$ satisfies

$$1 - \delta \sum_{k \notin J} \lambda_k = \sum_{k \in J} \lambda_k + (1 - \delta) \sum_{k \notin J} \lambda_k \geqslant \sum_{k=1}^{n} \lambda_k \operatorname{Re} a^*(Gx_k) > 1 - \delta^2.$$

Hence, we get that

$$\sum_{k \notin J} \lambda_k < \delta.$$

This, together with the right-hand side inequality of (3.2), implies that

$$\begin{aligned}
\left\|Tx_0 - \sum_{k \in J} \frac{\lambda_k}{\sum_{j \in J} \lambda_j} \theta_k T(x_k)\right\| &\leqslant \delta^2 + \left\|\sum_{k=1}^{n} \lambda_k \theta_k T x_k - \sum_{k \in J} \frac{\lambda_k}{\sum_{j \in J} \lambda_j} \theta_k T x_k\right\| \\
&\leqslant \delta^2 + \delta + \left\|\sum_{k \in J} \lambda_k \theta_k T x_k - \sum_{k \in J} \frac{\lambda_k}{\sum_{j \in J} \lambda_j} \theta_k T x_k\right\| \\
&\leqslant \delta^2 + \delta + \left|1 - \frac{1}{\sum_{k \in J} \lambda_k}\right| \\
&\leqslant \delta^2 + \delta + \frac{\delta}{1 - \delta} < \varepsilon.
\end{aligned}$$

(ii) $\Rightarrow$ (iii): statement (ii) claims that $\mathscr{D}_T^{\delta}(\mathscr{A}, x_0)$ intersects $\operatorname{Slice}(\mathscr{A}, y, \delta)$ for every $y \in S_Y$ and every $\delta > 0$. Since for every $\varepsilon > 0$, $\mathscr{D}_T^{\varepsilon}(\mathscr{A}, x_0)$ contains $\mathscr{D}_T^{\delta}(\mathscr{A}, x_0)$ for every $0 < \delta < \varepsilon$, we conclude the result.

(iii) $\Rightarrow$ (i): Given $y \in S_Y$ and $\varepsilon > 0$, $\mathscr{D}_T^{\varepsilon}(\mathscr{A}, x_0)$ intersects $\operatorname{Slice}(\mathscr{A}, y, \varepsilon)$. Taking an element y^* in such intersection, by definition of $\mathscr{D}_T^{\varepsilon}(\mathscr{A}, x_0)$, we can find a finite set $F \subset \operatorname{gSlice}(S_X, G^* y^*, \varepsilon)$ such that

$$\operatorname{dist}(Tx_0, T(\operatorname{aconv}(F))) < \varepsilon.$$

But the condition $F \subset \operatorname{gSlice}(S_X, G^* y^*, \varepsilon)$ yields that every $x \in \operatorname{conv}(F)$ satisfies

$$\|Gx + y\| \geqslant \operatorname{Re} y^*(Gx) + \operatorname{Re} y^*(y) > 2 - 2\varepsilon,$$

which finishes the proof. □

We can improve the previous Proposition when considering $\mathscr{A}$ as the set of extreme points of the dual unit ball, as the following result shows.

Theorem 3.11. *Let X, Y, Z be Banach spaces, let $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$ and let $T \in \mathcal{L}(X,Z)$. Then, an element $x_0 \in B_X$ satisfies (◇) if and only if the set*

$$\mathscr{D}_T(x_0) := \left\{ y^* \in \operatorname{ext} B_{Y^*} : Tx_0 \in \overline{T\left(\operatorname{aconv} \operatorname{gSlice}(S_X, G^* y^*, \varepsilon)\right)} \text{ for every } \varepsilon > 0 \right\}$$

is a dense (G_δ) subset of $(\operatorname{ext} B_{Y^*}, w^*)$.

Proof. Assume first that x_0 satisfies (◇). By the Krein-Milman theorem, we may use Proposition 3.10.iii with $\mathscr{A} = \operatorname{ext}(B_{Y^*})$. Thus, using Lemma 2.5.a, we have that $\mathscr{D}_T^{\varepsilon}(\operatorname{ext} B_{Y^*}, x_0)$ intersects every weak-star open subset of $\operatorname{ext} B_{Y^*}$. In other words, $\mathscr{D}_T^{\varepsilon}(\operatorname{ext} B_{Y^*}, x_0)$ is a dense subset of $(\operatorname{ext} B_{Y^*}, w^*)$. Since

$$\mathscr{D}_T(x_0) = \bigcap_{m \in \mathbb{N}} \mathscr{D}_T^{1/m}(\operatorname{ext} B_{Y^*}, x_0),$$

we just have to show that $\mathscr{D}_T^{\varepsilon}(\operatorname{ext} B_{Y^*}, x_0)$ is open and then apply Lemma 2.5.c. Indeed, notice that for every $a_0^* \in \mathscr{D}_T^{\varepsilon}(\operatorname{ext}(B_{Y^*}), x_0)$ we can find a finite subset F of $\operatorname{gSlice}(S_X, G^* a_0^*, \varepsilon)$ such that $\operatorname{dist}(Tx_0, T(\operatorname{aconv}(F))) < \varepsilon$. The set

$$U := \bigcap_{x \in F} \operatorname{gSlice}(\operatorname{ext}(B_{Y^*}), Gx, \varepsilon)$$

is a relatively weak-star open subset of $\operatorname{ext}(B_{Y^*})$, which contains y_0 by definition. Also $U \subset \mathscr{D}_T^{\varepsilon}(\operatorname{ext}(B_{Y^*}), x_0)$, since $F \subset \operatorname{gSlice}(S_X, G^* y^*, \varepsilon)$ for every $y^* \in U$. So, $\mathscr{D}_T^{\varepsilon}(\operatorname{ext} B_{Y^*}, x_0)$ is open as desired.

For the converse implication, notice that if $\mathscr{D}_T(x_0)$ is dense in $(\operatorname{ext} B_{Y^*}, w^*)$, then assertion (iii) of Proposition 3.10 immediately holds for $\mathscr{A} = \operatorname{ext}(B_{Y^*})$, and so x_0 satisfies (◇). □

One of the main applications of target operators is the following result.

Proposition 3.12. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. If $T \in \mathcal{L}(X,Y)$ is a target for G, then*

$$\|G + \mathbb{T}T\| = 1 + \|T\|.$$

Proof. By Remark 2.2, we can assume that $\|T\| = 1$. For $0 < \varepsilon < 1$, take $x_0 \in B_X$ with $\|Tx_0\| > 1 - \varepsilon$ and write $y_0 := Tx_0/\|Tx_0\|$. By Proposition 3.10.ii, there exists $y_0^* \in \operatorname{Slice}(S_{Y^*}, y_0, \varepsilon)$ with

$$\operatorname{dist}\big(Tx_0, T\left(\operatorname{aconv}\operatorname{gSlice}(S_X, G^* y_0^*, \varepsilon)\right)\big) < \varepsilon.$$

We can then find $n \in \mathbb{N}$ and elements $x_k \in \operatorname{gSlice}(S_X, G^* y_0^*, \varepsilon)$, $\theta_k \in \mathbb{T}$, $\lambda_k \geqslant 0$ for $k = 1, \ldots, n$, such that

$$\sum_{k=1}^{n} \lambda_k = 1 \qquad \text{and} \qquad \left\| Tx_0 - T\left(\sum_{k=1}^{n} \lambda_k \theta_k x_k\right)\right\| < \varepsilon.$$

Since $|y_0^*(Tx_0)| > 1 - \varepsilon$, a standard convexity argument leads to the existence of some $k \in \{1, \ldots, n\}$ with $|y_0^*(Tx_k)| \geqslant 1 - 2\varepsilon$. Therefore

$$\|G + \mathbb{T}T\| \geqslant |y_0^*(Gx_k)| + |y_0^*(Tx_k)| > 2 - 3\varepsilon,$$

and the proof finishes. □

The following observations can be proved directly from the definition of target, but they follow easier from Proposition 3.10.

Remark 3.13. Let X, Y, Z, Z_1, Z_2 be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator.

(a) *If $T \in \mathcal{L}(X,Z)$ is a target for G, then λT is a target for G for every $\lambda \in \mathbb{K}$.*
(b) *Let $T_1 \in \mathcal{L}(X,Z_1)$ and $T_2 \in \mathcal{L}(X,Z_2)$ be operators such that $\|T_2 x\| \leqslant \|T_1 x\|$ for every $x \in X$. If T_1 is a target for G, then so is T_2.*

Target operators are separably determined, and this fact will be crucial in the study of the relationship with the aDP and SCD.

Theorem 3.14. *Let X, Y, Z be Banach spaces, let $G \in \mathcal{L}(X,Y)$ be a norm-one operator and consider $T \in \mathcal{L}(X,Z)$. Then, T is a target for G if and only if for every separable subspaces $X_0 \subset X$ and $Y_0 \subset Y$, there exist separable subspaces X_∞, Y_∞ satisfying $X_0 \subset X_\infty \subset X$ and $Y_0 \subset Y_\infty \subset Y$ and such that $G|_{X_\infty} \in \mathcal{L}(X_\infty, Y_\infty)$ has norm one and $T|_{X_\infty} \in \mathcal{L}(X_\infty, Z)$ is a target for $G|_{X_\infty}$.*

Proof. Let us assume first that T is a target for G.

Claim: given separable subspaces $\widetilde{X} \subset X$ and $\widetilde{Y} \subset Y$, we can find a countable set $B \subset B_X$ with the following property (P): *given $x_0 \in B_{\widetilde{X}}$, $y_0 \in S_{\widetilde{Y}}$ and $\varepsilon > 0$, there exists $F \subset B$ with*

$$\operatorname{conv}(F) \subset \{x \in B_X \colon \|Gx+y\| > 2-\varepsilon\} \quad \textit{and} \quad \operatorname{dist}(Tx_0, T(\operatorname{aconv}(F))) < \varepsilon.$$

Indeed, fixing C_0 and D_0 countable dense subsets of $B_{\widetilde{X}}$ and $B_{\widetilde{Y}}$ respectively, we can apply the definition of target operator to construct a countable set $B \subset B_X$ satisfying the property (P) for all $x_0 \in C_0$, $y_0 \in D_0$ and $\varepsilon \in \mathbb{Q}^+$. But using the density of C_0 and D_0, it turns out that B has the same property for each $x_0 \in B_{\widetilde{X}}$, $y_0 \in B_{\widetilde{Y}}$ and $\varepsilon > 0$.

Now we prove the theorem. First, we may and do assume that $\|G|_{X_0}\| = \|G\| = 1$. Put $X_1 = X_0$ and $Y_1 = \overline{\operatorname{span}}(Y_0 \cup G(X_0))$, both separable Banach spaces. Using the claim, we deduce the existence of a countable set $B_1 \subset B_X$ with the property (P) for X_1 and Y_1. Define $X_2 = \overline{\operatorname{span}}(X_1 \cup B_1)$ and $Y_2 = \overline{\operatorname{span}}(Y_1 \cup G(X_2))$. Repeating this process inductively, we construct increasing sequences of closed separable subspaces $X_n \subset X$ and $G(X_n) \subset Y_n \subset Y$ such that $B_{X_{n+1}}$ has the property (P) above for X_n and Y_n. Taking $X_\infty := \overline{\bigcup_{n\in\mathbb{N}} X_n}$ and $Y_\infty := \overline{\bigcup_{n\in\mathbb{N}} Y_n}$, we conclude the result, as $G|_{X_\infty}$ and $T|_{X_\infty}$ satisfy the definition of target operator by construction.

Let us check the converse implication. Given $x_0 \in B_X$ and $y_0 \in S_Y$ we can find separable subspaces $x_0 \in X_\infty \subset X$ and $y_0 \in Y_\infty \subset Y$ with the properties above, so applying the definition of target for $T|_{X_\infty}$, $G|_{X_\infty}$ and the previous elements we get the result. □

We need one more ingredient to be able to present the main result about the relationship between target operators and SCD operators.

Proposition 3.15. *Let X, Y, Z be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Let $T \in \mathcal{L}(X,Z)$ be an operator such that the set*

$$\mathcal{D}_T := \left\{ y^* \in \operatorname{ext} B_{Y^*} \colon T(B_X) \subset \overline{T(\operatorname{aconv}\operatorname{gSlice}(B_X, G^*y^*, \varepsilon))} \textit{ for every } \varepsilon > 0 \right\}$$

is dense in $(\operatorname{ext} B_{Y^*}, w^*)$. Then, T is a target for G, and in the case of $Z = Y$, we have $\|G + \mathbb{T}T\| = 1 + \|T\|$.

Proof. In the notation of Theorem 3.11, the inclusion $\mathcal{D}_T \subset \mathcal{D}_T(x_0)$ holds for every $x_0 \in B_X$ and the same Theorem gives that T is a target for G. If $Z = Y$, an application of Proposition 3.12 implies that $\|G + \mathbb{T}T\| = 1 + \|T\|$. □

The converse of the above result holds for operators with separable image.

Proposition 3.16. *Let X, Y, Z be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Suppose that $T \in \mathcal{L}(X,Z)$ is a target for G such that $T(X)$ is separable. Then,*

$$\mathcal{D}_T := \left\{ y^* \in \operatorname{ext} B_{Y^*} \colon T(B_X) \subset \overline{T(\operatorname{aconv}\operatorname{gSlice}(B_X, G^*y^*, \varepsilon))} \textit{ for every } \varepsilon > 0 \right\}$$

is a dense G_δ subset of $(\operatorname{ext} B_{Y^*}, w^*)$.

Proof. Since T is a target for G, for every $x_0 \in B_X$ we have by Theorem 3.11 that the set

$$\mathscr{D}_T(x_0) := \left\{ y^* \in \operatorname{ext} B_{Y^*} \colon Tx_0 \in \overline{T\big(\operatorname{aconv}(\operatorname{gSlice}(S_X, G^*y^*, \varepsilon))\big)} \text{ for every } \varepsilon > 0 \right\}$$

is a dense G_δ subset of $(\operatorname{ext} B_{Y^*}, w^*)$. If we choose a sequence $(x_n)_{n\in\mathbb{N}}$ in B_X so that $(Tx_n)_{n\in\mathbb{N}}$ is dense in $T(B_X)$, then $\mathscr{D}_T = \bigcap_{n\in\mathbb{N}} \mathscr{D}_T(x_n)$. Since all $\mathscr{D}_T(x_n, \operatorname{ext} B_{Y^*})$ are dense G_δ subsets of $(\operatorname{ext} B_{Y^*}, w^*)$, so is $\mathscr{D}_T$ (see Lemma 2.5.c). □

Theorem 3.17. *Let X, Y, Z, Z_1 be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be an operator with the aDP. If for $T \in \mathcal{L}(X,Z)$ there is an SCD operator $T_1 \in \mathcal{L}(X,Z_1)$ such that $\|Tx\| \leqslant \|T_1 x\|$ for every $x \in X$, then T is a target for G. In the case of $Z = Y$, we have $\|G + \mathbb{T}T\| = 1 + \|T\|$.*

Proof. By Remark 3.13 we may assume that T is an SCD operator and that $\|T\| \leqslant 1$. Let $\{\widehat{S}_n \colon n \in \mathbb{N}\}$ be a determining family of slices for $T(B_X)$. Then,

$$S_n := T^{-1}(\widehat{S}_n) \cap S_X$$

is a slice of S_X for each $n \in \mathbb{N}$. Since G has the aDP, Theorem 3.6.ii tells us that $G(S_n)$ is a spear set for every $n \in \mathbb{N}$, which implies that $\operatorname{gFace}(\operatorname{ext} B_{Y^*}, \mathbb{T}G(S_n))$ is a dense G_δ set in $(\operatorname{ext} B_{Y^*}, w^*)$ by Corollary 2.6.iii. As $(\operatorname{ext} B_{Y^*}, w^*)$ is a Baire space (see Lemma 2.5.c), we deduce that the intersection

$$\bigcap_{n\in\mathbb{N}} \operatorname{gFace}(\operatorname{ext} B_{Y^*}, \mathbb{T}G(S_n))$$

is weak-star dense in $\operatorname{ext} B_{Y^*}$. Observe that, by Proposition 3.15, it suffices to show that this intersection is contained in $\mathscr{D}_T$. Given y_0^* belonging to this intersection, we have that for every $n \in \mathbb{N}$ and $\varepsilon > 0$, $G(S_n) \cap \mathbb{T}\operatorname{Slice}(B_Y, y_0^*, \varepsilon) \neq \emptyset$. Therefore, $S_n \cap \mathbb{T}\operatorname{gSlice}(B_X, G^*y_0^*, \varepsilon) \neq \emptyset$, and so $T(\mathbb{T}\operatorname{gSlice}(B_X, G^*y_0^*, \varepsilon)) \cap \widehat{S}_n \neq \emptyset$. Using that the family $\{\widehat{S}_n \colon n \in \mathbb{N}\}$ is determining for $T(B_X)$, we conclude that

$$T(B_X) \subset \overline{\operatorname{conv}\left(T(\mathbb{T}\operatorname{gSlice}(B_X, G^*y_0^*, \varepsilon))\right)} = \overline{T\left(\operatorname{aconv}\left(\operatorname{gSlice}(B_X, G^*y_0^*, \varepsilon)\right)\right)}$$

and, therefore, $y_0^* \in \mathscr{D}_T$. □

As a consequence of the previous results, we may present a class of operators which is a two-sided operator ideal consisting of operators T satisfying the condition $\|G + \mathbb{T}T\| = 1 + \|T\|$ whenever G has the aDP. Let us recall the needed definitions which we borrow from [9] and [70]. Let X, Y be Banach spaces, an operator $T \in \mathcal{L}(X,Y)$ is *hereditarily SCD* if $T(B_X)$ is a hereditarily SCD set, that is, if every convex subset B of $T(B_X)$ is SCD. Obviously, hereditarily SCD operators are SCD. The operator T is *HSCD-majorized* if there is a Banach space Z and a hereditarily SCD operator $\widetilde{T} \in \mathcal{L}(X,Z)$ such that $\|Tx\| \leqslant \|\widetilde{T}x\|$ for every $x \in X$. It is shown in [70, Theorem 3.1] that the class of HSCD-majorized operators is a two sided operator ideal. By Examples 1.56, this ideal contains those operators with separable range such that the image of the unit ball has the Radon-Nikodým Property, or the convex point of continuity property, or it is an Asplund set, and those operators with separable rank which do not fix copies of ℓ_1.

Corollary 3.18. *Let X, Y, Z be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be an operator with the aDP. If $T \in \mathcal{L}(X,Z)$ is a HSCD-majorized operator then T is a target for G. In the case of $Z = Y$, we have $\|G + \mathbb{T}T\| = 1 + \|T\|$.*

Corollary 3.19. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be an operator with the aDP. Then, the class of operators $T \in \mathcal{L}(X,Y)$ satisfying $\|G + \mathbb{T}T\| = 1 + \|T\|$ contains the component in $\mathcal{L}(X,Y)$ of the two-sided operator ideal formed by the HSCD-majorized operators.*

Even in the case when $G = \mathrm{Id}$, the result above is new.

Corollary 3.20. *Let X be a Banach space with the alternative Daugavet property. Then, the class of operators $T \in \mathcal{L}(X)$ satisfying $\|\mathrm{Id} + \mathbb{T}T\| = 1 + \|T\|$ contains the component in $\mathcal{L}(X)$ of the two-sided operator ideal formed by the HSCD-majorized operators.*

We can extend Theorem 3.17 to the non-separable setting in the following way.

Proposition 3.21. *Let X, Y, Z be Banach spaces and let $G \in \mathcal{L}(X,Y)$. If G has the aDP and $T \in \mathcal{L}(X,Z)$ satisfies that $T(B_{X_0})$ is an SCD set for every separable subspace X_0 of X, then T is a target for G. Therefore, if $Z = Y$ then $\|G + \mathbb{T}T\| = 1 + \|T\|$.*

Proof. Our aim is to use Theorem 3.14 to deduce that T is a target for G. To do so let us fix separable subspaces $X_0 \subset X$ and $Y_0 \subset Y$. By Proposition 3.7 we can find separable subspaces $X_0 \subset X_\infty \subset X$ and $Y_0 \subset Y_\infty \subset Y$ such that $G(X_\infty) \subset Y_\infty$ and $G|_{X_\infty} : X_\infty \longrightarrow Y_\infty$ has norm one and the aDP. Now, as $T|_{X_\infty} : X_\infty \longrightarrow Z$ is SCD, Theorem 3.17 tells us that $T|_{X_\infty}$ is a target for $G|_{X_\infty}$ and we can apply Theorem 3.14 to get that T is a target for G. □

Using the known results about SCD sets (see Examples 1.56), we may provide with the following consequence.

Corollary 3.22. *Let X, Y, Z be Banach spaces and let $G \in \mathcal{L}(X,Y)$. Suppose that G has the aDP and $T \in \mathcal{L}(X,Z)$ satisfies that $T(B_X)$ has one of the following properties: Radon-Nikodým Property, Asplund Property, convex point of continuity property or absence of ℓ_1-sequences. Then, T is a target for G. Therefore, if $Z = Y$ then $\|G + \mathbb{T}T\| = 1 + \|T\|$.*

Proof. In [9, §5] (see Examples 1.56) it is shown that any of the previous properties implies that the requirements of Proposition 3.21 are satisfied. □

To finish the discussion about which operators are targets for a given aDP operator, we may also extend Corollary 3.19 to operators with non separable range as follows. Given two Banach spaces X and Y, consider the class of those operators $T \in \mathcal{L}(X,Y)$ such that for every separable subspace X_0 of X, $T|_{X_0}$ is HSCD-majorized. As a consequence of the cited result [70, Theorem 3.1], this class is a two sided operator ideal. Therefore, extending straightforwardly the proof of Proposition 3.21, we get the following result.

Corollary 3.23. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be an operator with the aDP. The class of operators $T \in \mathcal{L}(X,Y)$ satisfying $\|G+\mathbb{T}T\| = 1+\|T\|$ contains the component in $\mathcal{L}(X,Y)$ of the two-sided operator ideal of those operators such that their restrictions to separable subspaces are HSCD-majorized. Moreover, this ideal contains those operators for which the image of the unit ball has one of the following properties: Radon-Nikodým Property, Asplund Property, convex point of continuity property, or absence of ℓ_1-sequences.*

We finish this section with two results concerning elements satisfying property ($\diamond$) in Definition 3.9.

Proposition 3.24. *Let X be a Banach space and let $x_0 \in B_X$. Then, x_0 is a spear vector if and only if x_0 belongs to $\operatorname{ext} B_{X^{**}}$ and satisfies ($\diamond$) with $G = T = \mathrm{Id}_X$.*

Proof. Using Theorem 3.11, x_0 has property ($\diamond$) for $G = T = \mathrm{Id}_X$ if and only if

$$\mathscr{D}(x_0) := \left\{x^* \in \operatorname{ext} B_{X^*} : x_0 \in \overline{\operatorname{aconv}}\big(\operatorname{gSlice}(S_X, x^*, \varepsilon)\big) \text{ for every } \varepsilon > 0\right\}$$

is dense in $(\operatorname{ext} B_{X^*}, w^*)$. If x_0 is a spear, then $x_0 \in \operatorname{ext} B_{X^{**}}$ by Proposition 2.11.b. Moreover, the definition of spear yields that $|x^*(x_0)| = 1$ for each $x^* \in \operatorname{ext} B_{X^*}$, and so $\mathscr{D}(x_0) = \operatorname{ext} B_{X^*}$.

Let us see the converse. If $x_0 \in \operatorname{ext} B_{X^{**}}$ then for each $x^* \in \mathscr{D}(x_0)$ and $\varepsilon > 0$, we have that x_0 is an extreme point of $\overline{\operatorname{aconv}}^{\sigma(X^{**},X^*)}\big(\operatorname{gSlice}(S_X, x^*, \varepsilon)\big)$. By Milman's Theorem (see Lemma 2.5.b), we deduce that $x_0 \in \overline{\operatorname{gSlice}(S_X, \mathbb{T}x^*, \varepsilon)}^{\sigma(X^{**},X^*)}$, and so $|x^*(x_0)| \geqslant 1-\varepsilon$. Since $\varepsilon > 0$ is arbitrary and $\mathscr{D}(x_0)$ is weak-star dense in $\operatorname{ext} B_{X^*}$, we conclude that $|x^*(x_0)| = 1$ for each $x^* \in \operatorname{ext} B_{X^*}$, which means that x_0 is a spear by Corollary 2.8.iv. □

Proposition 3.25. *Let X, Y, Z be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Given $T \in \mathcal{L}(X,Z)$, the set of points $x_0 \in B_X$ satisfying ($\diamond$) in Definition 3.9 is absolutely convex and closed.*

Proof. Let B be the set of points satisfying ($\diamond$) for G and T. Fixed $x_0 \in \overline{\operatorname{aconv}}(B)$, $\varepsilon > 0$, and $y \in S_Y$, there is a finite subset $F \subset B$ with $\operatorname{dist}(x_0, \operatorname{aconv}(F)) < \varepsilon$. By Theorem 3.11, we have that $\bigcap_{b\in F} \mathscr{D}_T(b, \operatorname{ext} B_{Y^*})$ is a dense G_δ subset of $(\operatorname{ext} B_{Y^*}, w^*)$. Take any

$$y^* \in \left[\bigcap_{b\in F} \mathscr{D}_T(b, \operatorname{ext} B_{Y^*})\right] \cap \operatorname{gSlice}(B_{Y^*}, y, \varepsilon).$$

Then, Tb belongs to $\overline{T\big(\operatorname{aconv} \operatorname{gSlice}(S_X, G^*y^*, \varepsilon)\big)}$ for each $b \in F$, and so

$$\operatorname{dist}\big(Tx_0, T\big(\operatorname{aconv} \operatorname{gSlice}(S_X, G^*y^*, \varepsilon)\big)\big) \leqslant \|T\| \operatorname{dist}(x_0, \operatorname{aconv}(F)) < \|T\|\varepsilon.$$

A straightforward normalization gives that x_0 satisfies ($\diamond$) for G and T, so $x_0 \in B$ and $B = \overline{\operatorname{aconv}}(B)$, as desired. □

3.4 Lush operators

We start with the definition of lush operator, which generalizes the concept of lush space when applied to the Identity.

Definition 3.26. Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. We say that G is *lush* if Id_X is a target for G.

From the definition of target (or better from Remark 3.13.b), it follows immediately the following observation.

Remark 3.27. G is lush if and only if every operator whose domain is X is a target for G. In particular, every lush G is a spear operator, that is,

$$\|G + \mathbb{T}T\| = 1 + \|T\|$$

for every $T \in \mathcal{L}(X,Y)$.

Let us summarize the results of the previous section when applied to lushness.

Proposition 3.28. *Let X, Y be Banach spaces, let $\mathscr{A} \subset B_{Y^*}$ with $\overline{\operatorname{conv}}^{w^*}(\mathscr{A}) = B_{Y^*}$ and let $\mathscr{B} \subset B_X$ with $\overline{\operatorname{aconv}}(\mathscr{B}) = B_X$. Then the following assertions are equivalent for a norm-one operator $G \in \mathcal{L}(X,Y)$:*

(i) G is lush.
(ii) For every $x_0 \in \mathscr{B}$, $y \in S_Y$ and $\varepsilon > 0$ there is $F \subset B_X$ such that

$$\operatorname{conv}(F) \subset \{x \in B_X : \|Gx + y\| > 2 - \varepsilon\} \text{ and } \operatorname{dist}(x_0, \operatorname{aconv}(F)) < \varepsilon.$$

(iii) For every $x_0 \in \mathscr{B}$, $y \in S_Y$ and $\varepsilon > 0$ there exists $y^* \in \operatorname{Slice}(\mathscr{A}, y, \varepsilon)$ such that

$$\operatorname{dist}\big(x_0, \operatorname{aconv}(\operatorname{gSlice}(S_X, G^*y^*, \varepsilon))\big) < \varepsilon.$$

(iv) For every $x_0 \in \mathscr{B}$, the set

$$\mathscr{D}(x_0) = \left\{y^* \in \operatorname{ext} B_{Y^*} : x_0 \in \overline{\operatorname{aconv}}\big(\operatorname{gSlice}(S_X, G^*y^*, \varepsilon)\big) \textit{ for every } \varepsilon > 0\right\}$$

is a dense (G_δ) subset of $(\operatorname{ext} B_{Y^*}, w^*)$.

(v) *For every $x_0 \in \mathscr{B}$, every $y \in S_Y$ and every $\varepsilon > 0$, there exists $y^* \in \operatorname{ext}(B_{Y^*})$ such that*

$$y \in \operatorname{Slice}(S_Y, y^*, \varepsilon) \quad \textit{and} \quad x_0 \in \overline{\operatorname{aconv}}\big(\operatorname{gSlice}(S_X, G^*y^*, \varepsilon)\big).$$

(vi) *For every separable subspaces $X_0 \subset X$ and $Y_0 \subset Y$, we can find separable subspaces $X_0 \subset X_\infty \subset X$ and $Y_0 \subset Y_\infty \subset Y$ such that $G(X_\infty) \subset Y_\infty$, $\|G|_{X_\infty}\| = 1$ and $G|_{X_\infty} : X_\infty \longrightarrow Y_\infty$ is lush.*

Proof. The equivalences are consequence of Proposition 3.10, Theorem 3.11 and Theorem 3.14, together with Proposition 3.25 to pass from $\mathscr{B}$ to $B_X = \overline{\operatorname{aconv}}(\mathscr{B})$. □

Next, we get from the previous sections some conditions for an operator having the aDP to be lush. The main result in this line is the next one, which follows from Theorem 3.17 applied to $T = \operatorname{Id}_X$.

Theorem 3.29. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Suppose that B_X is SCD. Then, G has the aDP if and only if G is lush.*

As all the properties involved in the above result are separably determined, we have the following generalization.

Corollary 3.30. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Suppose that B_{X_0} is SCD for every separable subspace $X_0 \subset X$. Then, G has the aDP if and only if G is lush.*

Proof. Since every lush operator is a spear, it has in particular the aDP. The converse is consequence of Proposition 3.21 applied to $T = \operatorname{Id}_X$. □

The most interesting particular cases of the above results are summarized in the next corollary, which uses the examples of SCD spaces provided in Examples 1.56.

Corollary 3.31. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Suppose that X has one of the following properties: Radon-Nikodým Property, Asplund Property, convex point of continuity property or absence of isomorphic copies of ℓ_1. Then, G has the aDP if and only if G is lush.*

A result of this kind for the codomain space will be given in Proposition 5.3: if $G \in \mathcal{L}(X,Y)$ has the aDP and Y is Asplund, then G is lush.

Our next aim is to provide the following sufficient conditions for an operator to be lush which will be used in the next chapters.

Proposition 3.32. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Then, each of the following conditions ensures G to be lush.*

(a) The set $\{y^* \in B_{Y^*} : G^*y^* \in \operatorname{Spear}(X^*)\}$ is norming for Y.
(b) The set $\{y^* \in \operatorname{ext} B_{Y^*} : G^*y^* \in \operatorname{Spear}(X^*)\}$ is dense in $(\operatorname{ext} B_{Y^*}, w^*)$.
(c) $B_X = \overline{\operatorname{conv}}\{x \in B_X : Gx \in \operatorname{Spear}(Y)\}$.

Proof. The fact that (a) implies lushness follows from Proposition 3.28.v, as Theorem 2.9 gives that $B_X = \overline{\operatorname{aconv}}\big(\operatorname{Face}(S_X, G^*y^*)\big)$ for every $y^* \in B_{Y^*}$ such that $G^*y^* \in \operatorname{Spear}(X^*)$. Condition (b) is a particular case of condition (a). Finally, by using Corollary 2.8.iv, condition (c) implies that every $y^* \in \operatorname{ext}(B_{Y^*})$ satisfies $B_X = \overline{\operatorname{aconv}}\big(\operatorname{Face}(S_X, G^*y^*)\big)$, so G^*y^* is a spear vector by (the easy part of) Theorem 2.9. □

We do not know whether the conditions (a) or (b) above are necessary for lushness in general, but they are when the domain space is separable as the following deep result shows. We will see later that they are also necessary when the codomain is an Asplund space (see Proposition 5.3).

Theorem 3.33. *Let X be a separable Banach space and let Y be a Banach space. If $G \in \mathcal{L}(X,Y)$ is lush, then the set $\Omega = \{y^* \in \operatorname{ext} B_{Y^*} : G^*y^* \in \operatorname{Spear}(X^*)\}$ is a G_δ dense subset of $(\operatorname{ext} B_{Y^*}, w^*)$. In other words, if G is lush, there exists a G_δ dense subset Ω of $(\operatorname{ext} B_{Y^*}, w^*)$ such that*

$$B_X = \overline{\operatorname{aconv}}\big(\operatorname{Face}(S_X, G^*y^*)\big)$$

for every $y^* \in \Omega$.

Proof. This is consequence of Proposition 3.16 and the characterization of spear vectors given in Corollary 2.8. The last part is a consequence of Theorem 2.9. □

On the other hand, condition (c) of Proposition 3.32 is not in general necessary for lushness: consider $X = Y = c_0$ and $G = \operatorname{Id}$, which is lush as c_0 is a lush space, but $\operatorname{Spear}(Y)$ is empty as B_{c_0} contains no extreme points. We will see later that condition (c) is necessary when the domain space has the Radon-Nikodým Property (see Proposition 5.2).

We finish the section with some elementary observations analogous to the ones given for spear operators and for operators with the aDP.

Remark 3.34. Let X, Y be Banach spaces and $G \in \mathcal{L}(X,Y)$.

(i) *The composition with isometric isomorphisms preserves lushness: If X_1, Y_1 are Banach spaces and $\Phi_1 \in \mathcal{L}(X_1, X)$, $\Phi_2 \in \mathcal{L}(Y, Y_1)$ are isometric isomorphisms, then, $G \in \mathcal{L}(X,Y)$ is lush if and only if $\Phi_2 G \Phi_1 \in \mathcal{L}(X_1, Y_1)$ is lush.*
(ii) *If G is lush and Z is a subspace of Y containing $G(X)$, then $G\colon X \longrightarrow Z$ is lush. However, lushness is not preserved by extending the codomain of the operator, as the same example of Remark 3.4 shows.*

(iii) *As an easy consequence of (i) and (ii), we have that the following statements are equivalent: (a) X is lush, (b) there exist a Banach space Z and an isometric isomorphism in $\mathcal{L}(X,Z)$ or in $\mathcal{L}(Z,X)$ which is lush, (d) there exist a Banach space W and an isometric embedding $G \in \mathcal{L}(X,W)$ which is lush.*

Chapter 4

Some examples in classical Banach spaces

Our aim here is to present examples of operators which are lush, spear, or have the aDP, defined in some classical Banach spaces. One of the most intriguing examples is the Fourier transform on L_1, which we prove that is lush. Next, we study a number of examples of operators arriving to spaces of continuous functions. In particular, it is shown that every uniform algebra is lush-embedded into a space of bounded continuous functions. Finally, examples of operators acting from spaces of integrable functions are studied.

4.1 Fourier transform

Let H be a locally compact Abelian group and let σ be the Haar measure on H. The dual group Γ of H is the set of all continuous homomorphisms $\gamma\colon H \longrightarrow \mathbb{T}$ endowed with a topology that makes it a locally compact group (see [111, §1.2] for the details). If $L_1(H)$ is the space of σ-integrable functions over H, and $C_0(\Gamma)$ is the space of continuous functions on Γ which vanish at infinity, then the *Fourier transform* $\mathscr{F}\colon L_1(H) \longrightarrow C_0(\Gamma)$ is defined as

$$\mathscr{F}(f)\colon \Gamma \longrightarrow \mathbb{C}, \qquad \big[\mathscr{F}(f)\big](\gamma) = \int_H f(x)\gamma(x^{-1})\, d\sigma(x).$$

Theorem 4.1. *Let H be a locally compact Abelian group and let Γ be its dual group. Then, the Fourier transform $\mathscr{F}\colon L_1(H) \longrightarrow C_0(\Gamma)$ is lush. In particular, $\mathscr{F}$ is a spear operator, that is,*

$$\|\mathscr{F} + \mathbb{T}T\| = 1 + \|T\|$$

for every $T \in \mathcal{L}(L_1(H), C_0(\Gamma))$.

Proof. For each $\gamma \in \Gamma$, $\mathscr{F}^*(\delta_\gamma)$ corresponds to the function $g \in L_\infty(H) \equiv L_1(H)^*$ given by $g(x) = \gamma(x^{-1})$ for every $x \in H$. Hence, $|g(x)| = 1$ for every $x \in H$, and so

$\mathscr{F}^*(\delta_\gamma)$ is a spear of $L_\infty(H) \equiv L_1(H)^*$ by Example 2.12.d. As $\mathbb{T}\{\delta_\gamma\colon \gamma \in \Gamma\}$ is the set of extreme points of $B_{C_0(\Gamma)^*}$, Proposition 3.32.b shows that $\mathscr{F}$ is lush. □

4.2 Operators arriving to sup-normed spaces

Our goal here is to study various families of operators arriving to spaces of continuous functions. We start with a general result.

Proposition 4.2. *Let X be a Banach space, let L be a locally compact Hausdorff topological space and let $G \in \mathcal{L}(X, C_0(L))$ be a norm-one operator. Consider the following statements:*

(i) The set $\{t \in L\colon G^*\delta_t \in \operatorname{Spear}(X^*)\}$ is dense in L.
(ii) G is lush.
(iii) G is a spear operator.
(iv) G has the aDP.
(v) $\{G^*\delta_t : t \in U\}$ is a spear set of B_{X^*} for every open subset $U \subset L$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Leftrightarrow$ (v).

Besides, we have the following:

(a) If L is scattered, then all of the statements are equivalent.
(b) If X is separable, then (i) $\Leftrightarrow$ (ii).

Proof. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are clear. Using that

$$\operatorname{ext}\left(B_{C_0(L)^*}\right) = \mathbb{T}\{\delta_t : t \in L\},$$

we conclude easily (i) $\Rightarrow$ (ii) from Proposition 3.32. Let us prove (iv) $\Rightarrow$ (v). Given an open subset $U \subset L$, take $h\colon L \longrightarrow [0,1]$ in $C_0(L)$ with $\|h\|_\infty = 1$ and $\operatorname{supp}(h) \subset U$. Then, for $0 < \varepsilon < 1$, the weak-star slice $S := \operatorname{Slice}(\operatorname{ext} B_{C_0(L)^*}, h, \varepsilon)$ is contained in $\mathbb{T}\{\delta_t : t \in U\}$, and since $G^*(S)$ is a spear set by Theorem 3.6, we deduce that so does $\{G^*\delta_t : t \in U\}$. To check that (v) $\Rightarrow$ (iv), notice that for every weak-star slice S of $\operatorname{ext} B_{C_0(L)^*}$, we can find an open subset $V \subset L$ such that $\{\delta_t : t \in V\} \subset \mathbb{T}S$, and thus $G^*(S)$ is a spear set whenever $\{G^*\delta_t : t \in V\}$ is.

(a). L is scattered if and only if $C_0(L)$ is Asplund (see [8, Comment after Corollary 2.6], for instance). Notice that $G^*\colon C_0(L)^* \longrightarrow X^*$ is weak-star to weak-star continuous, so we just have to prove that given an open set $U \subset L$, there exists $t \in U$ such that $G^*\delta_t$ is a spear. Since $G^*(U)$ is weak-star fragmentable (see [39, Theorem 11.8] for the definition and relation between Asplundness and fragmentability), for each $\varepsilon > 0$ there exists a weak-star open set V satisfying that $V \cap G^*(U)$ has diameter less than ε . Now, since G^* is weak-star continuous, we can find an open set $W \subset L$ with $G^*(\overline{W}) \subset V \cap G^*(U)$. Because of the local compactness of L, we can select W in such a way that $\overline{W}$ is compact. In particular $G^*(\overline{W})$ is a closed spear set with diameter less than ε. It is clear that we can iterate this process to construct a decreasing sequence $\big(W_n\big)_{n\in\mathbb{N}}$ of open subsets of U such that $\operatorname{diam}\big(G^*(\overline{W_n})\big)$ tends to zero. Since $G^*(\overline{W_n})$ is a spear set by (v), it follows then from Lemma 2.10 that an element $t\in\bigcap_n\overline{W_n}\subset U$ must satisfy that $G^*\delta_t$ is a spear.

(b). If X is separable, the result follows from Theorem 3.33. □

Let us mention that (a) is also a particular case of Proposition 5.3 in the next chapter, using the stated above equivalence between L being scattered and $C_0(L)$ being Asplund. In the more restrictive case in which L has the discrete topology (which is trivially scattered), the result will be also proved, with a different approach, in Example 5.5.

The next result characterizes lush spaces. We need some notation. Let Ω be a completely regular Hausdorff topological space and denote by $C_b(\Omega)$ the Banach space of all scalar bounded and continuous functions on Ω endowed with the supremum norm.

Theorem 4.3. *Let X be a Banach space. Then, X is lush if and only if the canonical inclusion $J\colon X\longrightarrow C_b(\operatorname{ext}B_{X^*})$ is lush.*

Recall that for a completely regular Hausdorff topological space Ω the corresponding $C_b(\Omega)$ can be canonically seen as a $C(K)$ space by taking $K=\beta\Omega$, the Stone-Čech compactification of Ω. Since Ω is a dense subset of $\beta\Omega$, we have that the set $\mathbb{T}\{\delta_t : t\in\Omega\}$ is dense in $(\operatorname{ext}(B_{C_b(\Omega)^*}),w^*)$.

Proof. Since J is an isometry, it easily follows that Id_X is lush whenever J is lush (see Remark 3.34). Let us see the converse implication. The space $\Omega:=\operatorname{ext}B_{X^*}$ endowed with the weak-star topology is completely regular. As we mentioned before, the set

$$\mathscr{A}:=\mathbb{T}\{\delta_{x^*} : x^*\in\Omega\}$$

is dense in $(\operatorname{ext}(C_b(\Omega)^*),w^*)$ so, in particular,

$$B_{C_b(\Omega)^*}=\overline{\operatorname{conv}}^{w^*}(\mathscr{A}).$$

By Proposition 3.10, J is lush if and only if for every $x_0\in B_X$ and $\varepsilon>0$ the set

$$\mathscr{D}^{\varepsilon}(\mathscr{A},x_0)=\{\theta\delta_{x^*}:\ \theta\in\mathbb{T},\,x^*\in\Omega,\,\operatorname{dist}(x_0,\operatorname{aconv}(\operatorname{gSlice}(B_X,J^*(\theta\delta_{x^*}),\varepsilon)))<\varepsilon\}$$

intersects every w^*-slice of $\mathscr{A}$, or equivalently, if it is dense in $(\mathscr{A},w^*)$ (as the slices form a basis of the weak-star topology by Lemma 2.5.a). Notice that $J^*(\theta\delta_{x^*})=\theta x^*$ and, moreover, $\operatorname{gSlice}(B_X,\theta x^*,\varepsilon)=\theta^{-1}\operatorname{gSlice}(B_X,x^*,\varepsilon)$, thus

$$\mathscr{D}^{\varepsilon}(\mathscr{A},x_0)=\mathbb{T}\{\delta_{x^*} : x^*\in\Omega,\,\operatorname{dist}(x_0,\operatorname{aconv}(\operatorname{gSlice}(B_X,x^*,\varepsilon)))<\varepsilon\}.$$

Using that X is lush, we have for every $\varepsilon>0$ that

$$\Omega \subset \overline{\{x^* \in \Omega :\ \operatorname{dist}(x_0, \operatorname{aconv}(\operatorname{gSlice}(B_X, x^*, \varepsilon))) < \varepsilon\}}^{w^*},$$

or equivalently

$$\{\delta_{x^*} : x^* \in \Omega\} \subset \overline{\{\delta_{x^*} : x^* \in \Omega\,,\ \operatorname{dist}(x_0, \operatorname{aconv}(\operatorname{gSlice}(B_X, x^*, \varepsilon))) < \varepsilon\}}^{w^*}.$$

Therefore, $\mathscr{D}^\varepsilon(\mathscr{A}, x_0)$ is dense in $(\mathscr{A}, w^*)$ for each $\varepsilon > 0$ and we can then conclude that J is lush. □

The following very general result will allow us to deduce many other interesting examples.

Theorem 4.4. *Let Γ be a non empty set and let $\Gamma \in \mathscr{A} \subset \mathscr{P}(\Gamma)$. Let $X \subset Y \subset \ell_\infty(\Gamma)$ be Banach spaces satisfying the following properties:*

(i) For every $y \in S_Y$, $\varepsilon > 0$ and $A \in \mathscr{A}$, there exist $b \in \mathbb{K}$ and $U \in \mathscr{A}$ such that $A \supset U$,

$$|b| = \sup_{t \in A} |y(t)| \quad \text{and} \quad |y(t) - b| < \varepsilon \quad \text{whenever } t \in U.$$

(ii) For each $A \in \mathscr{A}$ there is $h \in \ell_\infty(\Gamma)$ such that

$$h(\Gamma) \subset [0,1], \quad \operatorname{supp}(h) \subset A, \quad \|h\|_\infty = 1 \quad \text{and} \quad \operatorname{dist}(h, X) < \varepsilon.$$

Then, the inclusion $J : X \longrightarrow Y$ is lush.

Proof. Fix $x \in S_X$, $y \in S_Y$ and $0 < \varepsilon < 1$. Since $\Gamma \in \mathscr{A}$, we can find $A \in \mathscr{A}$ and $b \in \mathbb{T}$ such that

$$|y(t) - b| < \frac{\varepsilon}{9} \quad \text{for each } t \in A.$$

Again by (i) we can find $U \in \mathscr{A}$ such that $A \supset U$, and $a \in \overline{\mathbb{D}}$ with $|a| = \sup_{t \in A} |x(t)|$ such that

$$|x(t) - a| < \frac{\varepsilon}{9} \quad \text{for each } t \in U.$$

By (ii), there is $h \in \ell_\infty(\Gamma)$, with $h(\Gamma) \subset [0,1]$, $\operatorname{supp}(h) \subset U$, $\|h\|_\infty = 1$ and $\operatorname{dist}(h, X) < \varepsilon/9$. Let us fix $x_0 \in X$ with $\|h - x_0\|_\infty < \varepsilon/9$.

We claim that for each $\gamma \in \mathbb{K}$ with $|a + \gamma b| = 1$ one has that

$$\|x + \gamma b x_0\|_\infty \leqslant 1 + \frac{\varepsilon}{3}.$$

Indeed, the conditions $|a + \gamma b| = 1$, $|b| = 1$ and $|a| \leqslant 1$ imply that $|\gamma| \leqslant 2$. We distinguish two cases: if $t \notin U$ then $h(t) = 0$ which gives $|x_0(t)| < \varepsilon/9$, so we get that

$$|x(t) + \gamma b x_0(t)| \leqslant 1 + \frac{\varepsilon}{3}.$$

On the other hand, if $t \in U$ then

$$\begin{aligned}|x(t)+\gamma b x_0(t)| &\leqslant |x(t)-a|+|a(1-h(t))+h(t)(a+\gamma b)|+|b||\gamma||x_0(t)-h(t)|\\ &< \frac{\varepsilon}{9}+1+\frac{2\varepsilon}{9}=1+\frac{\varepsilon}{3}.\end{aligned}$$

This finishes the proof of the claim.

Observe that 0 belongs to the convex hull of the set $\{\gamma\in\mathbb{K}\colon |a+\gamma b|=1\}$ as $|a|\leqslant 1=|b|$. We can then find $\gamma_1,\gamma_2\in\{\gamma\in\mathbb{K}\colon |a+\gamma b|=1\}$ and $0\leqslant\lambda\leqslant 1$ such that $\lambda\gamma_1+(1-\lambda)\gamma_2=0$. Take $t_0\in U$ with $h(t_0)>1-\varepsilon/9$, pick $\theta_0,\theta_1,\theta_2\in\mathbb{T}$ satisfying $\theta_1\big(x(t_0)+\gamma_1 b x_0(t_0)\big)\geqslant 0$, $\theta_2\big(x(t_0)+\gamma_2 b x_0(t_0)\big)\geqslant 0$ and $\theta_0 y(t_0)\geqslant 0$. Define

$$x_1=\theta_1\frac{x+\gamma_1 b x_0}{1+\varepsilon/3}\qquad\text{and}\qquad x_2=\theta_2\frac{x+\gamma_2 b x_0}{1+\varepsilon/3}.$$

By the claim above, we have that $x_j\in B_X$. Besides, we can write

$$\begin{aligned}\left(1+\frac{\varepsilon}{3}\right)|x_j(t_0)| &= |x(t_0)+\gamma_j b x_0(t_0)|\\ &\geqslant |a+\gamma_j b|-|x(t_0)-a|-|\gamma_j||b||x_0(t_0)-1|\geqslant 1-\frac{5\varepsilon}{9}.\end{aligned}$$

Moreover, $x_j(t_0)\geqslant 0$ and for every $\mu\in[0,1]$ we have that

$$\mu x_1(t_0)+(1-\mu)x_2(t_0)\geqslant\frac{1-5\varepsilon/9}{1+\varepsilon/3}=1-\frac{8\varepsilon}{9+3\varepsilon}>1-\frac{8\varepsilon}{9}.$$

Therefore, we can estimate as follows

$$\begin{aligned}\big\|y+\theta_0^{-1}J\big(\mu x_1+(1-\mu)x_2\big)\big\| &\geqslant \big|\theta_0 y(t_0)+\mu x_1(t_0)+(1-\mu)x_2(t_0)\big)\big|\\ &= |y(t_0)|+\mu x_1(t_0)+(1-\mu)x_2(t_0)\\ &> 1-\frac{\varepsilon}{9}+1-\frac{8\varepsilon}{9}=2-\varepsilon\end{aligned}$$

and, moreover,

$$\operatorname{dist}(x,\operatorname{aconv}(\{x_1,x_2\}))\leqslant\|x-(\lambda\theta_1^{-1}x_1+(1-\lambda)\theta_2^{-1}x_2)\|=\left\|x-\frac{x}{1+\varepsilon/3}\right\|\leqslant\varepsilon.$$

This shows that J is lush by (ii) of Proposition 3.28 with $F=\{x_1,x_2\}$. □

For the convenience of the reader, let us recall next the concept of C-rich subspace from Definition 1.19 avoiding the use of C-narrow operators (see also Proposition 1.20).

Definition 4.5. Let Ω be a Hausdorff topological space. A closed subspace $X\subset C_b(\Omega)$ is called *C-rich* if for every $\varepsilon>0$ and every open subset $U\subset\Omega$, there exists $h\in S_{C_b(\Omega)}$ such that $h(\Omega)\subset[0,1]$, $\operatorname{supp}(h)\subset U$ and $d(h,X)<\varepsilon$. Moreover, if Ω is a normal Hausdorff space (in particular, if it is a compact Hausdorff space) then we can omit the condition $h(\Omega) \subset [0,1]$.

Remark. If Ω is a completely regular Hausdorff space, then $C_b(\Omega)$ is C-rich in itself. Indeed, we have by definition that for every open subset $U \subset \Omega$ and $x \in U$ there is a continuous function $h\colon \Omega \longrightarrow [0,1]$ such that $\operatorname{supp}(h) \subset U$ and $h(x) = 1$.

The main tool in the rest of the section will be the following.

Theorem 4.6. *Let Ω be a Hausdorff topological space. If $X \subset C_b(\Omega)$ is C-rich, then the inclusion $J\colon X \longrightarrow C_b(\Omega)$ is lush. In particular, we have $\|J + \mathbb{T}T\| = 1 + \|T\|$ for every $T \in \mathcal{L}(X, C_b(\Omega))$.*

Proof. We just have to check that the hypothesis of Theorem 4.4 are satisfied for $X \subset C_b(\Omega) \subset \ell_\infty(\Omega)$ and taking as $\mathscr{A}$ the family of all open subsets of Ω. Hypothesis (i) is satisfied by just using the continuity, while (ii) is consequence of the C-richness of X. □

Remark 4.7. Let us observe that there are natural inclusions $J\colon X \longrightarrow C_b(\Omega)$ which are lush without X being a C-rich subspace of $C_b(\Omega)$. For instance, using Theorem 4.3 we deduce that the inclusion

$$J\colon \ell_1 \longrightarrow C(\mathbb{T}^{\mathbb{N}}), \quad (a_n)_{n\in\mathbb{N}} \longmapsto \left[(z_n)_{n\in\mathbb{N}} \longmapsto \sum_{n=1}^{\infty} a_n z_n\right]$$

is lush. However, $J(\ell_1)$ is not C-rich in $C(\mathbb{T}^{\mathbb{N}})$. Indeed, we argue by contradiction. Let $\delta > 0$, consider the open set $U = \{z \in \mathbb{T}^{\mathbb{N}} \colon |z_1 - 1| < \delta\}$, and suppose that $h \in C(\mathbb{T}^{\mathbb{N}})$ and $a \in \ell_1$ satisfy that

$$h(\mathbb{T}^{\mathbb{N}}) \subset [0,1], \quad \|h\|_\infty = 1, \quad \operatorname{supp}(h) \subset U \quad \text{and} \quad \|J(a) - h\|_\infty < \delta.$$

Taking supremum over all $z \in \mathbb{T}^{\mathbb{N}} \setminus U$, we deduce that

$$(1 - \tfrac{\delta}{2})|a_1| + \sum_{n\leqslant 2} |a_n| \leqslant \sup_{|z_1-1|\geqslant\delta} \operatorname{Re}(a_1 z_1) + \sum_{n\leqslant 2} |a_n| \leqslant \sup_{|z_1-1|\geqslant\delta} |J(a)(z)| < \delta. \quad (4.1)$$

While $\|h\|_\infty = 1$ implies that

$$\|a\|_{\ell_1} = \sum_{n\geqslant 1} |a_n| > 1 - \delta. \quad (4.2)$$

Taking $\delta > 0$ small enough, (4.2) and (4.1) contradict each other.

Let us present some applications of Theorem 4.6. First, it was shown in [69, Proposition 1.2] that if K is a perfect compact Hausdorff topological space, then every finite-codimensional subspace of $C(K)$ is C-rich, but this is not always the case when K has isolated points. Actually, finite-codimensional C-rich subspaces of general $C(K)$ spaces were characterized in [22, Proposition 2.5] in terms of the supports of the functionals defining the subspace. We recall that the *support* of an element $F \in C(K)^*$ (represented by the regular measure μ_F) is

$$\operatorname{supp}(F) := \bigcap \{C \subset K \colon C \text{ closed}, \ |\mu_F|(K \setminus C) = 0\}.$$

We include the proof of this result for the convenience of the reader.

Corollary 4.8. *Let K be a compact Hausdorff topological space, consider functionals $F_1, \dots, F_n \in C(K)^*$ and let $Y = \bigcap_{i=1}^n \ker F_i$. If $\bigcup_{i=1}^n \operatorname{supp}(F_i)$ does not intersect the set of isolated points of K, then the natural inclusion $J \colon Y \longrightarrow C(K)$ is lush. In particular, if K is perfect, then for every finite-codimensional subspace Y of $C(K)$, the inclusion $J \colon Y \longrightarrow C(K)$ is lush.*

Proof. We only have to show that Y is C-rich and then apply Theorem 4.6. We fix a nonempty open subset U of K and $\varepsilon > 0$, and we may consider two cases. Case 1: U contains an isolated point of K (say, τ). Then $h = \mathbb{1}_{\{\tau\}} \in S_{C(K)}$ is a positive U-supported function which lies in Y, so the requirements of Definition 4.5 are fulfilled. Case 2: U does not contain isolated points of K. In this case one can find a sequence of disjoint open subsets $U_n \subset U$ and a sequence of functions $h_n \in S_{C(K)}$ with $\operatorname{supp}(h_n) \subset U_n$ for every $n \in \mathbb{N}$. Denote by $q \colon C(K) \longrightarrow C(K)/Y$ the natural quotient map. Since $\{h_n\}$ tends weakly to 0, $\{q(h_n)\}$ tends weakly to 0. But $C(K)/Y$ is finite-dimensional, so $\|q(h_n)\| \longrightarrow 0$ as well, and we can select $n \in \mathbb{N}$ with $\|q(h_n)\| < \varepsilon$ and $\operatorname{supp}(h_n) \subset U_n \subset U$, and Definition 4.5 is fulfilled. $\square$

For $C[0,1]$ we may even go to smaller subspaces, using a result of [69]: if X is a subspace of $C[0,1]$ such that $C[0,1]/X$ does not contain isomorphic copies of $C[0,1]$, then X is C-rich in $C[0,1]$ [69, Proposition 1.2 and Definition 2.1].

Corollary 4.9. *Let X be a subspace of $C[0,1]$ such that $C[0,1]/X$ does not contain isomorphic copies of $C[0,1]$ (in particular, if $C[0,1]/X$ is reflexive). Then, the inclusion $J \colon X \longrightarrow C[0,1]$ is lush.*

To get more results, we may use the spaces $C_E(K\|L)$ presented in section 1.4. We only have to call Corollary 1.21 to be able to apply Theorem 4.6.

Corollary 4.10. *Let K be a compact Hausdorff topological space and let L be a closed nowhere dense subset of K. Then, for every subspace E of $C(L)$, the inclusion $J \colon C_E(K\|L) \longrightarrow C(K)$ is lush.*

This applies, in particular, to the inclusion $J \colon X \longrightarrow \ell_\infty$ for every $c_0 \subset X \subset \ell_\infty$.

Let us now go to present the main part of this section. Recall that a *uniform algebra* (on a compact Hausdorff topological space K) is a closed subalgebra $A \subset C(K)$ that separates the points of K. We refer to [32] for background.

Theorem 4.11. *Let K be a compact Hausdorff topological space and let A be a uniform algebra on K. Then, there exists a subset $\Omega \subset K$ such that $A \subset C_b(\Omega)$ (isometrically) is C-rich, and so the inclusion $J\colon A \longrightarrow C_b(\Omega)$ is lush. Moreover, if A is unital, then Ω is just its Choquet boundary.*

Proof. If A is a unital uniform algebra, then consider $\Omega \subset K$ to be the Choquet boundary of A. Given $0 < \varepsilon < 1$ and $U \subset K$ with $U \cap \Omega \neq \emptyset$, take $0 < \eta < \varepsilon/4$ small enough so that every $z \in \mathbb{C}$ with $|z| + (1-\eta)|1-z| \leqslant 1$ satisfies that $|\operatorname{Im} z| < \varepsilon/2$. By [24, Lemma 2.5], there exists $f \in A$ and $t_0 \in U \cap \Omega$ such that $f(t_0) = \|f\|_\infty = 1$, $|f(t)| < \eta$ for each $t \in K \setminus U$ and

$$|f(t)| + (1-\eta)|1-f(t)| \leqslant 1 \text{ for each } t \in K.$$

Put $C := K \setminus U$ and $B = \{t \in U\colon |f(t)| \geqslant 2\eta\}$. These are disjoint compact subsets of K, so there exists $\varphi\colon K \longrightarrow [0,1]$ continuous such that $\varphi|_C \equiv 0$ and $\varphi|_B \equiv 1$. The element $h := |\operatorname{Re} f| \cdot \varphi\colon K \longrightarrow [0,1]$ belongs to $S_{C(K)}$ and satisfies $\operatorname{supp}(h) \subset U$. We just have to check that $\|h - f\|_\infty < \varepsilon$. Indeed, if $t \in B$ then $|1 - f(t)| < 1$, so $\operatorname{Re} f(t) > 0$ and so $|h(t) - f(t)| = |\operatorname{Im} f(t)| \leqslant \varepsilon$; on the other hand, if $t \in K \setminus B$ then $|h(t) - f(t)| \leqslant 4\eta < \varepsilon$. The restriction $h|_\Omega$ satisfies the definition of C-rich for the given $\varepsilon > 0$ and $U \cap \Omega \subset \Omega$.

If A is not unital, then we can repeat the same argument as above but now using [24, Lemma 2.7] and taking Ω as the set $\Gamma_0 \subset K$ that appears in the referenced lemma. □

The Choquet boundary of the disk algebra $A(\mathbb{D})$ is $\mathbb{T}$ (see [32, Proposition 4.3.13], for instance), so by the previous result we have the following consequence.

Corollary 4.12. *The natural inclusion $J\colon A(\mathbb{D}) \longrightarrow C(\mathbb{T})$ is lush.*

Another family of interesting C-rich subspaces is the following. Let H be an infinite compact Abelian group, let σ be the Haar measure on H, let Γ be the dual group of H, let $M(H)$ be the space of all regular Borel measures on H and, finally, let $\mathscr{F}\colon M(H) \longrightarrow C_b(\Gamma)$ be the Fourier-Stieltjes transform, which is the natural extension of the classical Fourier transform of section 4.1 (see [111, §1.3]). For $\Lambda \subset \Gamma$, the space of Λ-spectral continuous functions is defined by

$$C_\Lambda(H) = \big\{f \in C(H)\colon \big[\mathscr{F}(f)\big](\gamma) = 0 \ \forall \gamma \in \Gamma \setminus \Lambda\big\},$$

and similarly it is defined the space of Λ-spectral measures $M_\Lambda(H)$. These spaces are known to be precisely the closed translation invariant subspaces of $C(H)$ and $M(H)$, respectively. A subset Λ of Γ is said to be a *semi-Riesz set* [124, p. 126] if all elements of M_Λ are diffuse (i.e. if they map singletons to 0). Semi-Riesz sets include *Riesz sets*, defined as those $\Lambda \subset \Gamma$ such that $M_\Lambda \subset L_1(\sigma)$; the chief example of a Riesz subset of the dual group $\Gamma = \mathbb{Z}$ of $H = \mathbb{T}$ is $\Lambda = \mathbb{N}$. We refer to [54, §IV.4] for background. It is shown in [86, Theorem 4.13] that $\Gamma \setminus \Lambda^{-1}$ is a semi-Riesz set if and only if $C_\Lambda(H)$ is C-rich in $C(H)$. Therefore, we have the following consequence of Theorem 4.6.

Corollary 4.13. *Let H be an infinite compact Abelian group and let Λ be a subset of the dual group Γ of H. If $\Gamma \setminus \Lambda^{-1}$ is a semi-Riesz set, then the inclusion $J\colon C_\Lambda(H) \longrightarrow C(H)$ is lush.*

Remark 4.14. It is proved in [124, Theorem 3.7] that if $\Gamma \setminus \Lambda^{-1}$ is a semi-Riesz set, then $C_\Lambda(H)$ is *nicely embedded* into $C(H)$, that is, the natural isometric embedding $J\colon C_\Lambda(H) \longrightarrow C(H)$ satisfies that for every $t \in H$, $\|J^*\delta_t\| = 1$ and the linear span of $J^*\delta_t$ is an L-summand in X^* (this is actually a straightforward consequence of the definition of semi-Riesz set). Then, it follows immediately from Example 2.12.a that $J^*\delta_t \in \operatorname{Spear}(C_\Lambda(H)^*)$ for every $t \in H$, so J is lush by Proposition 3.32.a. This is thus an alternative elementary proof of Corollary 4.13 which does not need the more complicated [86, Theorem 4.13].

Let us also comment that it was proved in [124, Proof of Theorem 3.3] that unital function algebras are nicely embedded into $C_b(\Omega)$, where Ω is the Choquet boundary of the algebra, so Theorem 4.11, in the unital case, and Corollary 4.12, can be also proved by using the argument above.

We next provide with more applications of Theorem 4.4. The following definition appears in [60, Definition 3.2] for vector-valued spaces of continuous functions.

Definition 4.15. Let K be a compact Hausdorff topological space. We say that a closed subspace $X \subset \ell_\infty(K)$ is a $C(K)$-*superspace* if it contains $C(K)$ and for each $x \in X$, every open subset $U \subset K$ and each $\varepsilon > 0$, there are an open subset $V \subset U$ and an element $\theta \in \mathbb{K}$ such that

$$|\theta| = \sup_{t\in U} |x(t)| \quad \text{and} \quad |x(t) - \theta| < \varepsilon \quad \text{for each } t \in V.$$

The result for $C(K)$-superspaces is the following.

Corollary 4.16. *Let K be a compact Hausdorff topological space. If X is a $C(K)$-superspace, then the inclusion $J\colon C(K) \longrightarrow X$ is lush.*

Proof. Using Theorem 4.4 for the inclusions $C(K) \subset X \subset \ell_\infty(K)$ with $\mathscr{A}$ being the set of all open subsets of K, we have that (ii) is satisfied by Urysohn's Lemma, while (i) is just the definition of $C(K)$-superspace. □

An interesting application is given by the next example.

Example 4.17. Let $D[0,1]$ be the space of bounded functions on $[0,1]$ which are right-continuous, have left limits everywhere and are continuous at $t = 1$. It is shown in [60, Proposition 3.3] that $D[0,1]$ is a $C[0,1]$-superspace (this is because $D[0,1]$ is the closure in $\ell_\infty[0,1]$ of the span of the step functions $\mathbb{1}_{[a,b)}$, $0 \leqslant a \leqslant b < 1$ and $\mathbb{1}_{[a,1]}$, $0 \leqslant a \leqslant 1$). Therefore, the inclusion $J\colon C[0,1] \longrightarrow D[0,1]$ is lush.

4.3 Operators acting from spaces of integrable functions

Our aim here is to describe operators from $L_1(\mu)$ spaces which have the aDP. For commodity, we only deal with probability spaces, but this is not a mayor restriction as L_1-spaces associated to σ-finite measures are (up to an isometric isomorphism) L_1-spaces associated to probability measures (see [26, Proposition 1.6.1], for instance). We introduce some notation. Let (Ω,Σ,μ) be a probability space and let Y be a Banach space. We write $\Sigma^+ := \{B\in\Sigma\colon\ \mu(B)>0\}$ and for $A\in\Sigma^+$ we consider

$$\Sigma_A := \{B\in\Sigma\colon\ B\subset A\}, \qquad \Sigma_A^+ := \Sigma_A\cap\Sigma^+, \quad \text{and} \quad \Gamma_A := \left\{\frac{\mathbb{1}_B}{\mu(B)}: B\in\Sigma_A^+\right\}.$$

We recall that an operator $T\in\mathcal{L}(L_1(\mu),Y)$ is *(Riesz) representable* if there exists $g\in L_\infty(\mu,Y)$ (i.e. a strongly measurable and essentially bounded function) such that

$$T(f) = \int_\Omega f g\, d\mu \qquad \big(f\in L_1(\mu)\big).$$

Rank-one operators are representable by the classical Riesz representation theorem assuring that $L_1(\mu)^*\equiv L_\infty(\mu)$, and then so are all finite-rank operators. Actually, compact operators [37, p. 68, Theorem 2] and even weakly compact operators [37, p. 65, Theorem 12] are representable, but the converse result is not true [37, p. 79, Example 22]. Finally, let us say that the set of representable operators can be isometrically identified with $L_\infty(\mu,Y)$ [37, p. 62, Lemma 4]. We refer the reader to chapter III of [37] for more information and background on representable operators.

Here is the characterization of aDP operators which is the main result of this section.

Theorem 4.18. *Let (Ω,Σ,μ) be a probability space, let Y be a Banach space and let $G\in\mathcal{L}(L_1(\mu),Y)$ be a norm-one operator. The following assertions are equivalent:*

(i) G has the aDP.
(ii) $G(\Gamma_A)$ is a spear set for every $A\in\Sigma^+$.
(iii) $\|G+\mathbb{T}T\| = 1+\|T\|$ for every $T\in\mathcal{L}(L_1(\mu),Y)$ representable.

Let us recall the following exhaustion argument that we briefly prove here.

Remark 4.19. Let (Ω,Σ,μ) be a finite measure space. If for each $A\in\Sigma^+$ there is $B\in\Sigma_A^+$ satisfying a certain property (P), then we can find a countable family $\mathscr{A}\subset\Sigma^+$ of disjoint sets such that every $A\in\mathscr{A}$ satisfies property (P) and $\Omega\setminus\bigcup\mathscr{A}$ is μ-null.

Indeed, this follows from a simple argument: using Zorn's lemma we can take a maximal family $\mathscr{A}$ of disjoint sets in Σ^+ satisfying property (P), which must be countable as μ is finite. To see the last condition, notice that if $A:=\Omega\setminus\bigcup\mathscr{A}$ had positive measure, then we could use the hypothesis to find a subset $B\in\Sigma_A^+$ satisfying (P), and hence $\mathscr{A}\cup\{A\}$ would contradict the maximality of $\mathscr{A}$.

We need a preliminary result.

Lemma 4.20. *Let (Ω,Σ,μ) be a probability space and let Y be a Banach space. For every $T\in\mathcal{L}(L_1(\mu),Y)$ one has that*

$$\|T\| = \sup_{A\in\Sigma^+} \big\|T(\mathbb{1}_A)/\mu(A)\big\| = \sup_{A\in\Sigma^+}\inf_{B\in\Sigma_A^+} \big\|T(\mathbb{1}_B)/\mu(B)\big\|. \tag{4.3}$$

Proof. The inequalities $\geqslant$ are clear in both cases, so we just have to see that

$$\alpha := \sup_{A\in\Sigma_+}\inf_{B\in\Sigma_A^+} \|T(\mathbb{1}_B)/\mu(B)\|$$

is greater than or equal to $\|T\|$. Let $h=\sum_{A\in\pi} c_A\mathbb{1}_A$ be a simple function, where π is a finite partition of Ω into elements of Σ^+, so $\|h\|_1=\sum_{A\in\pi}|c_A|\mu(A)$. Given $\varepsilon>0$, we have that for each $A\in\Sigma^+$ there is $B\in\Sigma_A^+$ such that $\|T(\mathbb{1}_B)/\mu(B)\|<\alpha+\varepsilon$. Using Remark 4.19 in each set $A\in\pi$, we can find a countable partition $\mathscr{A}\subset\Sigma^+$ of Ω such that every $B\in\mathscr{A}$ of positive measure is contained in some element of π and satisfies $\|T(\mathbb{1}_B)/\mu(B)\|<\alpha+\varepsilon$. If we write $c_B=c_A$ whenever $B\subset A$, then

$$\|T(h)\| \leqslant \sum_{A\in\pi}|c_A|\|T(\mathbb{1}_A)\| \leqslant \sum_{B\in\mathscr{A}}\mu(B)|c_B|\,\|T(\mathbb{1}_B)/\mu(B)\| \leqslant (\alpha+\varepsilon)\|h\|_1.$$

Since $\varepsilon>0$ was arbitrary, we conclude that $\|T(h)\|\leqslant\alpha\|h\|_1$. As h runs on all simple functions, it follows that $\|T\|\leqslant\alpha$. □

Proof (of Theorem 4.18). (i) $\Rightarrow$ (ii): Fix $A\in\Sigma^+$, $y\in S_Y$, and $\varepsilon\in(0,1)$. Consider the rank-one operator $T\colon L_1(\mu)\longrightarrow Y$ given by

$$T(f)=y\int_A f\,d\mu \quad \big(f\in L_1(\mu)\big).$$

Then, $\|G+\mathbb{T}T\|=1+\|T\|=2$. By Lemma 4.20 we can find $B\in\Sigma_A^+$ such that

$$\left\|G\left(\frac{\mathbb{1}_B}{\mu(B)}\right)+\mathbb{T}y\right\| = \left\|\frac{G(\mathbb{1}_B)}{\mu(B)}+\mathbb{T}\frac{T(\mathbb{1}_B)}{\mu(B)}\right\| \geqslant 2-\varepsilon.$$

(ii) $\Rightarrow$ (iii): Let $T\in\mathcal{L}(L_1(\mu),Y)$ and $\varepsilon>0$. By Lemma 4.20, we can find $A\in\Sigma^+$ satisfying $\inf_{B\in\Sigma_A^+}\|T(\mathbb{1}_B)/\mu(B)\|>\|T\|-\varepsilon$. If T is representable, then there exists $B\in\Sigma_A^+$ such that $\operatorname{diam}\big(T(\Gamma_B)\big)<\varepsilon$ (see [37, p. 62, Lemma 4 and p. 135, Lemma 6]), so taking any $y\in T(\Gamma_B)$ and using that $G(\Gamma_B)$ is a spear set we obtain

$$\|G+\mathbb{T}T\| \geqslant \|G(\Gamma_B)+\mathbb{T}y\|-\|T(\Gamma_B)-y\| \geqslant 1+\|y\|-\varepsilon \geqslant 1+\|T\|-2\varepsilon.$$

(iii) $\Rightarrow$ (i) is obvious as rank-one operators are representable. □

Remark 4.21. A direct way to prove (i) $\Rightarrow$ (iii) in Theorem 4.18 is the following: every representable operator $T\colon L_1(\mu)\longrightarrow X$ factorizes through ℓ_1, i.e. there are operators $S\colon L_1(\mu) \longrightarrow \ell_1$ and $R\colon \ell_1 \longrightarrow X$ such that $T = R \circ S$ (see the proof of Theorem 8 in [37, p. 66]). Then, S is an SCD operator satisfying $\|Tf\| \leqslant \|R\|\|Sf\|$ for each $f \in L_1(\mu)$, and so Corollary 3.18 implies that T is a target for G. However, item (ii) in Theorem 4.18 gives an intrinsic characterization of aDP operators acting from an L_1 space which has its own interest.

As an obvious consequence of Theorem 4.18, if for a Banach space Y all bounded linear operators from $L_1(\mu)$ to Y are representable (i.e. when Y has the Radon-Nikodým Property with respect to μ [37, p. 63, Theorem 5]), then the aDP is equivalent to be spear for every $G \in \mathcal{L}(L_1(\mu),Y)$. We can actually give a much stronger result which characterizes the representable operators $G \in \mathcal{L}(L_1(\mu),Y)$ which are spears as those represented by a spear vector of $L_\infty(\mu,Y)$. As a consequence, we will describe the spear vectors of $L_\infty(\mu,Y)$ as those functions which take spear values almost everywhere, extending Example 2.12.d to the vector-valued case.

Corollary 4.22. *Let (Ω,Σ,μ) be a probability space and let Y be a Banach space. Let $G \in \mathcal{L}(L_1(\mu),Y)$ be a norm-one operator which is representable by $g \in L_\infty(\mu,Y)$. Then, the following are equivalent:*

(i) G is lush.
(ii) G is a spear.
(iii) G has the aDP.
(iv) $g(t) \in \operatorname{Spear}(Y)$ for a.e. $t \in \Omega$.
(v) $g \in \operatorname{Spear}\big(L_\infty(\mu,Y)\big)$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are known. Let us prove (iii) $\Rightarrow$ (iv). Fix $\varepsilon > 0$. Since g is strongly measurable, given any $A \in \Sigma^+$ there exists $B \in \Sigma_A^+$ with $\operatorname{diam}(g(B)) < \varepsilon$ (see [25, Proposition 2.2], for instance). By Remark 4.19 we can take a countable family $\mathscr{A} \subset \Sigma^+$ of disjoint sets with the property that $\operatorname{diam}(g(A)) < \varepsilon$ for each $A \in \mathscr{A}$ and $N_\varepsilon := \Omega \setminus \bigcup \mathscr{A}$ is μ-null. Given $t \in \Omega \setminus N_\varepsilon$, it must belong to some $A \in \mathscr{A}$, and since $G(\Gamma_A) \subset \overline{\operatorname{conv}}\big(g(A)\big)$ (see [37, p. 48, Corollary 8]), we deduce that $\operatorname{diam}(G(\Gamma_A)) < \varepsilon$. Using that $G(\Gamma_A)$ is a spear set by Theorem 4.18, it follows that for every $x \in X$,

$$\begin{aligned}\|g(t) + \mathbb{T}x\| &\geqslant \|G(\Gamma_A) + \mathbb{T}x\| - \|G(\Gamma_A) - g(t)\| \\ &\geqslant \|G(\Gamma_A) + \mathbb{T}x\| - \varepsilon = 1 + \|x\| - \varepsilon.\end{aligned}$$

Finally, if we take now a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive numbers converging to zero and consider the corresponding N_{ε_n} for each $n \in \mathbb{N}$, then every $t \in \Omega \setminus \bigcup_{n\in\mathbb{N}} N_{\varepsilon_n}$ satisfies $\|g(t) + \mathbb{T}x\| = 1 + \|x\|$ for each $x \in X$, i.e. $g(t)$ is a spear.

Let us prove (iv) $\Rightarrow$ (i). Consider the adjoint operator $G^*\colon Y^* \longrightarrow L_\infty(\mu)$ which is explicitly given by $G^*y^* = y^* \circ g$ for every $y^* \in Y^*$. By (iv) and Corollary 2.8, for every extreme point y^* of B_{Y^*} we have that $|y^*g(t)| = 1$ for a.e. $t \in \Omega$ so $y^* \circ g$ is a spear vector of $L_\infty(\mu)$ (see Example 2.12.d). By Theorem 3.32.b we deduce that G is lush.

Finally, the relations (ii) $\Rightarrow$ (v) $\Rightarrow$ (iii) are straightforward using the isometric identification of $L_\infty(\mu,X)$ with the subspace of $\mathcal{L}(L_1(\mu),X)$ consisting of all representable operators, which contains in particular those of rank-one. $\square$

Corollary 4.23. *Let (Ω,Σ,μ) be a probability space, let Y be a Banach space which has the Radon-Nikodým Property with respect to μ, and let $G\in\mathcal{L}(L_1(\mu),Y)$ be a norm-one operator. Then, the following assertions are equivalent: (i) G has the aDP, (ii) G is a spear operator, (iii) G is lush.*

It is easy to see that every Banach space Y has the Radon-Nikodým Property with respect to the counting measure on $\mathbb{N}$ (use [37, Theorem 8, p. 66], for instance) and so $G\in\mathcal{L}(\ell_1,Y)$ has the aDP if and only if G is lush (see Example 5.5 for another deduction of this fact using that ℓ_1 has the Radon-Nikodým Property).

On the other hand, let us observe that, if Y has the Radon-Nikodým Property (with respect to every probability measure, or equivalently, with respect to the Lebesgue measure on $[0,1]$), then for every Banach space X we have that each $G\in\mathcal{L}(X,Y)$ with the aDP is a spear operator by Corollary 3.22.

4.3.1 Examples of spear operators which are not lush

In this final part, we are going to make use of several notions of the chapter in order to construct examples of operators which are spear despite not being lush. Our approach is based on [64, Theorem 4.1] where it is constructed a subspace Y of $L[0,2]$ such that $L[0,2]/Y$ is not lush but its dual is. We extend this result to a more general family of L_1-spaces and operators.

Following the notation of the section, (Ω,Σ,μ) stands for a probability space. We start with the following definition.

Definition 4.24. Let (Ω,Σ,μ) be a probability space and let X be a subspace of $L_\infty(\mu)$. We say that X is a *C-rich* subspace of $L_\infty(\mu)$ if for every $A\in\Sigma^+$ and every $\varepsilon>0$ there is a norm-one function $h\in L_\infty(\mu)$ with $h\cdot\mathbb{1}_{\Omega\setminus A}=0$ and $\operatorname{dist}(h,X)<\varepsilon$.

As $L_\infty(\mu)$ can be identified with a space of continuous functions on a Hausdorff compact topological space, we have to see that this new definition is consistent with the previous one. Indeed, recall that $L_\infty(\mu)$ is a commutative Banach algebra (with unit) and so by Gelfand's theorem, there is an isometric algebra isomorphism

$$\phi\colon L_\infty(\mu)\longrightarrow C(K_\mu)$$

where K_μ is the space of maximal ideals. In particular, for every $A\in\Sigma$ it holds that $\phi(\mathbb{1}_A)=\mathbb{1}_{U_A}$ for some clopen subset U_A of K_μ, being moreover $\{U_A\colon A\in\Sigma\}$ a basis for the topology on K_μ (see [43, section I.9]). In particular, it follows that $U_\Omega=K_\mu$ and $U_{\Omega\setminus A}=U_\Omega\setminus U_A$ necessarily.

Lemma 4.25. *Let (Ω,Σ,μ) be a probability space and let X be a subspace of $L_\infty(\mu)$. Then, X is C-rich in $L_\infty(\mu)$ if and only if $\phi(X)$ is a C-rich subspace of $C(K_\mu)$.*

Proof. Assume first that $\phi(X)$ is C-rich and let $A\in\Sigma^+$ and $\varepsilon>0$. Then, there exists $F\in C(K_\mu)$ with $\|F\|_\infty=1$, $\operatorname{supp}(F)\subset U_A$ and $\operatorname{dist}(F,\phi(X))<\varepsilon$. Since ϕ is an isometry, we deduce that $F=\phi(h)$ with $\|h\|=1$ and $\operatorname{dist}(h,X)<\varepsilon$, and using moreover that it is an algebra homomorphism then we get that

$$0=\mathbb{1}_{U_{\Omega\setminus A}}\cdot\phi(h)=\phi(h\cdot\mathbb{1}_{\Omega\setminus A}),$$

so $h\cdot\mathbb{1}_{\Omega\setminus A}=0$. Conversely, suppose that X is C-rich in $L_\infty(\Omega,\Sigma,\mu)$, let $\varepsilon>0$ and a basic open subset U_A for some $A\in\Sigma^+$. By hypothesis, there exists $h\in L_\infty(\mu)$ with $\|h\|_\infty=1$ such that $h\cdot\mathbb{1}_{\Omega\setminus A}=0$. Then, $\|\phi(h)\|_\infty=1$ and

$$0=\phi(h\cdot\mathbb{1}_{\Omega\setminus A})=\phi(h)\cdot\mathbb{1}_{U_{\Omega\setminus A}}$$

which yields, in particular, that $\operatorname{supp}\phi(h)\subset U_A$. □

We now state the main result of the subsection.

Theorem 4.26. *Let (Ω,Σ,μ) be an atomless probability space such that $L_1(\mu)$ is separable. Then, there exists a (closed) subspace $Y\subset L_1(\mu)$ such that:*

(a) $(L_1(\mu)/Y)^*\equiv Y^\perp$ does not contain spear vectors;
(b) for every other subspace $X\subset Y$, the operator

$$\pi\colon L_1(\mu)/X\longrightarrow L_1(\mu)/Y$$

is not lush but π^* is lush; in particular, π^* and π are spear operators.

Let us introduce some notation that we will employ during the proof. For each $A\in\Sigma^+$ denote by $\mu_A:=\mu|_{\Sigma_A}$ the restriction of μ to Σ_A, so that (A,Σ_A,μ_A) is a finite measure space. We then have a natural isometric embedding $L_\infty(\mu_A)\longrightarrow L_\infty(\mu)$ which associates to each $f\in L_\infty(\mu_A)$ the (unique) element $\widetilde{f}\in L_\infty(\mu)$ satisfying that $\widetilde{f}|_A=f$ and $\widetilde{f}|_{\Omega\setminus A}=0$ almost everywhere.

Proof (of Theorem 4.26). We can find a partition $\Omega=\bigcup_{m=0}^\infty\Delta_m$ where $\Delta_m\in\Sigma^+$ for each $m\geqslant 0$ and $\mu(\Delta_0)>1/4$. Consider

$$\widetilde{\Delta}:=\bigcup_{m\geqslant 1}\Delta_m\quad\text{and}\quad W:=\left\{f\in L_\infty(\mu_{\widetilde{\Delta}})\colon\ \int_{\Delta_n}f\,d\mu=0\text{ for every }n\geqslant 1\right\}.$$

Let us fix a dense countable subset $\{f_m\colon m\in\mathbb{N}\}\subset S_{L_2(\mu_{\Delta_0})}$ and define the operator

$$J\colon L_\infty(\mu_{\Delta_0})\longrightarrow L_\infty(\mu_{\widetilde{\Delta}}),\quad g\longmapsto J(g)=\sum_{m\in\mathbb{N}}\left(\int_{\Delta_0}g\,f_m\,d\mu\right)\mathbb{1}_{\Delta_m}.$$

Notice that for every $g \in L_\infty(\mu_{\Delta_0})$

$$\|J(g)\|_\infty = \sup_{m\in\mathbb{N}} \left| \int_{\Delta_0} g f_m \, d\mu \right| = \|g\|_2. \tag{4.4}$$

In particular, J is weakly compact. Let us define

$$\mathcal{Z} := \{g + 2J(g) + f \colon g \in L_\infty(\mu_{\Delta_0}),\, f \in W\} \subset L_\infty(\mu).$$

We are going to collect now some properties of this space:

(I) *$\mathcal{Z}$ is weak-star closed.* We can rewrite $\mathcal{Z}$ as the subspace of all $h \in L_\infty(\mu)$ satisfying the system of linear equations

$$2\mu(\Delta_m) \int_{\Delta_0} h f_m \, d\mu = \int_{\Delta_m} h \, d\mu.$$

(II) *$\mathcal{Z}$ is C-rich in $L_\infty(\mu)$.* Let $A \in \Sigma^+$ and $\varepsilon > 0$. We have that $A_m := A \cap \Delta_m \in \Sigma^+$ for some $m \geqslant 0$. In the case $m = 0$, we have that J is weakly compact when restricted to $L_\infty(\mu_{A_0})$ (which is infinite-dimensional as μ_{A_0} is atomless), so there exists $g \in L_\infty(\mu_{A_0})$ with $\|g\|_\infty = 1$ and $\|J(g)\| < \varepsilon$; taking $h := g + 2J(g) \in \mathcal{Z}$ we then have that

$$\operatorname{dist}(g, \mathcal{Z}) \leqslant \|g - h\| \leqslant 2\|J(g)\| < \varepsilon.$$

Otherwise, $m \in \mathbb{N}$ and taking $f \in S_{L_\infty(\mu_{A_m})}$ with $\int_{\Delta_m} f \, d\mu = 0$ (which is possible as μ_{A_m} is atomless), we conclude that $f \in W \subset \mathcal{Z}$ and $\operatorname{supp}(f) \subset A_m \subset A$.

(III) *$\mathcal{Z}$ does not contain any modulus-one function.* Assume that there is a function $h = g + 2J(g) + f$ with $g \in L_\infty(\mu_{\Delta_0})$ and $f \in W$ satisfying that $|h| = 1$ μ-a.e. Then $g = h \cdot \mathbb{1}_{\Delta_0}$ has modulus-one on Δ_0 and so

$$\|J(g)\|_\infty = \|g\|_2 = \mu(\Delta_0)^{\frac{1}{2}} > 1/2$$

by (4.4). But, on the other hand

$$1 = \|h\|_\infty \geqslant \sup_{m\in\mathbb{N}} \left| \frac{1}{\mu(\Delta_m)} \int_\Omega h \cdot \mathbb{1}_{\Delta_m} \, d\mu \right| = 2\|J(g)\|_\infty > 1.$$

We then got a contradiction.

(IV) *$\mathcal{Z}$ has no spear vectors.* It is enough to check that a spear vector $f \in \mathcal{Z}$ would have modulus-one almost everywhere and reduce it to (III). Assume on the contrary that there is $0 < \alpha < 1$ such that $A := \{|f| < \alpha\} \in \Sigma^+$. Since $\mathcal{Z}$ is C-rich by (II), we can find $h \in \mathcal{Z}$ with $\|h\|_\infty = 1$ and $|h(t)| < \alpha$ for almost every $t \in \Omega \setminus A$. Thus

$$\max_{\theta\in\mathbb{T}} \|f + \theta h\| \leqslant \||f| + |h|\|_\infty \leqslant 1 + \alpha < 2,$$

which contradicts that f is a spear vector.

We can finish the proof: since $\mathcal{Z}$ is weak-star closed, we can fix a subspace $Y \subset L_1(\mu)$ such that $Y^\perp = \mathcal{Z}$ and (a) follows from (IV). Let us prove (b). If we take any subspace $X \subset Y$ and consider the operator $\pi\colon L_1(\mu)/X \longrightarrow L_1(\mu)/Y$, then its adjoint $\pi^*\colon Y^\perp \equiv \mathcal{Z} \longrightarrow X^\perp$ is just the (isometric) inclusion. We then have that π^* is lush by Remark 3.34.ii together with the fact that $\mathcal{Z}$ is C-rich in $L_\infty(\mu)$ by (II). On the other hand, π is not lush since, otherwise, we would have that there is an extreme point $z \in \mathcal{Z}$ (actually "many", by Theorem 3.33) such that $\pi^*(z)$ is a spear vector, and so z would be a spear in $\mathcal{Z}$, contradicting (a). □

The two extreme cases in the previous theorem, that is, $X = Y$ and $X = \{0\}$, produce interesting examples. The first case allows to recover [64, Theorem 4.1] taking $\mathcal{X} = L_1(\mu)/Y$ for any atomless measure μ.

Example 4.27. There is a separable Banach space $\mathcal{X}$ such that $G := \mathrm{Id}\colon \mathcal{X} \longrightarrow \mathcal{X}$ is not lush, while G^* is lush. Moreover, it satisfies that $\mathrm{Spear}(\mathcal{X}^*) = \emptyset$ although G is spear. As a consequence, there is no lush operator whose domain is $\mathcal{X}$ by Theorem 3.33.

The case when $X = \{0\}$ in Theorem 4.26.b produce the following result which shows that the equivalences in Corollaries 4.22 and 4.23 do not hold in general.

Example 4.28. Let (Ω, Σ, μ) be an atomless probability space such that $L_1(\mu)$ is separable. Then, there exists a separable Banach space $\mathcal{X}$ such that $\mathcal{L}(L_1(\mu), \mathcal{X})$ contains a spear operator which is not lush.

Chapter 5

Further results

Our goal here is to complement the previous chapter with some interesting results. We characterize lush operators when the domain space has the Radon-Nikodým Property or the codomain space is Asplund, and we get better results when the domain or the codomain is finite-dimensional or when the operator has rank one. Further, we study the behaviour of lushness, spearness and the aDP with respect to the operation of taking adjoint operators.

5.1 Radon-Nikodým Property in the domain or Asplund codomain

We first provide a result about the relationship of an operator with the aDP and spear vectors of the range space and the spear vectors of the dual space to the domain space.

Proposition 5.1. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be an operator with the aDP, then*

(a) $Gx \in \operatorname{Spear}(Y)$ *for every denting point x of B_X.*
(b) $G^*y^* \in \operatorname{Spear}(X^*)$ *for every w^*-denting point y^* of B_{Y^*}.*

Proof. We only illustrate the proof of (a), since the other one is completely analogous. If x is denting, then we can find a decreasing sequence $(S_n)_{n\in\mathbb{N}}$ of slices of B_X containing x and such that $\operatorname{diam} S_n$ tends to zero. Since G has the aDP, Theorem 3.6.iii gives that $\big(G(S_n)\big)_{n\in\mathbb{N}}$ is a decreasing sequence of spear sets whose diameters tend to zero, so $Gx \in \bigcap_n G(S_n)$ is a spear vector by Lemma 2.10. □

We now characterize spear operators acting from a Banach space with the Radon-Nikodým Property.

Proposition 5.2. *Let X be a Banach space with the Radon-Nikodým Property, let Y be a Banach space and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Then, the following assertions are equivalent:*

(i) G is lush.
(ii) G is a spear operator.
(iii) G has the aDP.
(iv) $|y^*(Gx)| = 1$ for every $y^* \in \operatorname{ext}(B_{Y^*})$ and every denting point x of B_X.
(v) $B_X = \overline{\operatorname{conv}}\big\{x \in B_X : Gx \in \operatorname{Spear}(Y)\big\}$ or, equivalently,

$$B_X = \overline{\operatorname{conv}}\big\{x \in B_X : |y^*(Gx)| = 1 \ \forall y^* \in \operatorname{ext}(B_{Y^*})\big\}$$
$$= \overline{\operatorname{conv}}\left(\bigcap_{y^* \in \operatorname{ext}(B_{Y^*})} \mathbb{T}\operatorname{Face}(S_X, G^*y^*)\right).$$

Proof. (i) ⇒ (ii) ⇒ (iii) are clear. (iii) ⇒ (iv) follows from Proposition 5.1 and Corollary 2.8.iv. (iv) ⇒ (v) is consequence of the fact that B_X is the closed convex hull of its denting points since X has the Radon-Nikodým Property (see [20, §2] for instance), and the equivalent reformulation is a consequence of Theorem 2.9. Finally, (v) ⇒ (i) follows from Proposition 3.32.c. □

For Asplund spaces as codomain, we have the following characterization.

Proposition 5.3. *Let X be a Banach space, let Y be an Asplund space and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Then, the following assertions are equivalent:*

(i) G is lush.
(ii) G is a spear operator.
(iii) G has the aDP
(iv) $|x^{**}(G^*y^*)| = 1$ for every $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and every w^*-denting point y^* of B_{Y^*}.
(v) The set $\big\{y^* \in \operatorname{ext}(B_{Y^*}) : G^*y^* \in \operatorname{Spear}(X^*)\big\}$ is dense in $(\operatorname{ext}(B_{Y^*}), w^*)$ or, equivalently, there is a dense subset K of $(\operatorname{ext} B_{Y^*}, w^*)$ such that

$$B_X = \overline{\operatorname{aconv}}\big(\operatorname{Face}(S_X, G^*y^*)\big)$$

for every $y^* \in K$.
(vi) $B_{Y^*} = \overline{\operatorname{conv}}^{w^*}\big\{y^* \in B_{Y^*} : G^*y^* \in \operatorname{Spear}(X^*)\big\}$.

Proof. (i) ⇒ (ii) ⇒ (iii) are clear. (iii) ⇒ (iv) follows from Proposition 5.1 and Corollary 2.8(iv). (iv) ⇒ (v): the set contains all w^*-denting points of B_{Y^*} by (iv), so it is weak-star dense since for Asplund spaces, w^*-denting points are weak-star dense in the set of extreme points of the dual ball (see [20, §2] for instance). The equivalent reformulation is consequence of Theorem 2.9. Finally, (v) ⇒ (vi) is clear and (vi) ⇒ (i) follows from Proposition 3.32.b. □

We do not know whether the above result extends to the case when Y is SCD. What is easily true, using Corollary 3.18, is that aDP and spearness are equivalent in this case.

Remark 5.4. Let X be a Banach space, let Y be an SCD Banach space, and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Then, G has the aDP if and only if G is a spear operator.

As a consequence of the results above, we may improve Proposition 3.3.

Example 5.5. Let Γ be an arbitrary set, let X, Y be Banach spaces and let $(e_\gamma)_{\gamma\in\Gamma}$ be the canonical basis of $\ell_1(\Gamma)$ (as defined in Example 2.12.b).

(a) For $G \in \mathcal{L}(\ell_1(\Gamma),Y)$ the following are equivalent: G is lush, G is a spear operator, G has the aDP, $G(e_\gamma) \in \operatorname{Spear}(Y)$ for every $\gamma \in \Gamma$, $|y^*(G(e_\gamma))| = 1$ for every $y^* \in \operatorname{ext}(B_{Y^*})$ and every $\gamma \in \Gamma$.
(b) For $G \in \mathcal{L}(X,c_0(\Gamma))$ the following are equivalent: G is lush, G is a spear operator, G has the aDP, $B_X = \overline{\operatorname{aconv}}\big(\operatorname{Face}(S_X, G^*e_\gamma)\big)$ for every $\gamma \in \Gamma$, $G^*(e_\gamma) \in \operatorname{Spear}(X^*)$ for every $\gamma \in \Gamma$.

Part of assertion (a) above also follows from Corollary 4.23; the whole assertion (b) also follows from Proposition 4.2.

5.2 Finite-dimensional domain or codomain

Our goal now is to discuss the situation about spear operators when the domain or the codomain is finite-dimensional. We start with the case in which the domain is finite-dimensional, where the result is just an improvement of Proposition 5.2. To get it, we only have to recall that for finite-dimensional spaces, the concepts of denting point and extreme point coincide thanks to the compactness of the unit ball and Choquet's Lemma (Lemma 2.5.a).

Proposition 5.6. *Let X be a finite-dimensional space, let Y be a Banach space and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Then, the following are equivalent:*

(i) G is lush.
(ii) G is a spear operator.
(iii) G has the aDP.
(iv) $|y^*(Gx)| = 1$ for every $y^* \in \operatorname{ext}(B_{Y^*})$ and every $x \in \operatorname{ext}(B_X)$.
(v) $Gx \in \operatorname{Spear}(Y)$ for every $x \in \operatorname{ext}(B_X)$.
(vi) $$B_X = \operatorname{conv}\left(\bigcap_{y^* \in \operatorname{ext}(B_{Y^*})} \mathbb{T}\operatorname{Face}(S_X, G^*y^*) \right).$$

The next example shows that even in the finite-dimensional case, bijective lush operators can be very far away from being isometries and that their domain and codomain are not necessarily spaces with numerical index 1.

Example 5.7. There exists a bijective lush operator between finite-dimensional Banach spaces such that neither its domain nor its codomain has the aDP.

Indeed, let X_1 be the real four-dimensional space whose unit ball is given by

$$B_{X_1} = \operatorname{conv}\big\{(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)\colon \varepsilon_k\in\{-1,1\} \text{ and } \varepsilon_i\neq\varepsilon_j \text{ for some } i,j\big\}.$$

Let Y_1 be the real space ℓ_∞^4, let $X_2 = Y_1^* = \ell_1^4$ and, finally, let $Y_2 = X_1^*$. Consider the operator $G_1 \in \mathcal{L}(X_1,Y_1)$ given by $G_1(x_1) = x_1$ for every $x_1 \in X_1$ and consider $G_2 = G_1^* \in \mathcal{L}(X_2,Y_2)$. Finally, calling $X = X_1 \oplus_\infty X_2$ and $Y = Y_1 \oplus_\infty Y_2$, the operator we are looking for is $G \in \mathcal{L}(X,Y)$ given by $G(x_1,x_2) = (G_1x_1, G_2x_2)$ for every $(x_1,x_2)\in X$.

We start showing that G is lush. To this end, by Proposition 5.6, all we have to do is to check that G carries extreme points of B_X to spear vectors of Y. By Example 2.12.g, this is equivalent to show that both G_1 and G_2 carry extreme points to spear elements. This is evident for G_1 and it is also straightforward to show for G_2 (alternatively, the first assertion gives that G_1 is lush by Proposition 5.6, so $G_2 = G_1^*$ is also lush by Corollary 5.19 in the next section, so G_2 carries extreme points of B_{X_2} to spear elements in Y_2 by using again Proposition 5.6).

Finally, let us show that X does not have the aDP (i.e. that Id_X does not have the aDP). By Proposition 5.6, it is enough to find an extreme point of B_X which is not a spear vector of X. By Example 2.12.g, it is enough to find an extreme point of B_{X_1} which is not a spear vector of X_1. Let us show that this happens for $x_1 = (1,1,-1,-1)\in X_1$. On the one hand, x_1 is clearly an extreme point of B_{X_1} by construction. On the other hand, if x_1 is a spear vector, we have $|x_1^*(x_1)| = 1$ for every $x_1^* \in \operatorname{ext}(B_{X_1^*})$ by Corollary 2.8.iv, so we will get a contradiction if we show that the functional $x_1^* = (\frac12,\frac12,\frac12,\frac12)\in X_1^*$ is an extreme point of $B_{X_1^*}$. Let us show this last assertion. First, x_1^* belongs to $B_{X_1^*}$ since for every $x_1 = (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)\in \operatorname{ext}(B_{X_1})$ we have that

$$|x^*(x)| = \frac12|\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4| \leqslant 1$$

(as necessarily $\varepsilon_i\neq\varepsilon_j$ for some i,j). Next, consider $y_1^*\in X_1^*$ such that both $x_1^*+y_1^*$ and $x_1^*-y_1^*$ lie in $B_{X_1^*}$. This, together with the fact that

$$x_1^*(-1,1,1,1) = x_1^*(1,-1,1,1) = x_1^*(1,1,-1,1) = x_1^*(1,1,1,-1) = 1,$$

implies that

$$y_1^*(-1,1,1,1) = y_1^*(1,-1,1,1) = y_1^*(1,1,-1,1) = y_1^*(1,1,1,-1) = 0,$$

so $y_1^* = 0$ since it vanishes on a basis of X_1. This gives that x_1^* is an extreme point, as desired.

When the codomain is finite-dimensional, we can improve Proposition 5.3 as follows, just taking into account that w^*-denting points and extreme points of the dual ball are the same for a finite-dimensional space.

Proposition 5.8. *Let X be a Banach space, let Y be a finite-dimensional space and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Then, the following assertions are equivalent:*

(i) G is lush.
(ii) G is a spear operator.
(iii) G has the aDP
(iv) $|x^{**}(G^*y^*)| = 1$ for every $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and every $y^* \in \operatorname{ext}(B_{Y^*})$.
(v) $G^*y^* \in \operatorname{Spear}(X^*)$ for every $y^* \in \operatorname{ext}(B_{Y^*})$.
(vi) $B_X = \overline{\operatorname{aconv}}\big(\operatorname{Face}(S_X, G^*y^*)\big)$ for every $y^* \in \operatorname{ext}(B_{Y^*})$.
(vii) $B_{Y^*} = \operatorname{conv}\big(\{y^* \in B_{Y^*} : G^*y^* \in \operatorname{Spear}(X^*)\}\big)$.

We do not know whether this result, or part of it, is also true when just the range of the operator G is finite-dimensional. But we can provide with the following result for rank-one operators.

Corollary 5.9. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one rank-one operator, and write $G = x_0^* \otimes y_0$ for suitable $x_0^* \in S_{X^*}$ and $y_0 \in S_Y$. Then, the following assertions are equivalent:*

(i) G is lush.
(ii) G is a spear operator.
(iii) G has the aDP
(iv) $x_0^* \in \operatorname{Spear}(X^*)$ and $y_0 \in \operatorname{Spear}(Y)$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear. Let us prove (iii) $\Rightarrow$ (iv). First, for every $y \in Y$, consider the rank-one operator $T = x_0^* \otimes y \in \mathcal{L}(X,Y)$ and observe that

$$\|y_0 + \mathbb{T}y\| = \|x_0^* \otimes y_0 + \mathbb{T}x_0^* \otimes y\| = \|G + \mathbb{T}T\| = 1 + \|T\| = 1 + \|y\|,$$

so $y_0 \in \operatorname{Spear}(Y)$. Next, we have that $G\colon X \longrightarrow G(X) = \mathbb{K}y_0 \equiv \mathbb{K}$, also has the aDP (use Remark 3.8) and we may use Proposition 5.1 to get that $G^*(1) = x_0^* \in \operatorname{Spear}(X^*)$.

(iv) $\Rightarrow$ (i). Observe that $G^*(y^*) = y^*(y_0)x_0^*$ for every $y^* \in Y^*$. Now, for each $y^* \in \operatorname{ext}(B_{Y^*})$ we have that $|y^*(y_0)| = 1$ by Corollary 2.8.iv (as $y_0 \in \operatorname{Spear}(Y)$), so $G^*(y^*) \in \mathbb{T}x_0^* \subset \operatorname{Spear}(X^*)$. Now, Proposition 3.32.b gives us that G is lush. □

5.3 Adjoint Operators

We would like to discuss here the relationship of the aDP, spearness and lushness with the operation of taking the adjoint.

As the norm of an operator and the one of its adjoint coincide, the following observation is immediate.

Remark 5.10. *Let X, Y be Banach spaces and let $G\in\mathcal{L}(X,Y)$ be a norm-one operator. If G^* is a spear operator, then G is a spear operator. If G^* has the aDP, then G has the aDP.*

With respect to lushness, the above result is not true, even for G equal to the Identity, as it is shown by Example 4.27.

We may give a positive result in this line: if the second adjoint of an operator is lush, then the operator itself is lush. This will be given in Corollary 5.14, but we need some preliminary work to get the result. We start with a general result which allows to restrict the domain of a lush operator.

Proposition 5.11. *Let X, Z be Banach spaces and let $H\in\mathcal{L}(X^{**},Z^*)$ be a weak-star to weak-star continuous norm-one operator. If H is lush, then $H\circ J_X: X\longrightarrow Z^*$ is lush.*

For the sake of clearness, we include the most technical part of the proof of this result in the following lemma.

Lemma 5.12. *Let X, Y, W be Banach spaces and let $G_1\in\mathcal{L}(X,Y)$ and $G_2\in\mathcal{L}(Y,W)$ be norm-one operators. Suppose that there is a subset $A_1\subset B_{Y^*}$ such that G_1 satisfies the following property*

$$\begin{aligned}&\text{For every slice } S \text{ of } B_X, \text{ every } y^*\in A_1, \text{ and every } \varepsilon>0,\\ &\big[G_1(S)\cap \operatorname{conv}\operatorname{gSlice}(B_Y,\mathbb{T}y^*,\varepsilon)\neq\emptyset\big] \Rightarrow \big[G_1(S)\cap \operatorname{gSlice}(B_Y,\mathbb{T}y^*,\varepsilon)\neq\emptyset\big].\end{aligned}\tag{P1}$$

Suppose also that G_2 is lush and there is a subset $A_2\subset S_{W^*}$ with $\overline{\operatorname{conv}}^{w^*}(A_2)=B_{W^*}$ such that $G_2^*(A_2)\subset A_1$. Then $G:=G_2\circ G_1$ is lush.

Proof. Fix $x_0\in S_X$, $w_0\in B_W$ and $\varepsilon>0$. Let $\delta\in(0,\varepsilon/3)$. Since G_2 is lush, applying Proposition 3.28.iii with $\mathscr{A}=A_2\subset S_{W^*}$, we can find $u^*\in A_2$ such that

$$\operatorname{Re} u^*(w_0)>1-\delta \quad\text{and}\quad \operatorname{dist}\big(G_1x_0,\operatorname{conv}\big(\operatorname{gSlice}(B_Y,\mathbb{T}G_2^*u^*,\delta)\big)\big)<\delta.\tag{5.1}$$

Then, there are $m\in\mathbb{N}$, $\lambda_j\in[0,1]$, $y_j\in\operatorname{gSlice}(B_Y,\mathbb{T}G_2^*u^*,\delta)$ for each $j=1,\dots,m$, with $\sum_j\lambda_j=1$ and

$$y:=G_1x_0-\sum_{j=1}^m\lambda_jy_j\in\delta B_Y.$$

Notice that for each j we have that $\|y_j+y\|\leqslant 1+\delta$, and

$$\left|G_2^*u^*\left(\frac{y_j+y}{1+\delta}\right)\right|\geqslant\frac{1-2\delta}{1+\delta}=1-\frac{3\delta}{1+\delta}.$$

This implies that

$$G_1\left(\frac{x_0}{1+\delta}\right)=\sum_{j=1}^{m}\lambda_j\frac{y_j+y}{1+\delta}\in\operatorname{conv}\left(\operatorname{gSlice}\left(B_Y,\mathbb{T}G_2^*u^*,\frac{3\delta}{1+\delta}\right)\right).$$

Hence, every slice S of B_X containing $x_0/(1+\delta)$ satisfies that

$$G_1(S)\cap\operatorname{conv}\left(\operatorname{gSlice}\left(B_Y,\mathbb{T}G_2^*u^*,\frac{3\delta}{1+\delta}\right)\right)\neq\emptyset.$$

As $G_2^*u^*\in A_1$, the hypothesis (P1) yields that

$$G_1(S)\cap\operatorname{gSlice}\left(B_Y,\mathbb{T}G_2^*u^*,\frac{3\delta}{1+\delta}\right)\neq\emptyset$$

and, therefore,

$$S\cap\operatorname{gSlice}\left(B_X,\mathbb{T}G_1^*G_2^*u^*,\frac{3\delta}{1+\delta}\right)\neq\emptyset.$$

Since S was arbitrary, we conclude that

$$\frac{x_0}{1+\delta}\in\operatorname{aconv}\left(\operatorname{gSlice}\left(B_X,G^*u^*,\frac{3\delta}{1+\delta}\right)\right),$$

and so

$$\operatorname{dist}\left(x_0,\operatorname{aconv}\operatorname{gSlice}\left(B_X,G^*u^*,\frac{3\delta}{1+\delta}\right)\right)<\frac{\delta}{1+\delta}.$$

As $0<\delta<\varepsilon/3$, we get that $\operatorname{dist}(x_0,\operatorname{aconv}\operatorname{gSlice}(B_X,G^*u^*,\varepsilon))<\varepsilon$. This, together with the first part of (5.1), gives that G is lush by Proposition 3.28.iii. □

Proof (of Proposition 5.11). We will use the above lemma with $X=X$, $Y=X^{**}$, $W=Z^*$, $G_1=J_X$ and $G_2=H$. To this end, we first show that $G_1=J_X$ satisfies condition (P1) of Lemma 5.12 with $A_1=J_{X^*}(B_{X^*})\subset B_{X^{***}}$. Indeed, let us fix a slice of the form $\operatorname{Slice}(B_X,x_1^*,\delta)$ of B_X, $J_{X^*}(x^*)\in A_1$ and $\varepsilon>0$, and suppose that

$$J_X\big(\operatorname{Slice}(B_X,x_1^*,\delta)\big)\cap\operatorname{conv}\big(\operatorname{gSlice}(B_{X^{**}},\mathbb{T}J_{X^*}(x^*),\varepsilon)\big)\neq\emptyset.$$

Since

$$\operatorname{conv}\big(\operatorname{gSlice}(B_{X^{**}},\mathbb{T}J_{X^*}(x^*),\varepsilon)\big)\subset\overline{\operatorname{conv}}^{\sigma(X^{**},X^*)}\big(J_X\big(\operatorname{gSlice}(B_X,\mathbb{T}x^*,\varepsilon)\big)\big),$$

we actually have that

$$J_X\big(\operatorname{Slice}(B_X,x_1^*,\delta)\big)\cap\overline{\operatorname{conv}}^{\sigma(X^{**},X^*)}\big(J_X\big(\operatorname{gSlice}(B_X,\mathbb{T}x^*,\varepsilon)\big)\big)\neq\emptyset$$

and so, a fortiori,

$$\operatorname{Slice}(B_{X^{**}},J_{X^*}(x_1^*),\delta)\cap\overline{\operatorname{conv}}^{\sigma(X^{**},X^*)}\big(J_X\big(\operatorname{gSlice}(B_X,\mathbb{T}x^*,\varepsilon)\big)\big)\neq\emptyset.$$

But it then follows that

$$\mathrm{Slice}(B_{X^{**}}, J_{X^*}(x_1^*), \delta) \cap J_X\left(\mathrm{gSlice}(B_X, \mathbb{T}x^*, \varepsilon)\right) \neq \emptyset.$$

This clearly implies that $J_X\left(\mathrm{Slice}(B_X, x_1^*, \delta)\right) \cap \mathrm{gSlice}(B_{X^{**}}, \mathbb{T}J_{X^*}(x^*), \varepsilon) \neq \emptyset$, as desired.

Now, let $G_2 = H \in \mathcal{L}(X^{**}, Z^*)$ and $A_2 = J_Z(S_Z) \subset Z^{**}$ which is norming for Z^*. As H is weak-star to weak-star continuous, we have that $G_2^*(A_2) \subset A_1$. Indeed, let $H_* \in \mathcal{L}(Z, X^*)$ such that $[H_*]^* = H$ and observe that

$$G_2^*(J_Z z) = [H_*]^{**}(J_Z z) = J_{X^*}(H_* z)$$

for every $z \in S_Z$.

Therefore, all the requirements of Lemma 5.12 are satisfied, so $G_2 \circ G_1 = H \circ J_X$ is lush. □

We get a couple of corollaries of this result. The first one deals with the natural inclusion of a lush Banach space into its bidual. It is an immediate consequence of the result above applied to $H = \mathrm{Id}_{X^{**}}$.

Corollary 5.13. *Let X be a Banach space. If X^{**} is lush, then the canonical inclusion $J_X : X \longrightarrow X^{**}$ is lush.*

The next consequence is the promised result saying that lushness passes from the biadjoint operator to the operator.

Corollary 5.14. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. If G^{**} is lush, then G is lush.*

Proof. Apply Proposition 5.11 to $H = G^{**} \in \mathcal{L}(X^{**}, Y^{**})$, which is weak-star to weak-star continuous, to get that $G^{**} \circ J_X : X \longrightarrow Y^{**}$ is lush. But, clearly, $G^{**} \circ J_X = J_Y \circ G$ and then, restricting the codomain and considering that Y and $J_Y(Y)$ are isometrically isomorphic, Remark 3.34 gives us that G is lush. □

These two corollaries improve [64, Proposition 4.3] where it is proved that a Banach space X is lush whenever X^{**} is lush.

Remark 5.15. The technical hypothesis $G_2^*(A_2) \subset A_1$ in Lemma 5.12 is fundamental to get the result. Indeed, consider the inclusion $J\colon c_0 \longrightarrow \ell_\infty$ and the projection $P\colon \ell_\infty \longrightarrow \ell_\infty/c_0$. Notice that $J = J_{c_0}$ satisfies the condition (P1) of Lemma 5.12 with $A_1 = J_{\ell_1}(B_{\ell_1}) \subset B_{\ell_\infty^*}$ (this is shown in the proof of Proposition 5.11). On the other hand P is lush since it carries every spear vector of ℓ_∞ into a spear vector of ℓ_∞/c_0. This can be easily seen using the canonical (isometric) identifications $\ell_\infty \equiv C(\beta\mathbb{N})$ and $\ell_\infty/c_0 = C(\beta\mathbb{N} \setminus \mathbb{N})$, so that P is just the restriction operator. On the other hand, $P \circ J = 0$, which clearly is not lush. The technical hypothesis of the lemma is not satisfied, since every $\mu \in C(\beta\mathbb{N} \setminus \mathbb{N})^*$ with $P^*\mu \in B_{\ell_1}$ must be zero.

The same example also shows the need of the operator H in Proposition 5.11 to be weak-star to weak-star continuous: indeed, just take $H = P\colon \ell_\infty \longrightarrow \ell_\infty/c_0$ and observe that $H \circ J_X = 0$.

Let us now discuss the more complicated direction: when lushness, spearness or the aDP passes from an operator to its adjoint. It is easy to provide examples of operators with the aDP whose adjoint do not share the property: for instance this is the case of the Identity operator on the space $C([0,1],\ell_2)$ (indeed, this space has the aDP by [71, Example in p. 858], while its dual contains ℓ_2 as L-summand and so it fails the aDP by [97, Proposition 3.1]). Providing with a counterexample showing that spearness does not pass from an operator to its adjoint is a more delicate issue, and took a long time to be solved. It was done in [22] as we showed in Example 1.42. Let us recall this example here, as it presents other interesting features. Recall that a *James boundary* for a Banach space X is a subset C of B_{X^*} such that $\|x\| = \max_{x^*\in C}|x^*(x)|$ for every $x \in X$. As a consequence of the Hanh-Banach and the Krein-Milman theorems, the set $\operatorname{ext}(B_{X^*})$ is a James boundary for X.

Example 5.16. Let us consider the countable compact subset of $\mathbb{R}$ given by

$$K = \left\{1-\tfrac{1}{n+1} : n\in\mathbb{N}\right\} \cup \left\{2-\tfrac{1}{n+1} : n\in\mathbb{N}\right\} \cup \left\{3-\tfrac{1}{n+1} : n\in\mathbb{N}\right\} \cup \{1,2,3\}$$

and define the Banach space

$$X = \big\{f \in C(K)\colon\, f(1)+f(2)+f(3) = 0\big\}.$$

It is proved in Example 1.42 that X is C-rich in $C(K)$ and that $X^* = Y \oplus_1 W$ where W has no spear vectors and Y consists of measures concentrated on isolated points of K. Thus $z \in \operatorname{Spear}(X^*)$ if and only if it has the form $z = (\delta_t, 0)$ where t is an isolated point of K (use Example 2.12.a).

(a) The inclusion $J\colon X \longrightarrow C(K)$ is lush but its adjoint J^* does not even have the aDP. Indeed, J is lush by Theorem 4.6. On the other hand, as 1 is an accumulation point, $J^*(\delta_1)$ is not an spear vector of X^*, and this shows that J^* does not have the aDP since $\delta_1 \in \operatorname{dent}(B_{C(K)^*})$ and we may use Proposition 5.1.

(b) Actually, it is routine to check that

$$\{y^* \in C(K)^*\colon\, J^*(y^*) \in \operatorname{Spear}(X^*)\} = \{\delta_t\colon\, t \text{ isolated point of } K\}.$$

Therefore, this set is norming for $C(K)$ but it is not a James boundary for $C(K)$. We deduce that

(c.1) Theorem 3.33 cannot be improved to get that the set Ω is the whole set of extreme points, nor a James boundary for X;

(c.2) the G_δ dense set in Proposition 5.3.v does not always coincide with the set of all extreme points of the dual ball, nor is always a James boundary for X.

Our next goal is to provide sufficient conditions which allow to pass the properties of an operator to its adjoint. The first of these conditions is that the domain space has the Radon-Nikodým Property.

Proposition 5.17. *Let X be a Banach space with the Radon-Nikodým Property, let Y be a Banach space and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. If G has the aDP, then G^* is lush. Therefore, the following six assertions are equivalent: G has the aDP, G is a spear operator, G is lush, G^* has the aDP, G^* is a spear operator, G^* is lush.*

Proof. If G has the aDP, $Gx \in \operatorname{Spear}(Y)$ for every $x \in \operatorname{dent}(B_X)$ by Proposition 5.1, and then Proposition 2.11.c gives that $J_Y(Gx) \in \operatorname{Spear}(Y^{**})$ for every $x \in \operatorname{dent}(B_X)$. Therefore, the set

$$\left\{x^{**} \in B_{X^{**}} : [G^*]^*(x^{**}) \in \operatorname{Spear}(Y^{**})\right\}$$

contains $J_X(\operatorname{dent}(B_X))$ which is norming for X^* as X has the Radon-Nikodým Property (see [20, §2] for instance). Then, Proposition 3.32.a gives that G^* is lush.

Finally, let us comment the proof of the last part. The three first assertions are equivalent by Proposition 5.2 since X has the Radon-Nikodým Property; G^* lush $\Rightarrow$ G^* spear $\Rightarrow$ G^* has the aDP $\Rightarrow$ G has the aDP by Remark 5.10. The remaining implication is just what we have proved above. □

Another result in this line is the following. We recall that a Banach space X is *M-embedded* if $J_X(X)^\perp$ is an L-summand in X^{***} (which is actually equivalent to the fact that the Dixmier projection on X^{***} is an L-projection). We refer the reader to the monograph [54] for more information and background. Examples of M-embedded spaces are reflexive spaces (trivial), c_0 and all of its closed subspaces, $K(H)$ (the space of compact operators on a Hilbert space H), $C(\mathbb{T})/A(\mathbb{D})$, the little Bloch space B_0, among others (see [54, Examples III.1.4]).

Proposition 5.18. *Let X be a Banach space, let Y be an M-embedded Banach space, and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. If G has the aDP, then G^* is lush. Therefore, the following nine assertions are equivalent: G has the aDP, G is a spear operator, G is lush, G^* has the aDP, G^* is a spear operator, G^* is lush, G^{**} has the aDP, G^{**} is a spear operator, G^{**} is lush.*

Proof. If G has the aDP, we use Proposition 5.1 to get that the set

$$\left\{y^* \in B_{Y^*} : G^*y^* \in \operatorname{Spear}(X^*)\right\}$$

contains the set D of those w^*-denting points of B_{Y^*}. By [54, Corollary III.3.2], we have that $B_{Y^*} = \overline{\operatorname{conv}}(D)$ so, a fortiori,

$$B_{Y^*} = \overline{\operatorname{conv}}\left\{y^* \in B_{Y^*} : G^*y^* \in \operatorname{Spear}(X^*)\right\}.$$

Then, Proposition 3.32.c gives that G^* is lush.

Finally, for the last part, the three first assertions are equivalent by Proposition 5.3 since Y is Asplund [54, Theorem III.3.2]. The middle three assertions are equivalent by Proposition 5.2 since Y^* has the Radon-Nikodým Property [54, Theorem III.3.2]. If G^* has the aDP, so does G (Remark 5.10) and this implies that G^* is lush by the above. As Y^* has the Radon-Nikodým Property, if G^* has the aDP, then G^{**} is lush by Proposition 5.17, and this gives the equivalence with the last three assertions. □

Even though part of what we have used in the proof above is Asplundness of M-embedded spaces, just this hypothesis on Y is not enough to get the result as Example 5.16 shows.

A consequence of the two results above is that lushness passes from an operator with reflexive domain or codomain to all of its successive adjoint operators.

Corollary 5.19. *Let X, Y be Banach spaces such that at least one of them is reflexive, and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. If G has the aDP, then G and all the successive adjoint operators of G are lush.*

Proof. If X is reflexive, then it has the Radon-Nikodým Property, so Proposition 5.17 gives that G and G^* are lush. If Y is reflexive, then it is clearly M-embedded, so Proposition 5.18 gives us that G and G^* are lush. For the successive adjoint operators, one of the above two arguments applies. □

The above result applies of course to operators with finite-dimensional codomain, but we do not know whether it can be extended to finite-rank operators. We may do when the operator has actually rank one.

Proposition 5.20. *Let X, Y be Banach spaces, and let $G \in \mathcal{L}(X,Y)$ be a rank-one norm-one operator. If G has the aDP, then G^* is lush. Therefore, all the successive adjoints of G are lush.*

Proof. If G has the aDP, we have that $G = x_0^* \otimes y_0$ with $x_0^* \in \operatorname{Spear}(X^*)$ and $y_0 \in \operatorname{Spear}(Y)$ by Corollary 5.9. Observe that $G^* = J_Y(y_0) \otimes x_0^* \colon Y^* \longrightarrow X^*$. Since $J_Y(y_0) \in \operatorname{Spear}(Y^{**})$ by Proposition 2.11.c, we get that G^* is lush by using again Corollary 5.9. The last assertion follows from the fact that the adjoint to a rank-one operator is again a rank-one operator, and so the argument can be iterated. □

The last result deals with L-embedded spaces. Recall that a Banach space Y is *L-embedded* if $Y^{**} = J_Y(Y) \oplus_1 Y_s$ for suitable closed subspace Y_s of Y^{**}. We refer to the monograph [54] for background. Examples of L-embedded spaces are reflexive spaces (trival), predual of von Neumann algebras so, in particular, $L_1(\mu)$ spaces, the Lorentz spaces $d(w,1)$ and $L^{p,1}$, the Hardy space H_0^1, the dual of the disk algebra $A(\mathbb{D})$, among others (see [54, Examples IV.1.1 and III.1.4]).

Proposition 5.21. *Let X be a Banach space, let Y be an L-embedded space, and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator.*

(a) If G is a spear operator, then G^* is a spear operator.
(b) If G has the aDP, then G^* has the aDP.

Proof. (a). Write $P_Y : Y^{**} \longrightarrow J_Y(Y)$ for the projection associated to the decomposition $Y^{**} = J_Y(Y) \oplus_1 Y_s$. We fix $T \in \mathcal{L}(Y^*, X^*)$ and consider the operators

$$A := P_Y \circ T^* \circ J_X : X \longrightarrow J_Y(Y) \qquad B := [\mathrm{Id} - P_Y] \circ T^* \circ J_X : X \longrightarrow Y_s,$$

and observe that $T^* \circ J_X = A \oplus B$. Given $\varepsilon > 0$, since $J_X(B_X)$ is dense in $B_{X^{**}}$ by Goldstine's Theorem and T^* is weak-star to weak-star continuous, we may find $x_0 \in S_X$ such that

$$\|T^* J_X(x_0)\| = \|Ax_0\| + \|Bx_0\| > \|T\| - \varepsilon.$$

Now, we may find $y_0 \in S_Y$ and $y_s^* \in S_{Y_s^*}$ such that

$$\|Ax_0\|\, y_0 = Ax_0 \qquad \text{and} \qquad y_s^*(Bx_0) = \|Bx_0\|.$$

We define $S\colon X \longrightarrow Y$ by $Sx = Ax + y_s^*(Bx)y_0$ for every $x \in X$, and observe that

$$\|S\| \geqslant \|Sx_0\| > \|T\| - \varepsilon.$$

As G is a spear operator, we have that $\|G + \mathbb{T}S\| > 1 + \|T\| - \varepsilon$, so we may find $x_1 \in S_X$, $\omega \in \mathbb{T}$, and $y_1^* \in S_{Y^*}$ such that

$$\left|y_1^*(Gx_1 + \omega Ax_1 + \omega y_s^*(Bx_1)y_0)\right| > 1 + \|T\| - \varepsilon.$$

Finally, consider $\Phi = (J_{Y^*}(y_1^*), y_1^*(y_0)y_s^*) \in Y^{***} = J_{Y^*}(Y^*) \oplus_\infty Y_s^*$ which has norm-one (here we use the L-embeddedness hypothesis) and observe that

$$\begin{aligned}
\|G^* + \mathbb{T}T\| = \|G^{**} + \mathbb{T}T^*\| &\geqslant \left|\left[\Phi\left(G^{**} + \omega T^*\right)\right](J_X(x_1))\right| \\
&= \left|y_1^*(Gx_1 + \omega Ax_1) + \omega y_1^*(y_0)y_s^*(Bx_1)\right| \\
&= \left|y_1^*(Gx_1 + \omega Ax_1 + \omega y_s^*(Bx_1)y_0)\right| \\
&> 1 + \|T\| - \varepsilon.
\end{aligned}$$

Moving $\varepsilon \downarrow 0$, we get that G^* is a spear operator, as desired.

(b). If G just has the aDP, we may repeat the above argument for rank-one operators $T \in \mathcal{L}(Y^*, X^*)$, and everything works fine as the operator $S \in \mathcal{L}(X,Y)$ constructed there has finite rank, so $\|G + \mathbb{T}S\| = 1 + \|S\|$ by Theorem 3.17 (as, clearly, finite-rank operators are SCD). □

Chapter 6
Isometric and isomorphic consequences

Our goal here is to present consequences on the Banach spaces X and Y of the fact that there is $G \in \mathcal{L}(X,Y)$ which is a spear operator, is lush, or has the aDP.

We first start with a deep structural consequence which generalizes [9, Corollary 4.10] where it was proved for real infinite-dimensional Banach spaces with the aDP.

Theorem 6.1. *Let X, Y be **real** Banach spaces and let $G \in \mathcal{L}(X,Y)$. If G has the aDP and has infinite rank, then X^* contains a copy of ℓ_1.*

Proof. Using Proposition 3.7, we can find separable subspaces $X_\infty \subset X$ and $Y_\infty \subset Y$ such that $G_\infty := G|_{X_\infty} : X_\infty \longrightarrow Y_\infty$ has the aDP, and still it has infinite rank. By Remark 3.8, we may and do suppose that $\overline{G_\infty(X_\infty)} = Y_\infty$. It is enough to show that X_∞^* contains a copy of ℓ_1 since, in this case, X^* also contains such a copy by the lifting property of ℓ_1 (see [83, Proposition 2.f.7] or [119, p. 11]). We have two possibilities. If X_∞ contains a copy of ℓ_1, then X_∞^* contains a quotient isomorphic to ℓ_∞ and so X_∞^* contains a copy of ℓ_1 again by the lifting property of ℓ_1. If X_∞ does not contain copies of ℓ_1, then B_{X_∞} is an SCD set by [9, Theorem 2.22] (see Examples 1.56), so Theorem 3.29 gives that G_∞ is lush. Then, by Theorem 3.33, the set $\left\{y^* \in \operatorname{ext} B_{Y_\infty^*} : G_\infty^*(y^*) \in \operatorname{Spear}(X_\infty^*)\right\}$ is weak-star dense in $\operatorname{ext} B_{Y_\infty^*}$. As G_∞ has dense range, G_∞^* is injective, and since Y_∞^* is infinite-dimensional, it follows that the set $\operatorname{Spear}(X_\infty^*)$ must be infinite. Now, Proposition 2.11.i gives us that X_∞^* contains a copy of c_0 or ℓ_1. But a dual space contains a copy of ℓ_1 whenever it contains a copy of c_0 [83, Proposition 2.e.8]. □

Another result in this line is the following.

Proposition 6.2. *Let X be a **real** Banach space with the Radon-Nikodým Property, let Y be a **real** Banach space, and let $G \in \mathcal{L}(X,Y)$. If G has the aDP and has infinite rank, then $Y \supset c_0$ or $Y \supset \ell_1$.*

Proof. By Proposition 5.2 we have that $B_X = \overline{\operatorname{conv}}\left\{x \in B_X : Gx \in \operatorname{Spear}(Y)\right\}$, so

$$G(B_X) \subset \overline{\text{conv}}\{Gx\colon x \in B_X,\, Gx \in \text{Spear}(Y)\} \subset \overline{\text{conv}}\,(\text{Spear}(Y))\,.$$

Now, if G has infinite rank, $\text{Spear}(Y)$ has to be infinite and so Proposition 2.11.j gives the result. □

Remark 6.3. Let us observe that both possibilities in the result above may happen. On the one hand, $G := \text{Id}_{\ell_1} : \ell_1 \longrightarrow \ell_1$ is lush by Example 5.5. On the other hand, the operator $G\colon \ell_1 \longrightarrow c$ given by $[G(e_n)](k) = -1$ if $k = n$ and $[G(e_n)](k) = 1$ if $k \neq n$ is also lush by Example 5.5 and it has infinite-rank.

We next deal with isometric consequences of the existence of operators with the aDP. The following result generalizes [63, Theorem 2.1] where it was proved for $G = \text{Id}$. Let us remark that the proof given there relied on a non-trivial result of the theory of numerical range: that the set of operators whose adjoint attain its numerical radius is norm dense in the space of operators.

Proposition 6.4. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be an operator with the aDP. Then*

(a) If X^* is strictly convex, then $X = \mathbb{K}$.
(b) If X^* is smooth, then $X = \mathbb{K}$.
(c) If Y^* is strictly convex, then $Y = \mathbb{K}$.

Proof. (a). We start showing that G^* has rank one. Using Theorem 3.6.iv we can find $y_0^* \in S_{Y^*}$ with $\|G^* y_0^*\| = 1$. By the same result, there is a weak-star dense subset of $\text{ext}\, B_{Y^*}$ whose elements y^* satisfy that

$$\|G^* y_0^* + \mathbb{T}\, G^* y^*\| = 2. \tag{6.1}$$

It follows from the definition of strict convexity that $G^* y^* \in \mathbb{T} G^* y_0^*$ for every such y^*, and we deduce by the Krein-Milman Theorem and the weak-star continuity of G^*, that $G^*(B_{Y^*})$ is contained in $\text{span}\{G^* y_0^*\}$. Hence, G^* has rank one. Therefore, G has rank one and $\text{Spear}(X^*)$ is non empty by Corollary 5.9. Finally, X^* is one-dimensional by Proposition 2.11.h.

(b). Given arbitrary elements $x_0^*, x_1^* \in S_{X^*}$ we use use Theorem 3.6.iv and the fact that $(\text{ext}\, B_{Y^*}, w^*)$ is a Baire space (see Lemma 2.5.c) to deduce the existence of some $y^* \in \text{ext}\, B_{Y^*}$ with $\|G^* y^* + \mathbb{T} x_i^*\| = 2$ for $i = 0,1$. Taking $x_i^{**} \in S_{X^{**}}$ with $x_i^{**}(G^* y^*) + |x_i^{**}(x_i^*)| = 2$ for each $i = 0,1$, we get that

$$x_0^{**}(G^* y^*) = x_1^{**}(G^* y^*) = 1 \quad \text{and} \quad |x_0^{**}(x_0^*)| = |x_1^{**}(x_1^*)| = 1.$$

Since X^* is smooth, it follows from the left hand side of the above formula that $x_0^{**} = x_1^{**}$ and hence, $\|x_0^* + \mathbb{T} x_1^*\| = 2$ by the right hand side of the above formula. So every element of S_{X^*} is a spear and then Proposition 2.11.e tells us that X^* is one-dimensional.

The proof of (c) follows the lines of the one of (a). Indeed, arguing like in (a), we find $y_0^* \in S_{Y^*}$ and a weak-star dense subset of $\operatorname{ext}(B_{Y^*})$ whose elements y^* satisfy (6.1) so, a fortiori, they satisfy that

$$\|y_0^* + \mathbb{T} y^*\| = 2.$$

Being Y^* strictly convex, we get that $y^* \in \mathbb{T} y_0^*$ for every such y^*, but this implies that Y^*, and so Y, is one-dimensional by the Krein-Milman Theorem. □

The following result generalizes [63, Proposition 2.5].

Proposition 6.5. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be an operator with the aDP.*

(a) If the norm of Y is Fréchet smooth, then $Y = \mathbb{K}$.
(b) If X and Y are real spaces and the norm of X is Fréchet smooth, then $X = \mathbb{R}$.

Proof. (a). By Proposition 5.1 we have that $G^*y^* \in \operatorname{Spear}(X^*)$ for every w^*-strongly exposed point y^* of B_{Y^*}. Since the norm of Y is Fréchet smooth, every functional in S_{Y^*} attaining its norm is a w^*-strongly exposed point of B_{Y^*} (see [35, Corollary I.1.5] for instance). As norm-one norm-attaining functionals are dense in S_{Y^*} by the Bishop-Phelps Theorem, and $\operatorname{Spear}(X^*)$ is norm closed by Proposition 2.11.d, we get in fact that $G^*y^* \in \operatorname{Spear}(X^*)$ for every $y^* \in S_{Y^*}$. So, given arbitrary elements $y_1^*, y_2^* \in S_{Y^*}$ we can write

$$2 = \|G^*(y_1^*) + \mathbb{T} G^*(y_2^*)\| \leqslant \|y_1^* + \mathbb{T} y_2^*\| \leqslant 2$$

which gives that every element in S_{Y^*} is a spear. Therefore, Y^* is one-dimensional by Proposition 2.11.e.

(b). Fixed $X_0 \subset X$ and $Y_0 \subset Y$ arbitrary separable subspaces we can use Proposition 3.7 to find separable subspaces $X_0 \subset X_\infty \subset X$ and $Y_0 \subset Y_\infty \subset Y$ such that $G(X_\infty) \subset Y_\infty$ and $G_\infty := G|_{X_\infty} : X_\infty \longrightarrow Y_\infty$ has norm one and the aDP. Next, we fix a countable dense subset $D \subset S_{X_\infty}$ and we consider $D^* \subset S_{X_\infty^*}$ given by

$$D^* = \{x^* \in S_{X_\infty^*} \ : \ \exists x \in D \text{ with } x^*(x) = 1\}$$

which is countable since D is countable and X_∞ is smooth. Therefore, we can use the fact that $(\operatorname{ext} B_{Y^*}, w^*)$ is a Baire space (see Lemma 2.5.c) and Theorem 3.6.iv to deduce the existence of some $y^* \in \operatorname{ext} B_{Y_\infty^*}$ with $\|G_\infty^* y^* + \mathbb{T} x^*\| = 2$ for every $x^* \in D^*$. We will show that $G_\infty^* y^* \in \operatorname{Spear}(X_\infty^*)$. To do so, fix $x^* \in S_{X_\infty^*}$ attaining its norm at $x \in S_{X_\infty}$ and recall that x strongly exposes x^* as X_∞ is Fréchet smooth. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in D converging to x and let $x_n^* \in D^*$ satisfying $x_n^*(x_n) = 1$ for every $n \in \mathbb{N}$. Then we have that

$$|x_n^*(x) - 1| = |x_n^*(x) - x_n^*(x_n)| \leqslant \|x - x_n\| \longrightarrow 0$$

so (x_n^*) converges in norm to x^* and, therefore,

$$2 = \|G_\infty^* y^* + \mathbb{T} x_n^*\| \longrightarrow \|G_\infty^* y^* + \mathbb{T} x^*\|$$

which gives $\|G_\infty^* y^* + \mathbb{T} x^*\| = 2$. Since norm-one norm-attaining functionals are dense in $S_{X_\infty^*}$ by the Bishop-Phelps Theorem, we deduce that $G_\infty^* y^*$ is a spear in X_∞^*. Finally, Proposition 2.11.k tells us that X_∞, and thus X_0, is one-dimensional as it is smooth. The arbitrariness of X_0 implies that X is one-dimensional. □

The next result deals with WLUR points. Given a Banach space X, a point $x \in S_X$ is said to be *LUR* (respectively *WLUR*) if for every sequence $(x_n)_{n\in\mathbb{N}}$ in B_X such that $\|x_n + x\| \longrightarrow 2$ one has that $(x_n) \longrightarrow x$ in norm (respectively weakly). It is clear that LUR points are WLUR, but the converse result is known to be false [116].

Proposition 6.6. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be an operator with the aDP. Then*

(a) If B_X contains a WLUR point, then $X = \mathbb{K}$.
(b) If B_Y contains a WLUR point, then $Y = \mathbb{K}$.

Proof. (a). Let x_0 be a WLUR point of B_X. We start showing that $\|Gx_0\| = 1$. To do so, take $x_0^* \in S_{X^*}$ with $x_0^*(x_0) = 1$ and use Theorem 3.6.iv to find $y^* \in \operatorname{ext} B_{Y^*}$ such that $\|G^* y^* + \mathbb{T} x_0^*\| = 2$. Therefore, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in B_X satisfying

$$\big|[G^* y^*](x_n) + \mathbb{T} x_0^*(x_n)\big| \longrightarrow 2$$

which clearly implies $|y^*(Gx_n)| = \big|[G^* y^*](x_n)\big| \longrightarrow 1$ and $|x_0^*(x_n)| \longrightarrow 1$. Hence, there is a sequence $(\theta_n)_{n\in\mathbb{N}}$ in $\mathbb{T}$ such that $\operatorname{Re} x_0^*(\theta_n x_n) \longrightarrow 1$ and so

$$\|\theta_n x_n + x_0\| \geqslant \operatorname{Re} x_0^*(\theta_n x_n + x_0) \longrightarrow 2.$$

Now, since x_0 is a WLUR point we get that $(\theta_n x_n)$ converges weakly to x_0. Therefore, $(G\theta_n x_n)$ converges weakly to Gx_0, and the fact that $|y^*(G\theta_n x_n)| \longrightarrow 1$ tells us that $|y^*(Gx_0)| = 1$ and, a fortiori, $\|Gx_0\| = 1$.

Suppose that X is not one-dimensional, then there is $x^* \in S_{X^*}$ with $x^*(x_0) = 0$. Consider the operator $T = x^* \otimes Gx_0 \in \mathcal{L}(X,Y)$ which satisfies $\|T\| = 1$. We have that $\|G + \mathbb{T}T\| = 2$ since G has the aDP, so there are sequences (z_n) in S_X and (y_n^*) in S_{Y^*} such that

$$|y_n^*(Gz_n) + \mathbb{T} y_n^*(Gx_0) x^*(z_n)| \longrightarrow 2$$

which implies $|x^*(z_n)| \longrightarrow 1$ and $|y_n^*(Gz_n) + \mathbb{T} y_n^*(Gx_0)| \longrightarrow 2$. Hence, we may find a sequence $(\omega_n)_{n\in\mathbb{N}}$ in $\mathbb{T}$ such that $|y_n^*(\omega_n Gz_n + Gx_0)| \longrightarrow 2$ and so

$$\|\omega_n z_n + x_0\| \geqslant \|G(\omega_n z_n + x_0)\| \geqslant |y_n^*(\omega_n Gz_n + Gx_0)| \longrightarrow 2.$$

Since x_0 is a WLUR point, we get that $(\omega_n z_n)$ converges weakly to x_0. This, together with $|x^*(z_n)| \longrightarrow 1$, tells us that $|x^*(x_0)| = 1$, which is a contradiction.

(b). Let y_0 be a WLUR point of B_Y. Since G has the aDP, Theorem 3.6.iv provides us with a dense G_δ set A in $(\operatorname{ext} B_{Y^*}, w^*)$ such that $\|G^* y^*\| = 1$ for every $y^* \in A$. We claim that $|y^*(y_0)| = 1$ for every $y^* \in A$. Indeed, fixed $y^* \in A$, consider the rank-one operator $T = G^*y^* \otimes y_0$ which satisfies $\|G + \mathbb{T}T\| = 2$. So there are sequences (x_n) in S_X and (y_n^*) in S_{Y^*} such that

$$2 \longleftarrow |y_n^*(Gx_n) + \mathbb{T}y_n^*(Tx_n)| = |y_n^*(Gx_n) + \mathbb{T}y_n^*(y_0)y^*(Gx_n)|.$$

This implies that $|y^*(Gx_n)| \longrightarrow 1$ and that there is a sequence $(\theta_n)_{n\in\mathbb{N}}$ in $\mathbb{T}$ such that

$$\|\theta_n Gx_n + y_0\| \geqslant |y_n^*(\theta_n Gx_n + y_0)| \longrightarrow 2.$$

Being y_0 a WLUR point, we deduce that $(\theta_n Gx_n)$ converges weakly to y_0 and, therefore, we get $|y^*(y_0)| = 1$, finishing the proof of the claim.

To finish the proof, fix $y \in S_Y$ and observe that

$$\begin{aligned}\|y_0 + \mathbb{T}y\| &\geqslant \sup_{y^*\in A} |y^*(y_0) + \mathbb{T}y^*(y)| \\ &= \sup_{y^*\in A} |y^*(y_0)| + |y^*(y)| = 1 + \sup_{y^*\in A} |y^*(y)| = 2.\end{aligned}$$

This, together with y_0 being a WLUR point, gives that $y \in \mathbb{T}y_0$. Therefore, Y is one-dimensional, as desired. □

Our next result improves Proposition 6.4 but only for lush operators. We do not know whether it is also true for operators with the aDP.

Proposition 6.7. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator which is lush. Then:*

(a) If X is strictly convex then $X = \mathbb{K}$.
(b) In the real case, if X is smooth then $X = \mathbb{R}$.
(c) If Y is strictly convex then $Y = \mathbb{K}$.

Proof. Given arbitrary separable subspaces $X_0 \subset X$ and $Y_0 \subset Y$, we can use Proposition 3.28.vi to get the existence of separable subspaces $X_0 \subset X_\infty \subset X$ and $Y_0 \subset Y_\infty \subset Y$ such that $G(X_\infty) \subset Y_\infty$, $\|G|_{X_\infty}\| = 1$, and $G_\infty := G|_{X_\infty} : X_\infty \longrightarrow Y_\infty$ is lush. Now Theorem 3.33 tells us that there exists a G_δ dense subset Ω of $(\operatorname{ext} B_{Y_\infty^*}, w^*)$ such that $G_\infty^*(\Omega) \subset \operatorname{Spear}(X_\infty^*)$ or, equivalently, that

$$B_{X_\infty} = \overline{\operatorname{aconv}}\big(\operatorname{Face}(S_{X_\infty}, G_\infty^* y^*)\big) \tag{6.2}$$

for every $y^* \in \Omega$.

(a). If X is strictly convex so is X_∞, and then Proposition 2.11.h tells us that X_∞ is one-dimensional as $\operatorname{Spear}(X_\infty^*)$ is non-empty. Thus, X_0 is one-dimensional and its arbitrariness gives that X is one-dimensional.

(b). If X is smooth so is X_∞. Using this time Proposition 2.11.l, we get that X_∞ is one-dimensional as $\operatorname{Spear}(X_\infty^*)$ is non-empty. Therefore, X_0 is one-dimensional and its arbitrariness tells us that X is one-dimensional.

(c). In this case, we have that Y_∞ is strictly convex. Observe that, fixed $y^* \in \Omega$, every element x in the set $\mathrm{Face}(S_{X_\infty}, G_\infty^* y^*)$ satisfies that $y^*(G_\infty x) = 1$, so by the strict convexity of Y_∞, the set $G_\infty\big(\mathrm{Face}(S_{X_\infty}, G_\infty^* y^*)\big)$ must consist of one point. This, together with (6.2), implies that G_∞ has rank one. Therefore, $\mathrm{Spear}(Y_\infty)$ is non-empty by Corollary 5.9 and so Y_∞ (and thus Y_0) is one-dimensional by Proposition 2.11.h. The arbitrariness of Y_0 tells us that Y is one-dimensional. □

Our last result in this chapter is an extension of Theorem 2.9 to arbitrary lush operators: every lush operator attains its norm (i.e. the supremum defining its norm is actually a maximum).

Proposition 6.8. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. If G is lush, then it is norm-attaining. Actually,*

$$B_X = \overline{\mathrm{conv}}\big(\{x \in S_X : \|Gx\| = 1\}\big).$$

Proof. Fix an arbitrary $x_0 \in B_X$. By Proposition 3.28, there are separable Banach spaces $x_0 \in X_\infty \subset X$ and $Y_\infty \subset Y$ satisfying that $G_\infty := G|_{X_\infty} : X_\infty \longrightarrow Y_\infty$ is lush. Using Theorem 3.33, there exists $y_0^* \in S_{Y_\infty^*}$ such that $G_\infty^* y_0^*$ is a spear, so Theorem 2.9 implies that

$$x_0 \in B_{X_\infty} = \overline{\mathrm{conv}}\left(\{x \in S_{X_\infty} : |[G_\infty^* y_0^*](x)| = 1\}\right) \subset \overline{\mathrm{conv}}\left(\{x \in S_X : \|Gx\| = 1\}\right),$$

giving thus the result. □

We will see in Example 8.7 that the aDP is not enough to get norm-attainment.

Chapter 7

Lipschitz spear operators

Let X, Y be Banach spaces. We denote by $\mathrm{Lip}_0(X,Y)$ the set of all Lipschitz mappings $F\colon X \longrightarrow Y$ such that $F(0)=0$. This is a Banach space when endowed with the norm

$$\|F\|_L = \sup\left\{ \frac{\|F(x)-F(y)\|}{\|x-y\|} : x,y \in X,\, x \neq y \right\}.$$

Observe that, clearly, $\mathcal{L}(X,Y) \subset \mathrm{Lip}_0(X,Y)$ with equality of norms.

Our aim in this chapter is to study those elements of $\mathrm{Lip}_0(X,Y)$ which are spears. First, let us give a name for this.

Definition 7.1. Let X,Y be Banach spaces. An element $G \in \mathrm{Lip}_0(X,Y)$ is a *Lipschitz spear operator* if $\|G+\mathbb{T}F\|_L = 1+\|F\|_L$ for every $F \in \mathrm{Lip}_0(X,Y)$.

We will prove here that every (linear) lush operator is a Lipschitz spear operator and present similar results for Daugavet centers and for operators with the aDP. To do so, we will use the technique of the Lipschitz-free space. We need some definitions and preliminary results. Let X be a Banach space. Observe that we can associate to each $x \in X$ an element $\delta_x \in \mathrm{Lip}_0(X,\mathbb{K})^*$ which is just the evaluation map $\delta_x(f) = f(x)$ for every $f \in \mathrm{Lip}_0(X,\mathbb{K})$. The *Lipschitz-free space* over X is the Banach space

$$\mathcal{F}(X) := \overline{\mathrm{span}}^{\|\cdot\|}\{\delta_x \colon x \in X\} \subset \mathrm{Lip}_0(X,\mathbb{K})^*.$$

It turns out that $\mathcal{F}(X)$ is an isometric predual of $\mathrm{Lip}_0(X,\mathbb{K})$ (which has been very recently shown to be the unique predual [122]). The map $\delta_X : x \longmapsto \delta_x$ establishes an isometric non-linear embedding $X \longrightarrow \mathcal{F}(X)$ since $\|\delta_x - \delta_y\|_{\mathcal{F}(X)} = \|x-y\|_X$ for all $x,y \in X$. The name Lipschitz-free space appeared for the first time in the paper [49] by G. Godefroy and N. Kalton, but the concept was studied much earlier and it is also known as the Arens-Ells space of X (see [121, §2.2]). The main features of the Lipschitz-free space which we are going to use here are contained in the following result. The first four assertions are nowadays considered folklore in the the theory of Lipschitz operators, and may be found in the cited paper [49] (written for the real case, but also working in the complex case), section 2.2 of the book [121] by

N. Weaver, and Lemma 1.1 of [58]. The fifth assertion was proved in [65, Lemma 2.4]. For background on Lipschitz-free spaces we refer the reader to the already cited [49, 58, 121] and the very recent survey [47] by G. Godefroy.

Lemma 7.2. *Let X, Y be Banach spaces.*

(a) For every $F \in \mathrm{Lip}_0(X,Y)$, there exists a unique linear operator $\widehat{F}\colon \mathcal{F}(X) \longrightarrow Y$ such that $\widehat{F} \circ \delta_X = F$ and $\|T_F\| = \|F\|_L$. Moreover, the application $F \longmapsto \widehat{F}$ is an isometric isomorphism from $\mathrm{Lip}_0(X,Y)$ onto $\mathcal{L}(\mathcal{F}(X),Y)$.

(b) There exists a norm-one $\mathbb{K}$-linear quotient map $\beta_X\colon \mathcal{F}(X) \longrightarrow X$ which is a left inverse of δ_X, that is, $\beta_X \circ \delta_X = \mathrm{Id}_X$. It is called the barycenter map *in [49], and is given by the formula*

$$\beta_X\left(\sum_{x\in X} a_x\delta_x\right) = \sum_{x\in X} a_x x.$$

(c) From the uniqueness in item (a), it follows that $\widehat{F} = F \circ \beta_X$ for every $F \in \mathcal{L}(X,Y)$.

(d) The set

$$\mathscr{B}_X = \left\{\frac{\delta_x - \delta_y}{\|x-y\|} : x,y\in X,\, x\neq y\right\} \subset \mathcal{F}(X)$$

is norming for $\mathcal{F}(X)^* = \mathrm{Lip}_0(X,\mathbb{K})$, i.e. $B_{\mathcal{F}(X)} = \overline{\mathrm{aconv}}(\mathscr{B}_X)$.

(e) Given $C \subset S_X$ and a slice $\mathcal{S}$ of $\mathscr{B}_X$,

$$\Big[\beta_X(\mathcal{S})\cap\overline{\mathrm{conv}}(C)\neq\emptyset\Big] \quad\Longrightarrow\quad \Big[\beta_X(\mathcal{S})\cap C\neq\emptyset\Big].$$

A comment on item (e) above could be clarifying. Let X be a Banach space. As $\mathscr{B}_X \subset \mathcal{F}(X)$ and $\mathcal{F}(X)^* = \mathrm{Lip}_0(X,\mathbb{K})$, a slice $\mathcal{S}$ of $\mathscr{B}_X$ has the form

$$\mathcal{S} = \mathrm{Slice}(\mathscr{B}_X, f, \alpha) = \left\{\frac{\delta_x-\delta_y}{\|x-y\|} : x,y\in X,\, x\neq y,\, \mathrm{Re}\left\langle f, \frac{\delta_x-\delta_y}{\|x-y\|}\right\rangle > 1-\alpha\right\},$$

where $f \in \mathrm{Lip}_0(X,\mathbb{K})$ has norm one and α is a positive real number. Then, we have that

$$\beta_X(\mathcal{S}) = \left\{\frac{x-y}{\|x-y\|} : x,y\in X,\, x\neq y,\, \frac{\mathrm{Re}\, f(x) - \mathrm{Re}\, f(y)}{\|x-y\|} > 1-\alpha\right\}$$

is what is called in [65] a *Lipschitz slice* of S_X. Then, item (e) above means that if a Lipschitz slice of S_X does not intersect a subset $C \subset S_X$, then it does not intersect $\overline{\mathrm{conv}}(C)$ either. This was proved in [65, Lemma 2.4] with a completely elementary proof. Let us also say that assertion (e) is equivalent to the following fact [10, Lemma 2.3]: *given a Lipschitz slice $\beta_X(\mathcal{S})$ of S_X and a point $x_0 \in \beta_X(\mathcal{S})$, there is a linear slice S of S_X such that $x_0 \in S \subset \beta_X(\mathcal{S})$* (indeed, one direction is obvious and for the non trivial one, let $C = S_X \setminus \beta_X(\mathcal{S})$ which clearly satisfies that $\beta_X(\mathcal{S})\cap C = \emptyset$; then, $\beta_X(\mathcal{S})\cap\overline{\mathrm{conv}}(C) = \emptyset$ and so the Hahn-Banach theorem gives the result). The proof of this last result given in [10] is independent of the above one and uses generalized derivatives and the Fundamental Theorem of Calculus for them.

The next one is the main result of this chapter. It is an application of our theory to Lipschitz-free spaces from which we will deduce the commented result about Lipschitz spear operators.

Theorem 7.3. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. If G is lush, then $\widehat{G}\colon \mathcal{F}(X) \longrightarrow Y$ is lush.*

We need the following general technical result.

Lemma 7.4. *Let X, Y, Z be Banach spaces and let $G_1 \in \mathcal{L}(Z,X)$ and $G_2 \in \mathcal{L}(X,Y)$ be norm-one operators. Suppose that there is a subset $\mathscr{B} \subset B_Z$ norming for Z^* (i.e. $B_Z = \overline{\text{aconv}}(B)$) such that G_1 satisfies the following property*

$$\begin{aligned}&\text{For every slice } S \text{ of } \mathscr{B}, \text{ every } x^* \in S_{X^*}, \text{ and every } \varepsilon > 0,\\ &\big[G_1(S) \cap \overline{\text{conv}}\,\text{gSlice}(S_X, \mathbb{T}x^*, \varepsilon) \neq \emptyset\big] \Rightarrow \big[G_1(S) \cap \text{gSlice}(S_X, \mathbb{T}x^*, \varepsilon) \neq \emptyset\big].\end{aligned} \tag{P2}$$

If G_2 is lush, then $G := G_2 \circ G_1$ is lush.

Proof. Fix $z_0 \in \mathscr{B}$, $y_0 \in S_Y$, and $\varepsilon > 0$. As G_2 is lush, by Proposition 3.28.v we may find $y^* \in \text{ext}(B_{Y^*})$ such that

$$y_0 \in \text{Slice}(S_Y, y^*, \varepsilon) \quad \text{and} \quad G_1(z_0) \in \overline{\text{conv}}\big(\text{gSlice}(S_X, \mathbb{T}G_2^*y^*, \varepsilon)\big).$$

Therefore, for every slice S of $\mathscr{B}$ containing z_0, we have that

$$G_1(S) \cap \overline{\text{conv}}\big(\text{gSlice}(S_X, \mathbb{T}G_2^*y^*, \varepsilon)\big) \neq \emptyset,$$

and so (P2) gives us that

$$G_1(S) \cap \text{gSlice}(S_X, \mathbb{T}G_2^*y^*, \varepsilon) \neq \emptyset.$$

Therefore, we have that

$$S \cap \text{gSlice}(S_Z, \mathbb{T}G_1^*G_2^*y^*, \varepsilon) \neq \emptyset.$$

This has been proved for every slice S of $\mathscr{B}$ containing $z_0 \in \mathscr{B}$, but it is a fortiori also true for every slice S of S_Z containing z_0, so it follows that

$$z_0 \in \overline{\text{conv}}\big(\text{gSlice}(S_Z, \mathbb{T}G^*y^*, \varepsilon)\big).$$

As $\mathscr{B}$ is norming for Z^*, Proposition 3.28.v gives us the result. □

Proof (of Theorem 7.3). By Lemma 7.2.e, it follows that $G_1 := \beta_X\colon \mathcal{F}(X) \longrightarrow X$ satisfies condition (P2) of Lemma 7.4 for $\mathscr{B} = \mathscr{B}_X$. As $G_2 := G\colon X \longrightarrow Y$ is lush, it follows from this lemma that $G_2 \circ G_1 \colon \mathcal{F}(X) \longrightarrow Y$ is lush. But $G_2 \circ G_1 = G \circ \beta_X = \widehat{G}$ by Lemma 7.2.c. □

The identification of $\mathcal{L}(\mathcal{F}(X), Y)$ with $\mathrm{Lip}_0(X,Y)$ given in Lemma 7.2.a allows to deduce the promised result about Lipschitz spear operators from Theorem 7.3.

Corollary 7.5. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. If G is lush, then G is a Lipschitz spear operator, i.e. $\|G + \mathbb{T}F\|_L = 1 + \|F\|_L$ for every $F \in \mathrm{Lip}_0(X,Y)$.*

A first particular case of this result follows when we consider a lush Banach space X and $G = \mathrm{Id}_X$. This result appeared previously in [65, 120]

Corollary 7.6 **([65, Theorem 4.1] and [120, Theorem 2.6]).** *Let X be a lush Banach space. Then, Id_X is a Lipschitz spear operator, i.e. $\|\mathrm{Id}_X + \mathbb{T}F\|_L = 1 + \|F\|_L$ for every $F \in \mathrm{Lip}_0(X,Y)$.*

As we commented, this result is already known, as it is contained in [120, Theorem 2.6] and [65, Theorem 4.1]. But to get it from those references, the concept of Lipschitz numerical index of a Banach space is needed. Let X be a Banach space. For $F \in \mathrm{Lip}_0(X,X)$, the *Lipschitz numerical range* of F [120] is

$$W_L(F) := \left\{ \frac{\xi^*(Fx - Fy)}{\|x-y\|} \colon \xi^* \in S_{X^*},\, \xi^*(x-y) = \|x-y\|,\, x,y \in X,\, x \neq y \right\},$$

the Lipschitz numerical radius of F is just $w_L(F) := \sup\{|\lambda| \colon \lambda \in W_L(F)\}$, and the *Lipschitz numerical index* of X is

$$\begin{aligned} n_L(X) &:= \inf\{w_L(F) \colon F \in \mathrm{Lip}_0(X,X),\, \|F\|_L = 1\} \\ &= \max\{k \geqslant 0 \colon k\|F\|_L \leqslant w_L(F)\ \forall F \in \mathrm{Lip}_0(X,X)\}. \end{aligned}$$

It is shown in [120, Corollary 2.3] that Id_X is a Lipschitz spear operator if and only if $n_L(X) = 1$. With this in mind, Corollary 7.6 is just [120, Theorem 2.6] in the real case and [65, Theorem 4.1] in the complex case. Let us comment that the main difficulty of the proofs in [120] and [65] is to deal with Lipschitz operators. With our approach using the Lipschitz-free spaces, we avoid this.

Theorem 7.3 and Corollary 7.5 apply to all the lush operators presented in this manuscript. We would like to emphasise the following two particular ones, which follow from Theorem 4.1 and Corollary 4.12, respectively.

Example 7.7. Let H be a locally compact Abelian group and let Γ be its dual group. Then, the Fourier transform $\mathscr{F} \colon L_1(H) \longrightarrow C_0(\Gamma)$ is a Lipschitz spear operator, that is,

$$\|\mathscr{F} + \mathbb{T}F\|_L = 1 + \|F\|_L$$

for every $F \in \mathrm{Lip}_0(L_1(H), C_0(\Gamma))$.

Example 7.8. The inclusion $J\colon A(\mathbb{D}) \longrightarrow C(\mathbb{T})$ is a Lipschitz spear operator, that is,

$$\|J + \mathbb{T}F\|_L = 1 + \|F\|_L$$

for every $F \in \mathrm{Lip}_0(A(\mathbb{D}), C(\mathbb{T}))$.

The last consequence of Theorem 7.3 (actually, of Corollary 7.5) we would like to present here is the following.

Corollary 7.9. *Let X be a Banach space. Then,*

$$\mathrm{Spear}(X^*) \subset \mathrm{Spear}(\mathrm{Lip}_0(X, \mathbb{K})).$$

That is, every spear functional is actually a Lipschitz spear functional.

Proof. Let $x^* \in \mathrm{Spear}(X^*)$. It follows from Corollary 5.8 that $g = x^* \in X^* \equiv \mathcal{L}(X, \mathbb{K})$ is lush. Then, Corollary 7.5 gives that $\|g + \mathbb{T}f\|_L = 1 + \|f\|_L$ for every $f \in \mathrm{Lip}_0(X, \mathbb{K})$, that is, $g \in \mathrm{Spear}\left(\mathrm{Lip}_0(X, \mathbb{K})\right)$ as desired. □

We would like next to deal with operators with the aDP. The main result here is that we may extend the aDP of a (linear) operator to its linearization to the Lipschitz-free space.

Theorem 7.10. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X, Y)$ be a norm-one operator. If G has the aDP, then $\widehat{G}\colon \mathcal{F}(X) \longrightarrow Y$ has the aDP.*

Proof. We fix $y_0 \in S_Y$ and $\varepsilon > 0$. As G has the aDP, we have that

$$B_X = \overline{\mathrm{conv}}\{x \in S_X : \|Gx + \mathbb{T}y_0\| > 2 - \varepsilon\}$$

by Theorem 3.6.iii. Then, if S is an arbitrary slice of $\mathscr{B}_X$, we obviously have that

$$\beta_X(S) \cap \overline{\mathrm{conv}}\{x \in S_X : \|Gx + \mathbb{T}y_0\| > 2 - \varepsilon\} \neq \emptyset,$$

and so Lemma 7.2.e gives us that

$$\beta_X(S) \cap \{x \in S_X : \|Gx + \mathbb{T}y_0\| > 2 - \varepsilon\} \neq \emptyset. \tag{7.1}$$

Therefore, $S \cap \left\{\xi \in S_{\mathcal{F}(X)} : \left\|\left[G \circ \beta_X\right](\xi) + \mathbb{T}y_0\right\| > 2 - \varepsilon\right\} \neq \emptyset$, that is, using that $G \circ \beta_X = \widehat{G}$ by Lemma 7.2.c,

$$S \cap \left\{\xi \in S_{\mathcal{F}(X)} : \left\|\widehat{G}(\xi) + \mathbb{T}y_0\right\| > 2 - \varepsilon\right\} \neq \emptyset.$$

The arbitrariness of S gives then that

$$\begin{aligned}\mathscr{B}_X &\subset \overline{\mathrm{conv}}\left\{\xi \in S_{\mathcal{F}(X)} : \|\widehat{G}(\xi) + \mathbb{T}y_0\| > 2 - \varepsilon\right\} \\ &= \overline{\mathrm{aconv}}\left\{\xi \in S_{\mathcal{F}(X)} : \|\widehat{G}(\xi) + \mathbb{T}y_0\| > 2 - \varepsilon\right\}.\end{aligned} \tag{7.2}$$

As $B_{\mathcal{F}(X)} = \overline{\text{aconv}}\,(\mathscr{B}_X)$, we actually have that

$$\begin{aligned} B_{\mathcal{F}(X)} &= \overline{\text{aconv}}\big\{\xi \in S_{\mathcal{F}(X)} \colon \|\widehat{G}(\xi) + \mathbb{T}y_0\| > 2 - \varepsilon\big\} \\ &= \overline{\text{conv}}\big\{\xi \in S_{\mathcal{F}(X)} \colon \|\widehat{G}(\xi) + \mathbb{T}y_0\| > 2 - \varepsilon\big\}, \end{aligned} \tag{7.3}$$

and Theorem 3.6.iii gives that $\widehat{G}$ has the aDP, as desired. □

The identification of $\mathcal{L}(\mathcal{F}(X),Y)$ with $\text{Lip}_0(X,Y)$ given in Lemma 7.2.a allows to write Theorem 7.10 in terms of the Lipschitz norm of Lipschitz operators. We need some preliminary work. Let X, Y be Banach spaces. For $F \in \text{Lip}_0(X,Y)$, we define the *slope* of F [65] as the set

$$\text{slope}(F) := \left\{ \frac{F(x_1) - F(x_2)}{\|x_1 - x_2\|} : x_1 \neq x_2 \in X \right\}.$$

Observe that if $T \in \mathcal{L}(X,Y)$, then $\text{slope}(T) = T(S_X)$. On the other hand, it is clear that $\text{slope}(F) = \widehat{F}\big(\mathscr{B}_X\big)$ and, in particular,

$$\overline{\text{aconv}}\,\big(\text{slope}(F)\big) = \overline{\text{aconv}}\,\big(\widehat{F}\big(\mathscr{B}_X\big)\big) = \overline{\widehat{F}\big(B_{\mathcal{F}(X)}\big)}.$$

With this in mind, we get that if $G \in \mathcal{L}(X,Y)$ has the aDP and $F \in \text{Lip}_0(X,Y)$ satisfies that $\overline{\text{aconv}}\,(\text{slope}(F))$ is SCD, then $\|\,\text{Id} + \mathbb{T}F\|_L = 1 + \|F\|_L$ by Theorems 7.10 and 3.17. But, actually, we can go further and avoid to use the absolutely closed convex hull in the assumption.

Corollary 7.11. *Let X, Y be Banach spaces and let $G \in \text{Lip}_0(X,Y)$ be a norm-one operator with the aDP. If $F \in \text{Lip}_0(X,Y)$ satisfies that $\text{slope}(F)$ is SCD, then $\|G + \mathbb{T}F\|_L = 1 + \|F\|_L$.*

The result will follow from (7.1) in the proof of Theorem 7.10 and the following general result.

Lemma 7.12. *Let X, Y, Z be Banach spaces, let $\mathscr{B} \subset B_X$ such that $\overline{\text{aconv}}\,(\mathscr{B}) = B_X$, and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator such that $G(S)$ is a spear set for every slice S of $\mathscr{B}$. Then, if $T \in \mathcal{L}(X,Z)$ satisfies that $T(\mathscr{B})$ is SCD, then T is a target for G. In the case of $Z = Y$, we have $\|G + \mathbb{T}T\| = 1 + \|T\|$.*

Proof. The proof of this lemma is an easy adaptation of the one of Theorem 3.17. Let $\{\widehat{S}_n \colon n \in \mathbb{N}\}$ be a determining family of slices for $T(\mathscr{B})$, then $S_n := T^{-1}(\widehat{S}_n) \cap \mathscr{B}$ is a slice of $\mathscr{B}$ satisfying that $G(S_n)$ is a spear set by hypothesis. The same argument from Theorem 3.17 shows that

$$\bigcap_{n \in \mathbb{N}} \text{gFace}(\text{ext}\, B_{Y^*}, \mathbb{T}G(S_n)) \tag{7.4}$$

is weak-star dense in $\text{ext}\, B_{Y^*}$. For every y_0^* belonging to this intersection, we have that for every $n \in \mathbb{N}$ and $\varepsilon > 0$, $G(S_n) \cap \mathbb{T}\,\text{Slice}(B_Y, y_0^*, \varepsilon) \neq \emptyset$. Therefore,

$S_n \cap \mathbb{T}\,\mathrm{gSlice}(\mathscr{B}, G^* y_0^*, \varepsilon) \neq \emptyset$, and so $T(\mathbb{T}\,\mathrm{gSlice}(\mathscr{B}, G^* y_0^*, \varepsilon)) \cap \widehat{S}_n \neq \emptyset$. Using that the family $\{\widehat{S}_n \colon n \in \mathbb{N}\}$ is determining for $T(\mathscr{B})$, we conclude that

$$T(B_X) \subset \overline{\mathrm{aconv}}\,(T(\mathscr{B})) \subset \overline{T\big(\mathrm{aconv}\big(\mathrm{gSlice}(\mathscr{B}, G^* y_0^*, \varepsilon)\big)\big)}$$

and, therefore, $y_0^* \in \mathscr{D}_T$. This shows that the intersection (7.4) is contained in $\mathscr{D}_T$, and thus $\mathscr{D}_T$ is also weak-start dense in $\mathrm{ext}\, B_{Y^*}$. We then conclude, by Proposition 3.15, that T is a target for G.

Proof (of Corollary 7.11). If G has the aDP, it follows from (7.1) in the proof of Theorem 7.10 that $\widehat{G} \colon \mathcal{F}(X) \longrightarrow Y$ satisfies the hypothesis of the above lemma with $\mathscr{B} = \mathscr{B}_X$. Now, if $\mathrm{slope}(F) = \widehat{F}(\mathscr{B}_X)$ is SCD, we have that $\|\widehat{G} + \mathbb{T}\widehat{F}\| = 1 + \|\widehat{F}\|$, that is, $\|G + \mathbb{T}F\|_L = 1 + \|F\|_L$ as desired. □

If we apply this result in the particular case when $X = Y$ and $G = \mathrm{Id}_X$, we get the following result which already appeared in [65].

Corollary 7.13 ([65, Theorem 3.7]). *Let X be a Banach space with the aDP. Then $\|\mathrm{Id}_X + \mathbb{T}F\|_L = 1 + \|F\|_L$ for every $F \in \mathrm{Lip}_0(X, X)$ such that* $\mathrm{slope}(F)$ *is SCD.*

Our next result extends Corollary 7.11 to the non-separable case.

Corollary 7.14. *Let X, Y be Banach spaces and let $G \in \mathrm{Lip}_0(X, Y)$ be a norm-one operator with the aDP. If $F \in \mathrm{Lip}_0(X, Y)$ satisfies that for every separable subspace X_0 of X,* $\mathrm{slope}(F|_{X_0})$ *is SCD, then $\|G + \mathbb{T}F\|_L = 1 + \|F\|_L$.*

The result follows immediately from Corollary 7.11 and the following lemma.

Lemma 7.15. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X, Y)$ be an operator with the aDP. Given $F \in \mathrm{Lip}_0(X, Y)$, there are separable subspaces X_∞ of X and Y_∞ of Y such that $G|_{X_\infty} \colon X_\infty \longrightarrow Y_\infty$ has the aDP, $F(X_\infty) \subset Y_\infty$ and $\|F|_{X_\infty}\|_L = \|F\|_L$.*

Proof. Consider two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ with $x_n \neq y_n$ for every $n \in \mathbb{N}$ and

$$\lim \frac{\|F(x_n) - F(y_n)\|}{\|x_n - y_n\|} = \|F\|_L,$$

let X_0 be the closed linear span in X of the elements of the two sequences and let Y_0 be the closed linear span of $F(X_0)$. By Proposition 3.7, there are separable subspaces X_1 of X and Y_1 of Y such that $G|_{X_1} \colon X_1 \longrightarrow Y_1$ has the aDP. By construction, we have that $\|F|_{X_1}\|_L = \|F\|_L$. Now, we may apply again Proposition 3.7 starting with X_1 and the closed linear span of $Y_1 \cup F(X_1)$ to get separable subspaces X_2, Y_2 such that $G|_{X_2} \colon X_2 \longrightarrow Y_2$ has the aDP, $\|F_{X_2}\|_L = \|F\|_L$, and $F(X_1) \subset Y_2$. Repeating the process, it is straightforward to check that the separable subspaces $X_\infty := \overline{\bigcup_{n\in\mathbb{N}} X_n}$ of X and $Y_\infty := \overline{\bigcup_{n\in\mathbb{N}} Y_n}$ of Y work. □

The main particular cases in which Corollary 7.14 applies are the following.

Corollary 7.16. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be an operator with the aDP. If $F \in \mathrm{Lip}_0(X,Y)$ satisfies that $\overline{\mathrm{conv}}(\mathrm{slope}(F))$ has the Radon-Nikodým property, or the convex point of continuity property, or it is an Asplund set, or it does not contain ℓ_1-sequences, then $\|G+\mathbb{T}F\|_L = 1+\|F\|_L$.*

To get the result, we need to know that a set is SCD when its closed convex hull is, a result which was proved in [65, Lemma 3.1]. Actually, this implication reverses and we will use this fact later. This result appeared first in the preliminary ArXiv version of the current book, and then was modified a little in the ArXiv version of [68]. We give the modified version with proof instead of referencing to [68], because in the journal version of [68] the proof is going to be substituted by a reference to this book ☺.

Proposition 7.17. *Let X be a Banach space. A bounded set $A \subset X$ is SCD if and only if $\mathrm{conv}(A)$ is SCD, and if and only if $\overline{\mathrm{conv}}(A)$ is SCD.*

Proof. It follows readily from the definition that a bounded set is SCD if and only if its closure is, see [9, Remark 2.7]. It is also true that A is SCD whenever $\mathrm{conv}(A)$ is. Let us give the easy argument from [65, Lemma 3.1]: let $\{S_n \colon n \in \mathbb{N}\}$ be a determining sequence of slices of $\mathrm{conv}(A)$; then, the sets $S_n' = S_n \cap A$ are not empty and are slices of A and the family $\{S_n' \colon n \in \mathbb{N}\}$ is determining for A (indeed, if B intersects all the S_n', then B intersects all the S_n, so $\mathrm{conv}(A) \subset \mathrm{conv}(B)$ and, a fortiori, $A \subset \mathrm{conv}(B)$).

Let us prove the more intriguing reverse implication. Suppose now that A is SCD, and let $\{\mathrm{Slice}(A, x_n^*, \varepsilon_n) \colon n \in \mathbb{N}\}$ be a family of slices determining for A. We consider the following (countable) family of slices of $\mathrm{conv}(A)$:

$$\mathscr{S} := \{\mathrm{Slice}(\mathrm{conv}(A), x_n^*, \varepsilon_n/k) \colon n,k \in \mathbb{N}\}.$$

Given $\mathrm{Slice}(\mathrm{conv}(A), x^*, \varepsilon)$ of $\mathrm{conv}(A)$, where $\|x^*\| = 1$ without loss of generality, we will show that it contains an element of $\mathscr{S}$, thus proving that $\mathrm{conv}(A)$ is SCD by Lemma 1.54. Now, for the slice of A given by $\mathrm{Slice}(A, x^*, \varepsilon/2)$ we know that there is $n_0 \in \mathbb{N}$ such that $\mathrm{Slice}(A, x_{n_0}^*, \varepsilon_{n_0}) \subset \mathrm{Slice}(A, x^*, \varepsilon/2)$. Taking $k \in \mathbb{N}$ big enough we will argue that

$$\mathrm{Slice}(\mathrm{conv}(A), x_{n_0}^*, \varepsilon_{n_0}/k) \subset \mathrm{conv}\Big(\mathrm{Slice}(A, x_{n_0}^*, \varepsilon_{n_0})\Big) + \frac{\varepsilon}{2} B_X. \tag{7.5}$$

To prove this inclusion, we let

$$r := \sup_{a\in A} \mathrm{Re}\, x_{n_0}^*(a) = \sup_{a\in \mathrm{conv}(A)} \mathrm{Re}\, x_{n_0}^*(a) \qquad \text{and} \qquad M := \sup_{a\in A} \|a\|.$$

Consider a convex combination $a = \sum_{i=1}^n \lambda_i a_i \in \mathrm{conv}(A)$ such that

$$\operatorname{Re} x_{n_0}^*(a) > r - \varepsilon_{n_0}/k,$$

where k is not yet specified. Write

$$I = \{i\colon \operatorname{Re} x_{n_0}^*(a_i) > r - \varepsilon_{n_0}\} \qquad \text{and} \qquad J = \{i\colon \operatorname{Re} x_{n_0}^*(a_i) \leqslant r - \varepsilon_{n_0}\}.$$

We then have

$$r - \frac{\varepsilon_{n_0}}{k} < \sum_{i\in I} \lambda_i \operatorname{Re} x_{n_0}^*(a_i) + \sum_{i\in J} \lambda_i \operatorname{Re} x_{n_0}^*(a_i) \leqslant r \sum_{i\in I} \lambda_i + \sum_{i\in J} \lambda_i (r - \varepsilon_{n_0}),$$

which implies

$$\sum_{i\in J} \lambda_i < \frac{1}{k} \qquad \text{and} \qquad \Lambda := \sum_{i\in I} \lambda_i > 1 - \frac{1}{k}.$$

Now, put $\mu_i := \lambda_i/\Lambda$ for $i \in I$ and consider the element

$$a' = \sum_{i\in I} \mu_i a_i \in \operatorname{conv}\left(\operatorname{Slice}(A, x_{n_0}^*, \varepsilon_{n_0})\right).$$

The estimate

$$\|a - a'\| = \left\| (\Lambda - 1) \sum_{i\in I} \mu_i a_i + \sum_{i\in J} \lambda_i a_i \right\| \leqslant |\Lambda - 1| M + \sum_{i\in J} \lambda_i M < \frac{2M}{k}$$

shows that the needed inclusion (7.5) holds true whenever $k \geqslant 4M/\varepsilon$.

It now follows for this choice of k that

$$\begin{aligned} \operatorname{Slice}(\operatorname{conv}(A), x_{n_0}^*, \varepsilon_{n_0}/k) &\subset \operatorname{conv}\left(\operatorname{Slice}(A, x_{n_0}^*, \varepsilon_{n_0})\right) + \frac{\varepsilon}{2} B_X \\ &\subset \operatorname{conv}\left(\operatorname{Slice}(A, x^*, \varepsilon/2)\right) + \frac{\varepsilon}{2} B_X \\ &\subset \operatorname{Slice}(\operatorname{conv}(A), x^*, \varepsilon/2) + \frac{\varepsilon}{2} B_X. \end{aligned}$$

Since, trivially, $\operatorname{Slice}(\operatorname{conv}(A), x_{n_0}^*, \varepsilon_{n_0}/k) \subset \operatorname{conv}(A)$, we finally get

$$\begin{aligned} \operatorname{Slice}(\operatorname{conv}(A), x_{n_0}^*, \varepsilon_{n_0}/k) &\subset \left(\operatorname{Slice}(\operatorname{conv}(A), x^*, \varepsilon/2) + \frac{\varepsilon}{2} B_X\right) \cap \operatorname{conv}(A) \\ &\subset \operatorname{Slice}(\operatorname{conv}(A), x^*, \varepsilon), \end{aligned}$$

as desired. □

Proof (of Corollary 7.16). If the set $\overline{\operatorname{conv}}(\operatorname{slope}(F))$ satisfies any of the aforementioned conditions, then $\overline{\operatorname{conv}}\big(\operatorname{slope}(F|_{X_0})\big)$ is SCD for every separable subspace X_0 of X (use Examples 1.56). By Proposition 7.17, it follows that $\operatorname{slope}(F|_{X_0})$ is SCD for every separable subspace X_0 of X, and so Corollary 7.14 gives the result. □

It is immediate that the above result applies to the recently introduced Lipschitz compact and Lipschitz weakly compact operators [58, Definition 2.1]: F

in $\mathrm{Lip}_0(X,Y)$ is Lipschitz compact (respectively, Lipschitz weakly compact) if $\mathrm{slope}(F)$ is relatively compact (respectively, relatively weakly compact).

The last aim in this chapter is to give for Daugavet centers analogous results to the ones we have for the aDP. We recall that a bounded linear operator G between two Banach spaces X and Y is said to be a *Daugavet center* [18] if

$$\|G+T\| = 1+\|T\|$$

for every rank-one operator $T \in \mathcal{L}(X,Y)$. In this case, we have to deal with the real version of the Lipschitz-free space. Given a (real or complex) Banach space X, we write $\mathcal{F}_{\mathbb{R}}(X)$ to denote the Lipschitz-free space of X in the sense of the real scalars, that is, $\mathcal{F}_{\mathbb{R}}(X)$ it is the canonical predual of $\mathrm{Lip}_0(X,\mathbb{R})$. If X and Y are real Banach spaces, nothing changes, but if they are complex spaces, we are considering only their real structure and so Lemma 7.2 is only valid for real scalars.

The main result for Daugavet centers is the following one.

Theorem 7.18. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. If G is a Daugavet center, then $\widehat{G_{\mathbb{R}}}: \mathcal{F}_{\mathbb{R}}(X) \longrightarrow Y_{\mathbb{R}}$ is a Daugavet center.*

We need the following characterization of Daugavet centers which follows immediately from [18] and which will play the role of our Theorem 3.6.iii. In particular, it follows from it that to be a Daugavet center only depends on the real structure of the Banach spaces involved.

Lemma 7.19 (see [18, Theorem 2.1])**.** *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Then G is a Daugavet center if and only if*

$$B_X = \overline{\mathrm{conv}}\big(\{x \in S_X : \|Gx+y_0\| > 2-\varepsilon\}\big)$$

for every $y_0 \in S_X$ and every $\varepsilon > 0$.

Proof (of Theorem 7.18). We just have to adapt mutatis mutandis the proof of Theorem 7.10, using the above lemma instead of Theorem 3.6.iii. But observe that the game played in (7.2) and (7.3) is not valid here as the set

$$\{\xi \in S_{\mathcal{F}_{\mathbb{R}}(X)} : \|\widehat{G}(\xi)+y_0\| > 2-\varepsilon\}$$

is not rounded. At this point is where we have to go to the real version of the Lipschitz-free space, since the set $\mathscr{B}_X$ is clearly $\mathbb{R}$-rounded and then we actually have $B_{\mathcal{F}_{\mathbb{R}}(X)} = \overline{\mathrm{conv}}\big(\mathscr{B}_X\big)$. With this in mind, everything works. □

Our next aim is to get consequences of Theorem 7.18 just in terms of the Lipschitz norm and the slope of Lipschitz operators.

Corollary 7.20. *Let X, Y be Banach spaces and let $G \in \mathrm{Lip}_0(X,Y)$ be a Daugavet center. If $F \in \mathrm{Lip}_0(X,Y)$ satisfies that $\mathrm{slope}(F)$ is SCD, then $\|G+F\|_L = 1+\|F\|_L$.*

Proof. If G is a Daugavet center, then $\widehat{G_{\mathbb{R}}}\colon \mathcal{F}_{\mathbb{R}}(X) \longrightarrow Y$ is also a Daugavet center by Theorem 7.18. Now, if $F \in \mathrm{Lip}_0(X,Y)$ satisfies that $\mathrm{slope}(F)$ is SCD, so is $\widehat{F_{\mathbb{R}}}(B_{\mathcal{F}_{\mathbb{R}}(X)}) = \overline{\mathrm{conv}}(\mathrm{slope}(F))$. Therefore, $\|\widehat{G_{\mathbb{R}}} + \widehat{F}\| = 1 + \|\widehat{F}\|$ by [17, Corollary 1]. Finally, this is equivalent to $\|G+F\|_L = 1 + \|F\|_L$ by Lemma 7.2. □

We may extend Corollary 7.20 to the non-separable case as we did for the aDP.

Corollary 7.21. *Let X, Y be Banach spaces and let $G \in \mathrm{Lip}_0(X,Y)$ be a Daugavet center. If $F \in \mathrm{Lip}_0(X,Y)$ satisfies that for every separable subspace X_0 of X, $\mathrm{slope}(F|_{X_0})$ is SCD, then $\|G+F\|_L = 1 + \|F\|_L$.*

The result follows immediately from Corollary 7.20 and the following lemma which allows a reduction to the separable case and which is completely analogous to Lemma 7.15. Its proof follows from [57, Theorem 1] in the same manner that the proof of Lemma 7.15 follows from Proposition 3.7.

Lemma 7.22. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a Daugavet center. Given $F \in \mathrm{Lip}_0(X,Y)$, there are separable subspaces X_∞ of X and Y_∞ of Y such that $G|_{X_\infty}\colon X_\infty \longrightarrow Y_\infty$ is a Daugavet center, $F(X_\infty) \subset Y_\infty$ and $\|F|_{X_\infty}\|_L = \|F\|_L$.*

The most interesting particular cases of Corollary 7.21 are summarized in the following result, whose proof is completely analogous to the one of Corollary 7.16.

Corollary 7.23. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a Daugavet center. If $F \in \mathrm{Lip}_0(X,Y)$ satisfies that $\overline{\mathrm{conv}}(\mathrm{slope}(F))$ has the Radon-Nikodým property, the convex point of continuity property or it is an Asplund set, or it does not contain ℓ_1-sequences, then $\|G+F\|_L = 1 + \|F\|_L$.*

Chapter 8
Some stability results

Our aim here is to provide several results on the stability of our properties by several operations like absolute sums, vector-valued function spaces, and ultraproducts.

8.1 Elementary results

The first result shows that we may produce an injective operator with the aDP from any operator with the aDP.

Proposition 8.1. *Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be an operator with the aDP and let $P\colon X \longrightarrow X/\ker G$ be the quotient map. Then, the quotient operator $\widetilde{G} \in \mathcal{L}\big(X/\ker G, Y\big)$ satisfying $\widetilde{G} \circ P = G$ has the aDP.*

Proof. By Theorem 3.6 it suffices to show that $\widetilde{G}(\widetilde{S})$ is a spear set in Y for every slice $\widetilde{S}$ of $B_{X/\ker G}$. So, we fix an arbitrary slice $\widetilde{S}$ of $B_{X/\ker G}$ and find $z^* \in S_{(X/\ker G)^*}$ and $\alpha > 0$ such that

$$\widetilde{S} = \big\{x + \ker G \in B_{X/\ker G}\colon\ \operatorname{Re} z^*(x + \ker G) > 1 - \alpha\big\}.$$

Since $P^*z^* \in S_{X^*}$, the set $S = \big\{x \in B_X\colon\ \operatorname{Re}[P^*z^*](x) > 1 - \alpha\big\}$ is a slice of B_X. Observe that if $x \in S$ then $\operatorname{Re} z^*(P(x)) > 1 - \alpha$ and $P(x) \in B_{X/\ker G}$ which give that $P(x) \in \widetilde{S}$. Therefore, we have that $P(S) \subset \widetilde{S}$ and so

$$G(S) = \widetilde{G}\big(P(S)\big) \subset \widetilde{G}(\widetilde{S}).$$

Now, as $G(S)$ is a spear set by Theorem 3.6, so is a fortiori $\widetilde{G}(\widetilde{S})$, as desired. □

The reciprocal result is not true: consider $G\colon \ell_2^2 \longrightarrow \mathbb{K}$ given by $G(x,y)=x$ and observe that $\widetilde{G}\equiv \mathrm{Id}_{\mathbb{K}}$ is clearly lush, while G does not even have the aDP (use Proposition 5.6, for instance).

Our next aim is to provide a way to extend the domain and the codomain keeping the properties of being spear, lush, or the aDP. The first result deals with extending the domain.

Proposition 8.2. *Let X, Y, Z be Banach spaces, let $G\in\mathcal{L}(X,Y)$ be a norm-one operator, and consider the operator $\widetilde{G}\colon X\oplus_\infty Z\longrightarrow Y$ given by $\widetilde{G}(x,z)=G(x)$ for every $(x,z)\in X\oplus_\infty Z$. Then:*

(a) if G is a spear operator, so is $\widetilde{G}$;
(b) if G has the aDP, so does $\widetilde{G}$;
(c) if G is lush, so is $\widetilde{G}$.

Proof. (a). Fix $T\in\mathcal{L}(X\oplus_\infty Z,Y)$ with $\|T\|>0$ and $\|T\|>\varepsilon>0$. Take $x_0\in S_X$ and $z_0\in S_Z$ satisfying $\|T(x_0,z_0)\|>\|T\|-\varepsilon$. Now pick $x^*\in S_{X^*}$ so that $x^*(x_0)=1$ and define the operator $S\in\mathcal{L}(X,Y)$ by

$$S(x)=T(x,x^*(x)z_0) \qquad (x\in X)$$

which satisfies $S(x_0)=T(x_0,z_0)$ and so $\|S\|>\|T\|-\varepsilon$. Now we can estimate as follows

$$\begin{aligned}\|\widetilde{G}+\mathbb{T}T\| &= \sup_{x\in B_X}\sup_{z\in B_Z}\|\widetilde{G}(x,z)+\mathbb{T}T(x,z)\|\\ &\geqslant \sup_{x\in B_X}\|\widetilde{G}(x,x^*(x)z_0)+\mathbb{T}T(x,x^*(x)z_0)\|\\ &= \sup_{x\in B_X}\|G(x)+\mathbb{T}S(x)\| = \|G+\mathbb{T}S\| = 1+\|S\| > 1+\|T\|-\varepsilon.\end{aligned}$$

The arbitrariness of ε gives $\|\widetilde{G}+\mathbb{T}T\|\geqslant 1+\|T\|$.

(b). Just observe that if T is a rank-one operator in the argument above, then S also is rank-one.

(c). Consider $(x_0,z_0)\in B_{X\oplus_\infty Z}=B_X\times B_Z$, $y\in S_Y$ and $\varepsilon>0$. As G is lush, Proposition 3.28.iii allows to find $y^*\in S_{Y^*}$ such that

$$\operatorname{Re} y^*(y)>1-\varepsilon \quad\text{and}\quad \operatorname{dist}\big(x_0,\operatorname{aconv}(\operatorname{gSlice}(S_X,G^*y^*,\varepsilon))\big)<\varepsilon.$$

Now, observe that $\widetilde{G}^*y^*=(G^*y^*,0)\in[X\oplus_\infty Z]^*$ and it is then immediate that

$$\operatorname{dist}\big((x_0,z_0),\operatorname{aconv}(\operatorname{gSlice}(S_{X\oplus_\infty Z},\widetilde{G}^*y^*,\varepsilon))\big)<\varepsilon.$$

Now, Proposition 3.28.iii gives that $\widetilde{G}$ is lush. □

We can get an analogous result to extend the codomain space.

Proposition 8.3. *Let X, Y, Z be Banach spaces, let $G \in \mathcal{L}(X,Y)$ be a norm-one operator, and consider the operator $\widetilde{G}\colon X \longrightarrow Y \oplus_1 Z$ given by $\widetilde{G}(x) = (G(x), 0)$ for every $x \in X$. Then:*

(a) if G is a spear operator, so is $\widetilde{G}$;
(b) if G has the aDP, so does $\widetilde{G}$;
(c) if G is lush, so is $\widetilde{G}$.

Proof. (a). Fix $T \in \mathcal{L}(X, Y \oplus_1 Z)$ with $\|T\| > 0$, $\|T\| > \varepsilon > 0$, and $x_0 \in S_X$ such that $\|Tx_0\| \geqslant \|T\| - \varepsilon$. Denote by P_Y and P_Z the respective projections from $Y \oplus_1 Z$ to Y and Z. Take $z^* \in S_{Z^*}$ satisfying $z^*(P_Z T x_0) = \|P_Z T x_0\|$ and pick $y_0 \in S_Y$ so that $P_Y T x_0 = \|P_Y T x_0\| y_0$. Now define $S \in \mathcal{L}(X,Y)$ by

$$Sx = P_Y T x + z^*(P_Z T x) y_0 \qquad (x \in X)$$

which satisfies

$$\|S\| \geqslant \|Sx_0\| = \big\|P_Y T x_0 + \|P_Z T x_0\| y_0\big\| = \|P_Y T x_0\| + \|P_Z T x_0\| \geqslant \|T\| - \varepsilon.$$

Finally, using the triangle inequality and the fact that G is a spear operator, we can estimate as follows:

$$\begin{aligned}\|\widetilde{G} + \mathbb{T} T\| &= \sup_{x \in B_X} \|Gx + \mathbb{T} P_Y T x\| + \|P_Z T x\| \\ &\geqslant \sup_{x \in B_X} \big\|Gx + \mathbb{T}\big(P_Y T x + z^*(P_Z T x) y_0\big)\big\| \\ &= \sup_{x \in B_X} \|Gx + \mathbb{T} S x\| = \|G + \mathbb{T} S\| = 1 + \|S\| \geqslant 1 + \|T\| - \varepsilon.\end{aligned}$$

The arbitrariness of ε finishes the proof.

(b). Observe that if T is a rank-one operator in the argument above, then S also is rank-one.

(c). Fix $x_0 \in B_X$, $(y,z) \in S_{Y \oplus_1 Z}$ and $\varepsilon > 0$. As G is lush, we may find $y^* \in S_{Y^*}$ such that

$$\operatorname{Re} y^*(y) > \|y\| - \varepsilon \quad \text{and} \quad \operatorname{dist}\big(x_0, \operatorname{aconv}(\operatorname{gSlice}(S_X, G^* y^*, \varepsilon))\big) < \varepsilon.$$

Indeed, if $y \neq 0$, apply Proposition 3.28.iii to $y/\|y\|$; if $y = 0$, we may apply that proposition to any vector in S_Y. Now, pick $z^* \in S_{Z^*}$ such that $z^*(z) = \|z\|$ and consider $(y^*, z^*) \in S_{Y^*} \times S_{Z^*} \subset S_{[Y \oplus_1 Z]^*}$. Observe that, on the one hand,

$$\operatorname{Re}[(y^*, z^*)](y,z) > \|y\| - \varepsilon + \|z\| = 1 - \varepsilon$$

and, on the other hand, $\widetilde{G}(y^*, z^*) = G^*(y^*)$, so we have that

$$\operatorname{dist}\big(x_0, \operatorname{aconv}(\operatorname{gSlice}(S_X, \widetilde{G}^*(y^*, z^*), \varepsilon))\big) < \varepsilon.$$

We then get that $\widetilde{G}$ is lush by Proposition 3.28.iii. □

8.2 Absolute sums

We show in this section the stability of our properties by c_0, ℓ_1, and ℓ_∞ sums of Banach spaces. The following result borrows the ideas from [99, Proposition 1] and [21, §5].

Proposition 8.4. *Let $\{X_\lambda : \lambda \in \Lambda\}$, $\{Y_\lambda : \lambda \in \Lambda\}$ be two families of Banach spaces and let $G_\lambda \in \mathcal{L}(X_\lambda, Y_\lambda)$ be a norm-one operator for every $\lambda \in \Lambda$. Let E be one of the Banach spaces c_0, ℓ_∞, or ℓ_1, let $X = [\bigoplus_{\lambda\in\Lambda} X_\lambda]_E$ and $Y = [\bigoplus_{\lambda\in\Lambda} Y_\lambda]_E$, and define the operator $G\colon X \longrightarrow Y$ by*

$$G\big[(x_\lambda)_{\lambda\in\Lambda}\big] = (G_\lambda x_\lambda)_{\lambda\in\Lambda}$$

for every $(x_\lambda)_{\lambda\in\Lambda} \in [\bigoplus_{\lambda\in\Lambda} X_\lambda]_E$. Then

(a) G is a spear operator if and only if G_λ is a spear operator for every $\lambda \in \Lambda$;
(b) G has the aDP if and only if G_λ has the aDP for every $\lambda \in \Lambda$;
(c) G is lush if and only if G_λ is lush for every $\lambda \in \Lambda$.

Proof. (a). We suppose first that G is a spear operator and, fixed $\kappa \in \Lambda$, we have to show that G_κ is a spear operator. Observe that calling $W = \big[\bigoplus_{\lambda\neq\kappa} X_\lambda\big]_E$ and $Y = \big[\bigoplus_{\lambda\neq\kappa} Y_\lambda\big]_E$, we can write $X = X_\kappa \oplus_\infty W$ and $Y = Y_\kappa \oplus_\infty Z$ when E is ℓ_∞ or c_0 and $X = X_\kappa \oplus_1 W$ and $Y = Y_\kappa \oplus_1 Z$ when E is ℓ_1. Given $T_\kappa \in \mathcal{L}(X_\kappa, Y_\kappa)$ non-zero, define $T \in \mathcal{L}(X,Y)$ by $T(x_\kappa, w) = (T_\kappa x_\kappa, 0)$ which obviously satisfies $\|T\| = \|T_\kappa\|$. Let P_κ and P_Z denote the projections from Y onto Y_κ and Z respectively. When E is ℓ_∞ or c_0, we can write

$$\begin{aligned}
1 + \|T_\kappa\| = 1 + \|T\| = \|G + \mathbb{T}\,T\| &= \sup_{(x_\kappa,w)\in B_X} \|G(x_\kappa,w) + \mathbb{T}\,T(x_\kappa,w)\| \\
&\overset{(*)}{=} \sup_{(x_\kappa,w)\in B_X} \max\big\{\|P_\kappa G(x_\kappa,w) + \mathbb{T}\,P_\kappa T(x_\kappa,w)\|, \\
&\qquad\qquad \|P_Z G(x_\kappa,w) + \mathbb{T}\,P_Z T(x_\kappa,w)\|\big\} \\
&= \max\left\{\|G_\kappa + \mathbb{T}\,T_\kappa\|, \sup_{(x_\kappa,w)\in B_X} \|P_Z G(x_\kappa,w)\|\right\} \\
&\leqslant \max\big\{\|G_\kappa + \mathbb{T}\,T_\kappa\|, \|G\|\big\}.
\end{aligned}$$

Since $\|G\| = 1$, it follows that $1 + \|T_\kappa\| \leqslant \|G_\kappa + \mathbb{T}\,T_\kappa\|$ and so G_κ is a spear operator.

When E is ℓ_1, the equality $(*)$ can be continued as follows

$$
\begin{aligned}
1+\|T_\kappa\| &\overset{(*)}{=} \sup_{(x_\kappa,w)\in B_X} \|P_\kappa G(x_\kappa,w)+\mathbb{T}P_\kappa T(x_\kappa,w)\| \\
&\qquad\qquad\qquad + \|P_Z G(x_\kappa,w)+\mathbb{T}P_Z T(x_\kappa,w)\| \\
&= \sup_{(x_\kappa,w)\in B_X} \|G_\kappa x_\kappa+\mathbb{T}T_\kappa x_\kappa\|+\|P_Z G(0,w)\| \\
&\leqslant \sup_{(x_\kappa,w)\in B_X} \|G_\kappa+\mathbb{T}T_\kappa\|\|x_\kappa\|+\|G\|\|w\| = \max\{\|G_\kappa+\mathbb{T}T_\kappa\|,\ \|G\|\}.
\end{aligned}
$$

Analogously to the previous case, it follows that G_κ is a spear operator.

We prove now the sufficiency when E is ℓ_∞ or c_0. Given an operator $T\in\mathcal{L}(X,Y)$ and fixed $\varepsilon>0$, we find $\kappa\in\Lambda$ such that $\|P_\kappa T\|>\|T\|-\varepsilon$ and write $X=X_\kappa\oplus_\infty W$ where $W=\left[\bigoplus_{\lambda\neq\kappa}X_\lambda\right]_E$. Since B_X is the convex hull of $S_{X_\kappa}\times S_W$ we may find $x_0\in S_{X_\kappa}$ and $w_0\in S_W$ such that

$$\|P_\kappa T(x_0,w_0)\|>\|T\|-\varepsilon.$$

Now fix $x^*\in S_{X_\kappa^*}$ with $x^*(x_0)=1$ and define the operator $S\in\mathcal{L}(X_\kappa,Y_\kappa)$ given by

$$S(x)=P_\kappa T(x,x^*(x)w_0) \qquad (x\in X_\kappa)$$

which satisfies $\|S\|\geqslant\|Sx_0\|=\|P_\kappa T(x_0,w_0)\|>\|T\|-\varepsilon$. Observe finally that

$$
\begin{aligned}
\|G+\mathbb{T}T\| &\geqslant \|P_\kappa G+\mathbb{T}P_\kappa T\| \geqslant \sup_{x\in X_\kappa}\|[P_\kappa G](x,x^*(x)w_0)+\mathbb{T}[P_\kappa T](x,x^*(x)w_0)\| \\
&= \sup_{x\in X_\kappa}\|G_\kappa(x)+\mathbb{T}S(x)\| = \|G_\kappa+\mathbb{T}S\| = 1+\|S\| > 1+\|T\|-\varepsilon.
\end{aligned}
$$

So, the arbitrariness of ε gives that $\|G+\mathbb{T}T\|\geqslant 1+\|T\|$, finishing the proof for $E=c_0,\ell_\infty$.

Suppose now that $E=\ell_1$. Fix an operator $T\in\mathcal{L}(X,Y)$ and observe that it may be seen as a family $(T_\lambda)_{\lambda\in\Lambda}$ of operators where $T_\lambda\in\mathcal{L}(X_\lambda,Y)$ for every $\lambda\in\Lambda$, and $\|T\|=\sup_\lambda\|T_\lambda\|$. Given $\varepsilon>0$, find $\kappa\in\Lambda$ such that $\|T_\kappa\|>\|T\|-\varepsilon$, and write $X=X_\kappa\oplus_1 W$, $Y=Y_\kappa\oplus_1 Z$, and $T_\kappa=(A,B)$ where $A\in\mathcal{L}(X_\kappa,Y_\kappa)$ and $B\in\mathcal{L}(X_\kappa,Z)$. Now we choose $x_0\in S_{X_\kappa}$ such that

$$\|T_\kappa x_0\|=\|Ax_0\|+\|Bx_0\|>\|T\|-\varepsilon,$$

we find $a_0\in S_{Y_\kappa}$, $z^*\in S_{Z^*}$ satisfying

$$\|Ax_0\|a_0=Ax_0 \qquad \text{and} \qquad z^*(Bx_0)=\|Bx_0\|,$$

and define the operator $S\in\mathcal{L}(X_\kappa,Y_\kappa)$ by

$$Sx=Ax+z^*(Bx)a_0 \qquad (x\in X_\kappa).$$

Then

$$\|S\| \geqslant \|Sx_0\| = \Big\|Ax_0 + \|Bx_0\|a_0\Big\| = \|Ax_0\| + \|Bx_0\| > \|T\| - \varepsilon.$$

Moreover, since G_κ is a spear operator, fixed $\varepsilon > 0$ we may find $x_\kappa \in S_{X_\kappa}$ and $y_\kappa^* \in S_{Y_\kappa^*}$ such that

$$|y_\kappa^*(G_\kappa x_\kappa + \mathbb{T}Sx_\kappa)| = \|G_\kappa x_\kappa + \mathbb{T}Sx_\kappa\| \geqslant 1 + \|S\| - \varepsilon.$$

Now take $x = (x_\kappa, 0) \in S_X$ and $y^* = (y_\kappa^*, y_\kappa^*(a_0)z^*) \in S_{Y^*}$, and observe that

$$\begin{aligned}\|G + \mathbb{T}T\| &\geqslant |y^*(Gx + \mathbb{T}Tx)| = \left|y_\kappa^*(G_\kappa x_\kappa) + \mathbb{T}[y_\kappa^*(Ax_\kappa) + y_\kappa^*(a_0)z^*(Bx_\kappa)]\right| \\ &= |y_\kappa^*(G_\kappa x_\kappa + \mathbb{T}Sx_\kappa)| = \|G_\kappa x_\kappa + \mathbb{T}Sx_\kappa\| \\ &\geqslant 1 + \|S\| - \varepsilon \geqslant 1 + \|T\| - 2\varepsilon.\end{aligned}$$

So, the arbitrariness of ε gives that $\|G + \mathbb{T}T\| \geqslant 1 + \|T\|$, finishing the proof for $E = \ell_1$.

(b). For the aDP, the arguments above apply just taking into account that when one starts with rank-one operators, the constructed operators are also rank-one.

(c). We assume first that G is lush. Fixed $\kappa \in \Lambda$, $x_\kappa \in S_{X_\kappa}$, $y_\kappa \in S_{Y_\kappa}$, $\varepsilon > 0$ we consider the elements $(z_\lambda)_{\lambda\in\Lambda} \in B_X$ and $(w_\lambda)_{\lambda\in\Lambda} \in S_Y$ given by

$$z_\lambda = 0, \quad w_\lambda = 0 \quad \text{for} \quad \lambda \neq \kappa \qquad \text{and} \qquad z_\kappa = x_\kappa, \quad w_\kappa = y_\kappa.$$

Now Proposition 3.28.iii provides with $y^* \in \operatorname{Slice}(B_{Y^*}, (w_\lambda)_{\lambda\in\Lambda}, \varepsilon)$ such that

$$\operatorname{dist}\big((z_\lambda)_{\lambda\in\Lambda}, \operatorname{aconv}(\operatorname{gSlice}(S_X, G^*y^*, \varepsilon))\big) < \varepsilon^2. \tag{8.1}$$

From this point we have to distinguish two cases depending on the space E. Suppose first that $E = c_0$ or $E = \ell_\infty$ and observe that $y^*|_{Y_\kappa} \in \operatorname{Slice}(B_{Y_\kappa^*}, y_\kappa, \varepsilon)$. In this case, given $(\widetilde{z}_\lambda)_{\lambda\in\Lambda} \in \operatorname{gSlice}(S_X, G^*y^*, \varepsilon)$, it follows that $\widetilde{z}_\kappa \in \operatorname{gSlice}(S_{X_\kappa}, G_\kappa^*(y^*|_{Y_\kappa}), 2\varepsilon)$ which, together with (8.1), allows us to deduce that

$$\operatorname{dist}\big(x_\kappa, \operatorname{aconv}(\operatorname{gSlice}(S_{X_\kappa}, G_\kappa^*(y^*|_{Y_\kappa}), 2\varepsilon))\big) < \varepsilon.$$

We consider now the more bulky case in which $E = \ell_1$. Using (8.1) we can find scalars $\lambda_i \in \mathbb{K}$ with $\sum_{i=1}^n |\lambda_i| = 1$ and elements $x^i \in \operatorname{gSlice}(S_X, G^*y^*, \varepsilon)$ such that

$$\left\|x_\kappa - \sum_{i=1}^n \lambda_i x_\kappa^i\right\| \leqslant \sum_{\lambda\in\Lambda} \left\|z_\lambda - \sum_{i=1}^n \lambda_i x_\lambda^i\right\|_{X_\lambda} < \varepsilon^2.$$

Since $\|x_\kappa\| = 1$, we deduce that $1 - \varepsilon^2 \leqslant \sum_{i=1}^n |\lambda_i| \|x_\kappa^i\|$ so Lemma 8.14 tells us that the set $I := \{i \colon \|x_\kappa^i\| > 1 - \varepsilon\}$ satisfies that $\sum_{i\in I} |\lambda_i| > 1 - \varepsilon$. Hence

$$\left\|x_\kappa - \sum_{i\in I} \lambda_i x_\kappa^i\right\| < 2\varepsilon. \tag{8.2}$$

But every $i \in I$ satisfies that

$$1-\varepsilon < \sum_{\lambda\in\Lambda} \operatorname{Re} y^*|_{Y_\lambda}(G_\lambda x_\lambda^i) \leqslant \operatorname{Re} y^*|_{Y_\kappa}(G_\kappa x_\kappa^j) + \sum_{\lambda\neq\kappa} \|x_\lambda^i\| < \operatorname{Re} y^*|_{Y_\kappa}(G_\kappa x_\kappa^i) + \varepsilon,$$

from where it follows that

$$x_\kappa^i \in \operatorname{gSlice}(B_X, G_\kappa^*(y^*|_{Y_\kappa}), 2\varepsilon)$$

for each $i \in I$. This, together with (8.2), tells us that

$$\operatorname{dist}\big(x_\kappa, \operatorname{aconv}(\operatorname{gSlice}(B_X, G_\kappa^*(y^*|_{Y_\kappa}), 2\varepsilon))\big) < 2\varepsilon,$$

finishing the proof of the necessity for $E = \ell_1$.

Let us prove the sufficiency when $E = \ell_\infty$ or $E = c_0$. Fixed $(x_\lambda)_{\lambda\in\Lambda} \in B_X$, $y = (y_\lambda)_{\lambda\in\Lambda} \in S_Y$, and $\varepsilon > 0$, there is $\kappa \in \Lambda$ such that $\|y_\kappa\| > 1-\varepsilon$. Using that G_κ is lush, we may find $y_\kappa^* \in \operatorname{gSlice}(S_{Y_\kappa^*}, y_\kappa, \varepsilon)$ satisfying

$$\operatorname{dist}\big(x_\kappa, \operatorname{aconv}(\operatorname{gSlice}(S_{X_\kappa}, G_\kappa^* y_\kappa^*, \varepsilon))\big) < \varepsilon.$$

Defining $y^* \in S_{Y^*}$ by $y^*[(z_\lambda)_{\lambda\in\Lambda}] = y_\kappa^*(z_\kappa)$ for every $(z_\lambda)_{\lambda\in\Lambda} \in X$, we clearly have $y^* \in \operatorname{Slice}(B_{Y^*}, y, \varepsilon)$. Observe that, fixed $\widetilde{x}_\kappa \in \operatorname{gSlice}(S_{X_\kappa}, G_\kappa^* y_\kappa^*, \varepsilon)$ and $\theta \in \mathbb{T}$, the element $(z_\lambda)_{\lambda\in\Lambda} \in B_X$ given by $z_\lambda = \overline{\theta} x_\lambda$ for $\lambda \neq \kappa$ and $z_\kappa = \widetilde{x}_\kappa$ belongs to $\operatorname{gSlice}(S_X, G^* y^*, \varepsilon)$. Using this it is easy to deduce that

$$\operatorname{dist}\big((x_\lambda)_{\lambda\in\Lambda}, \operatorname{aconv}(\operatorname{gSlice}(S_X, G^* y^*, \varepsilon))\big) < \varepsilon,$$

which tells us that G is lush by Proposition 3.28.iii.

Suppose that $E = \ell_1$. We take the set $\mathscr{B} = \big\{(x_\lambda)_{\lambda\in\Lambda} \in B_X : \#\operatorname{supp}(x_\lambda) = 1\big\}$ which is norming for X^*. Fixed $(x_\lambda)_{\lambda\in\Lambda} \in \mathscr{B}$, there is $\kappa \in \Lambda$ so that $x_\kappa \in B_{X_\kappa}$ and $x_\lambda = 0$ for $\lambda \neq \kappa$. Given $(y_\lambda)_{\lambda\in\Lambda} \in S_Y$ and $\varepsilon > 0$, we may and do assume that $y_\kappa \neq 0$ and, since G_κ is lush, we may use Proposition 3.28.iii for $x_\kappa \in B_{X_\kappa}$ and $\frac{y_\kappa}{\|y_\kappa\|} \in S_{Y_\kappa}$ to find $y_\kappa^* \in \operatorname{Slice}(B_{Y_\kappa^*}, \frac{y_\kappa}{\|y_\kappa\|}, \varepsilon)$ such that

$$\operatorname{dist}\big(x_\kappa, \operatorname{aconv}(\operatorname{gSlice}(S_{X_\kappa}, G_\kappa^* y_\kappa^*, \varepsilon))\big) < \varepsilon.$$

For each $\lambda \in \operatorname{supp}(y_\lambda) \setminus \{\kappa\}$ we take $y_\lambda^* \in S_{Y_\lambda^*}$ satisfying $y_\lambda^*(y_\lambda) = \|y_\lambda\|$ and we define $y^* \in S_{Y^*}$ by

$$y^*\big[(w_\lambda)_{\lambda\in\Lambda}\big] = \sum_{\lambda\in\operatorname{supp}(y_\lambda)} y_\lambda^*(w_\lambda) \qquad \big((w_\lambda) \in Y\big).$$

Then it obviously follows that $y^* \in \operatorname{Slice}(B_{Y^*}, (y_\lambda), \varepsilon)$ and, thanks to the shape of $(x_\lambda)_{\lambda\in\Lambda}$, one can easily deduce that

$$\operatorname{dist}\big((x_\lambda)_{\lambda\in\Lambda}, \operatorname{aconv}(\operatorname{gSlice}(S_X, G^* y^*, \varepsilon))\big) < \varepsilon.$$

This finishes the proof by using Proposition 3.28.iii since $\overline{\text{aconv}}(\mathscr{B}) = B_X$. □

8.3 Vector-valued function spaces

Our next aim is to present several results concerning the behaviour of our properties for vector-valued function spaces. We start analysing the situation for spaces of continuous functions.

Theorem 8.5. *Let X,Y be Banach spaces, let K be a compact Hausdorff topological space and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Consider the norm-one composition operator $\widetilde{G}\colon C(K,X) \longrightarrow C(K,Y)$ given by $\widetilde{G}(f) = G \circ f$ for every $f \in C(K,X)$. Then:*

(a) $\widetilde{G}$ is a spear operator if and only if G is a spear operator.
(b) $\widetilde{G}$ is lush if and only if G is lush.
(c) If K contains isolated points, then $\widetilde{G}$ has the aDP if and only if G does.
(d) If K is perfect, then $\widetilde{G}$ has the aDP if and only if $G(B_X)$ is a spear set.

Remark 8.6. All the information given in the above result was previously known for the case of the identity (see [64, 97, 98]).

Previously to present its proof, we use Theorem 8.5 to produce an example of an operator with the aDP which does not attain its norm. Recall that it was proved in Proposition 6.8 that this cannot happen if the operator is actually lush.

Example 8.7. Let X be a non-reflexive Banach space, let $G: X \longrightarrow \mathbb{K}$ be a norm-one functional which does not attain its norm, and let K be a perfect compact Hausdorff topological space. Then, the operator $\widetilde{G} \in \mathcal{L}(C(K,X),C(K))$ defined by $\widetilde{G}(f) = G \circ f$ for every $f \in C(K,X)$, has the aDP and it does not attain its norm.

Proof. As $\|G\| = 1$, $G(B_X)$ contains the open unit ball of $\mathbb{K}$, which is clearly a spear set, and so Theorem 8.5.d gives us that $\widetilde{G}$ has the aDP. On the other hand, as G does not attains its norm, for every $f \in C(K,X)$ with $\|f\| = 1$, it follows that

$$\left|[\widetilde{G}(f)](t)\right| = |G(f(t))| < \|f(t)\| \leq 1$$

for every $t \in K$, so $\widetilde{G}$ does not attain its norm, as claimed. □

Observe that G in the example above is not a spear operator (use Theorem 2.9), so neither is $\widetilde{G}$ by Theorem 8.5.a. Actually, the operator G does not have the aDP by Corollary 5.9.

Let us now present the proof of Theorem 8.5 which, for the reader convenience, will be divided in four parts, one for each item (a), (b), (c), and (d).

Proof (of Theorem 8.5.a). This is an easy adaptation of [98, Theorem 5]. Suppose first that G is a spear operator. Fixed $T \in \mathcal{L}(C(K,X),C(K,Y))$ with $\|T\| = 1$ and $\varepsilon > 0$, find $f_0 \in C(K,X)$ with $\|f_0\| = 1$ and $t_0 \in K$ such that

$$\|[T f_0](t_0)\| > 1 - \varepsilon. \tag{8.3}$$

Define $z_0 = f_0(t_0)$ and find a continuous function $\varphi\colon K \longrightarrow [0,1]$ such that $\varphi(t_0) = 1$ and $\varphi(t) = 0$ if $\|f_0(t) - z_0\| \geqslant \varepsilon$. Now write $z_0 = \lambda x_1 + (1-\lambda)x_2$ with $0 \leqslant \lambda \leqslant 1$, $x_1, x_2 \in S_X$, and consider the functions

$$f_j = (1-\varphi)f_0 + \varphi x_j \in C(K,X) \qquad (j = 1,2).$$

Then $\|\varphi f_0 - \varphi z_0\| < \varepsilon$ meaning that

$$\|f_0 - (\lambda f_1 + (1-\lambda) f_2)\| < \varepsilon,$$

and, using (8.3), we must have

$$\|[T f_1](t_0)\| > 1 - 2\varepsilon \qquad \text{or} \qquad \|[T f_2](t_0)\| > 1 - 2\varepsilon.$$

By making the right choice of $x_0 = x_1$ or $x_0 = x_2$ we get $x_0 \in S_X$ such that

$$\|\left[T\left((1-\varphi)f_0 + \varphi x_0\right)\right](t_0)\| > 1 - 2\varepsilon. \tag{8.4}$$

Next, we fix $x_0^* \in S_{X^*}$ with $x_0^*(x_0) = 1$, denote

$$\Phi(x) = x_0^*(x)(1-\varphi)f_0 + \varphi x \in C(K,X) \qquad (x \in X),$$

and consider the operator $S \in \mathcal{L}(X,Y)$ given by

$$Sx = [T(\Phi(x))](t_0) \qquad (x \in X)$$

which, by (8.4), obviously satisfies $\|S\| \geqslant \|Sx_0\| > 1 - 2\varepsilon$. Now, we use that G is a spear operator to find $x \in S_X$ satisfying $\|Gx + \mathbb{T}Sx\| > 1 + \|S\| - \varepsilon$, and observe that

$$\|\widetilde{G} + \mathbb{T}T\| \geqslant \left\|\left[(\widetilde{G} + \mathbb{T}T)(\Phi(x))\right](t_0)\right\| = \|Gx + \mathbb{T}Sx\| > 1 + \|S\| - \varepsilon > 2 - 3\varepsilon.$$

The arbitrariness of ε gives that $\|\widetilde{G} + \mathbb{T}T\| \geqslant 2$ and so $\widetilde{G}$ is a spear operator.

Suppose conversely that $\widetilde{G}$ is a spear operator. Fix $S \in \mathcal{L}(X,Y)$, $\varepsilon > 0$ and define the operator $T \in \mathcal{L}\big(C(K,X),C(K,Y)\big)$ by

$$[T(f)](t) = S(f(t)) \qquad (t \in K, f \in C(K,X))$$

which satisfies $\|T\| = \|S\|$. Since $\widetilde{G}$ is a spear operator we may find $f_0 \in C(K,X)$ and $t_0 \in K$ such that

$$\left\|\left[(\widetilde{G} + \mathbb{T}T)(f_0)\right](t_0)\right\| > 1 + \|T\| - \varepsilon$$

and we can write

$$1+\|S\|-\varepsilon = 1+\|T\|-\varepsilon < \left\| \left[(\widetilde{G}+\mathbb{T}T)(f_0) \right] (t_0) \right\| \\ = \|G(f_0(t_0))+\mathbb{T}S(f_0(t_0))\| \leqslant \|G+\mathbb{T}S\|.$$

The arbitrariness of ε tells us that G is a spear operator. □

We next deal with lushness for spaces of vector-valued continuous functions.

Proof (of Theorem 8.5.b). Suppose that G is lush and let us show that $\widetilde{G}$ is lush. This part of the proof is an easy adaptation of [64, Proposition 5.1]. Let $f \in S_{C(K,Y)}$, $g \in S_{C(K,X)}$, and $\varepsilon > 0$ be fixed. Then, we take $t_0 \in K$ with $\|f(t_0)\| = 1$ and, using that G is lush together with Proposition 3.28.iii, we find $y^* \in \operatorname{Slice}(B_{Y^*}, f(t_0), \varepsilon)$ such that

$$\operatorname{dist}\big(g(t_0), \operatorname{aconv}(\operatorname{gSlice}(S_X, G^*y^*, \varepsilon))\big) < \frac{\varepsilon}{2}.$$

So, there are $\theta_1, \ldots, \theta_n \in \mathbb{T}$, $\lambda_1, \ldots, \lambda_n \in [0,1]$ with $\sum_{k=1}^n \lambda_k = 1$, and $x_1, \ldots, x_n \in \operatorname{gSlice}(S_X, G^*y^*, \varepsilon)$ such that

$$\left\| g(t_0) - \sum_{k=1}^n \lambda_k \theta_k x_k \right\| < \frac{\varepsilon}{2}.$$

Next, we take an open set $U \subset K$ such that $t_0 \in U$ and

$$\|g(t)-g(t_0)\| < \frac{\varepsilon}{2} \qquad (t \in U),$$

and we fix a continuous function $\varphi\colon K \longrightarrow [0,1]$ with $\varphi(t_0)=0$ and $\varphi|_{K\setminus U} \equiv 1$. Now we consider the functional $\xi^* \in B_{C(K,Y)^*}$ given by $\xi^*(h) = y^*(h(t_0))$ for $h \in C(K,Y)$ which clearly satisfies $\xi^* \in \operatorname{Slice}(B_{C(K,Y)^*}, f, \varepsilon)$. Finally, for each $k = 1, \ldots, n$, we define $g_k \in C(K,X)$ by

$$g_k(t) = x_k + \varphi(t)(\theta_k^{-1} g(t) - x_k) \qquad (t \in K)$$

and we observe that

$$\widetilde{G}^*\xi^*(g_k) = \xi^*(G \circ g_k) = y^*(G(g_k(t_0))) = G^*y^*(x_k).$$

Therefore, we deduce that $g_k \in \operatorname{gSlice}(S_{C(K,X)}, \widetilde{G}^*\xi^*, \varepsilon)$ for every $k = 1, \ldots, n$. On the other hand, for an arbitrary $t \in K$ we have that

$$\left\| g(t) - \sum_{k=1}^n \lambda_k \theta_k g_k(t) \right\| = \left\| (1-\varphi(t)) \left(g(t) - \sum_{k=1}^n \lambda_k \theta_k x_k \right) \right\|.$$

So, if $t \in U$, then $\|g(t)-g(t_0)\| \leqslant \frac{\varepsilon}{2}$ and, therefore,

$$\left\| g(t) - \sum_{k=1}^{n} \lambda_k \theta_k g_k(t) \right\| \leqslant \|g(t) - g(t_0)\| + \left\| g(t_0) - \sum_{k=1}^{n} \lambda_k \theta_k x_k \right\| \leqslant \varepsilon.$$

If, otherwise, $t \notin U$, then $\varphi(t) = 1$ and thus $g(t) - \sum_{k=1}^{n} \lambda_k \theta_k g_k(t) = 0$. All this tells us that

$$\operatorname{dist}\big(g, \operatorname{aconv}(\operatorname{gSlice}(S_{C(K,X)}, \widetilde{G}^* \xi^*, \varepsilon))\big) < \varepsilon$$

and shows that $\widetilde{G}$ is lush.

Suppose now that $\widetilde{G}$ is lush and let us show that G is lush. To do so, we will use Proposition 3.28.iii with the set

$$\mathscr{A} = \{y^* \otimes \delta_t : y^* \in S_{Y^*}, t \in K\}$$

where $[y^* \otimes \delta_t](f) = y^*(f(t))$ for $f \in C(K,Y)$. Observe that $\mathscr{A}$ is norming and rounded, so $\overline{\operatorname{conv}}^{w^*}(\mathscr{A}) = B_{C(K,Y)^*}$. Fixed $x_0 \in S_X$, $y_0 \in S_Y$, and $\varepsilon > 0$, we consider $f \in S_{C(K,Y)}$ and $g \in S_{C(K,X)}$ given respectively by

$$f(t) = y_0 \qquad \text{and} \qquad g(t) = x_0 \qquad (t \in K).$$

Now we use that $\widetilde{G}$ is lush and Proposition 3.28.iii to find $y_0^* \otimes \delta_{t_0} \in \operatorname{Slice}(\mathscr{A}, f, \varepsilon)$ such that

$$\operatorname{dist}\big(g, \operatorname{aconv}(\operatorname{gSlice}(S_{C(K,X)}, \widetilde{G}^*(y_0^* \otimes \delta_{t_0}), \varepsilon))\big) < \varepsilon.$$

Therefore, as $f(t_0) = y_0$, we clearly get that $y_0^* \in \operatorname{Slice}(B_{Y^*}, y_0, \varepsilon)$. Moreover, using that if $h \in \operatorname{gSlice}(S_{C(K,X)}, \widetilde{G}^*(y_0^* \otimes \delta_{t_0}), \varepsilon)$ then $h(t_0) \in \operatorname{gSlice}(S_X, G^* y_0^*, \varepsilon)$, we easily deduce that

$$\operatorname{dist}\big(x_0, \operatorname{aconv}(\operatorname{gSlice}(S_X, G^* y_0^*, \varepsilon))\big) < \varepsilon$$

which gives that G is lush. □

Proof (of Theorem 8.5.c). We start showing that $\widetilde{G}$ has the aDP when G does. Indeed, observe that in the first part of the proof of Theorem 8.5.a, if the operator T has rank one then so does the operator $S \in \mathcal{L}(X,Y)$ constructed there.

To prove the reversed implication, fix an isolated point $t_0 \in K$ and observe that we can identify $C(K,X) \equiv X \oplus_\infty C(K \setminus \{t_0\}, X)$ and $C(K,Y) \equiv Y \oplus_\infty C(K \setminus \{t_0\}, Y)$. Now, if for $g \in C(K \setminus \{t_0\}, X)$ we write

$$\widehat{g}(t) = \begin{cases} 0 & \text{if } t = t_0 \\ g(t) & \text{if } t \neq t_0 \end{cases} \qquad (t \in K),$$

and we consider the operator $\widehat{G}\colon C(K \setminus \{t_0\}, X) \longrightarrow C(K \setminus \{t_0\}, Y)$ given by

$$\widehat{G}(g) = [\widetilde{G}(\widehat{g})]|_{K\setminus\{t_0\}} \qquad \big(g \in C(K \setminus \{t_0\}, X)\big)$$

then, we can write

$$\widetilde{G}(x,g) = (Gx, \widehat{G}(g)) \qquad \big(x \in X, g \in C(K \setminus \{t_0\}, X)\big).$$

Therefore, as $\widetilde{G}$ has the aDP, Proposition 8.4 gives us that G has aDP. □

Proof (of Theorem 8.5.d). We prove first the sufficiency. We will use Theorem 3.6.iii to show that $\widetilde{G}$ has the aDP. So, fixed $f \in S_{C(K,Y)}$ and $\varepsilon > 0$, we write

$$\Delta_\varepsilon(f) = \{g \in B_{C(K,X)} \colon \|\widetilde{G}(g) + \mathbb{T}f\| > 2 - \varepsilon\}$$

and we have to show that $\overline{\mathrm{conv}}(\Delta_\varepsilon(f)) = B_X$. The argument follows the lines of [125, p. 81]: let U be the open set $\{t \in K \colon \|f(t)\| > 1 - \varepsilon/2\}$ and pick, given $n \in \mathbb{N}$, open pairwise disjoint non-void subsets $U_1, \ldots, U_n \subset U$ and points $t_j \in U_j$. Next, we use the hypothesis to find $x_j \in B_X$ and $\theta_j \in \mathbb{T}$ such that $\|G(x_j) + \theta_j f(t_j)\| > 2 - \varepsilon$. Now, fixed $h \in B_{C(K,X)}$, we may choose functions $g_j \in B_{C(K,X)}$ such that $g_j \equiv h$ in $K \setminus U_j$ and $g_j(t_j) = x_j$. Indeed, take Urysohn functions $\varphi_j \colon K \longrightarrow [0,1]$ such that

$$\varphi_j|_{K\setminus U_j} \equiv 1 \quad \text{and} \quad \varphi_j(t_j) = 0 \qquad (j = 1, \ldots, n),$$

and define

$$g_j(t) = \varphi_j(t)h(t) + (1 - \varphi_j(t))x_j \qquad (t \in K,\ j = 1, \ldots, n).$$

On the one hand, observe that $g_j \in \Delta_\varepsilon(f)$:

$$\|\widetilde{G}(g_j) + \theta_j f\| \geqslant \|G(g_j(t_j)) + \theta_j f(t_j)\| = \|G(x_j) + \theta_j f(t_j)\| > 2 - \varepsilon.$$

On the other hand, for $t \in U_k$ we have that

$$\left\| h(t) - \frac{1}{n}\sum_{j=1}^{n} g_j(t) \right\| = \left\| h(t) - \frac{n-1}{n}h(t) - \frac{1}{n}g_k(t) \right\| = \frac{1}{n}\|h(t) - g_k(t)\| \leqslant \frac{2}{n};$$

and, for $t \notin \bigcup_j U_j$, it follows that $h(t) - \frac{1}{n}\sum_{j=1}^{n} g_j(t) = 0$. This proves that $h \in \overline{\mathrm{conv}}(\Delta_\varepsilon(f))$ and so $\widetilde{G}$ has the aDP.

Suppose now that $\widetilde{G}$ has the aDP. Fixed $\varepsilon > 0$ and a non-zero $y \in B_Y$, we take the constant function $f \in C(K,Y)$ given by $f \equiv \frac{y}{\|y\|}$ and we use Theorem 3.6.iii to find $g \in B_{C(K,X)}$ such that $\|\widetilde{G}(g) + \mathbb{T}f\| > 2 - \varepsilon$. So, there is $t_0 \in K$ satisfying $\|G(g(t_0)) + \mathbb{T}\frac{y}{\|y\|}\| > 2 - \varepsilon$ and we claim that $g(t_0)$ is the element in B_X we are looking for; indeed,

$$\begin{aligned}\|G(g(t_0)) + \mathbb{T}y\| &\geqslant \left\| G(g(t_0)) + \mathbb{T}\frac{y}{\|y\|} \right\| - \left\| \frac{y}{\|y\|} - y \right\| \\ &> 2 - \varepsilon - (1 - \|y\|) = 1 + \|y\| - \varepsilon,\end{aligned}$$

as desired. □

We next deal with spaces of essentially bounded measurable functions.

Theorem 8.8. *Let X,Y be a Banach spaces, let (Ω,Σ,μ) be a σ-finite measure space and let $G\in\mathcal{L}(X,Y)$ be a norm-one operator. Consider the norm-one composition operator $\widetilde{G}\colon L_\infty(\mu,X)\longrightarrow L_\infty(\mu,Y)$ given by $\widetilde{G}(f)=G\circ f$ for every $f\in L_\infty(\mu,X)$. Then:*

(a) $\widetilde{G}$ is a spear operator if and only if G is a spear operator.
(b) $\widetilde{G}$ is lush if and only if G is lush.
(c) If μ has an atom, then $\widetilde{G}$ has the aDP if and only if G does.
(d) If μ is atomless, then $\widetilde{G}$ has the aDP if and only if $G(B_X)$ is a spear set.

Remark 8.9. The results in items (a), (c), and (d) of the above theorem were known for the case of the identity (see [97, 100]). The content of (b) is completely new even for the identity.

Corollary 8.10. *Let X be a Banach space and let (Ω,Σ,μ) be a σ-finite measure space. Then, $L_\infty(\mu,X)$ is lush if and only if X is lush.*

We will use the following notation:

$$\Sigma^+_{\text{fin}}:=\{A\in\Sigma\colon 0<\mu(A)<+\infty\}.$$

Observe that when the measure is finite, Σ^+_{fin} equals the family Σ^+ given in section 4.3. Moreover, we will also use the subset of $S_{L_1(\mu,Y^*)}$ given by

$$\mathscr{A}:=\left\{y^*\frac{\mathbb{1}_A}{\mu(A)}\colon y^*\in S_{Y^*}, A\in\Sigma^+_{\text{fin}}\right\}$$

which clearly satisfies that

$$\overline{\operatorname{conv}}^{w^*}(\mathscr{A})=B_{L_\infty(\mu,Y)^*}.$$

Proof (of Theorem 8.8.a). This is an easy adaptation of [100, Theorem 2.3]. Suppose first that G is a spear operator. We fix $T\in\mathcal{L}\big(L_\infty(\mu,X),L_\infty(\mu,Y)\big)$ with $\|T\|=1$. Given $\varepsilon>0$ we may follow the first part of the proof of [100, Theorem 2.3] to find $f\in S_{L_\infty(\mu,X)}$, $x_0\in S_X$, and $A,B\in\Sigma$ with $0<\mu(B)<\infty$, such that

$$B\subset A \qquad\text{and}\qquad \left\|\frac{1}{\mu(B)}\int_B T(x_0\mathbb{1}_A+f\mathbb{1}_{\Omega\setminus A})\,d\mu\right\|>1-\varepsilon. \tag{8.5}$$

Now we fix $x_0^*\in S_{X^*}$ with $x_0^*(x_0)=1$, we write

$$\Phi(x)=x_0\mathbb{1}_A+x_0^*(x)f\mathbb{1}_{\Omega\setminus A}\qquad (x\in X)$$

and we define the operator $S\in\mathcal{L}(X,Y)$ given by

$$S(x) = \frac{1}{\mu(B)} \int_B T(\Phi(x))d\mu \qquad (x \in X)$$

which, by (8.5), satisfies $\|S\| \geqslant \|Sx_0\| > 1-\varepsilon$. Next, we use that G is a spear operator to find $x \in S_X$ such that $\|Gx + \mathbb{T}Sx\| > 2-\varepsilon$, so we can take $y^* \in S_{Y^*}$ satisfying

$$|y^*(Gx + \mathbb{T}Sx)| > 2-\varepsilon.$$

Finally, define the functional $g^* \in S_{L_\infty(\mu,Y)^*}$ by

$$g^*(h) = y^*\left(\frac{1}{\mu(B)} \int_B h\, d\mu\right) \qquad (h \in L_\infty(\mu,Y))$$

and observe that

$$\begin{aligned}\|\widetilde{G} + \mathbb{T}T\| &\geqslant \left|g^*\left(\widetilde{G}(\Phi(x)) + \mathbb{T}T(\Phi(x))\right)\right| \\ &= \left|g^*(G(x)\mathbb{1}_A) + \mathbb{T}g^*\left(T(\Phi(x))\right)\right| = |y^*(G(x)) + \mathbb{T}y^*(S(x))| > 2-\varepsilon.\end{aligned}$$

The arbitrariness of ε gives that $\widetilde{G}$ is a spear operator.

Assume now that $\widetilde{G}$ is a spear operator. Fix $S \in \mathcal{L}(X,Y)$ and define the operator $T \in \mathcal{L}\big(L_\infty(\mu,X), L_\infty(\mu,Y)\big)$ by

$$[T(f)](t) = S(f(t)) \qquad (t \in \Omega, f \in L_\infty(\mu,X))$$

which clearly satisfies $\|T\| = \|S\|$. As we mentioned at the beginning of the proof, we can find $f \in S_{L_\infty(\mu,X)}$, $x_0 \in S_X$, and $A, B \in \Sigma$ with $0 < \mu(B) < \infty$, such that $B \subset A$ and

$$\begin{aligned}\left\|\frac{1}{\mu(B)} \int_B (\widetilde{G} + \mathbb{T}T)(x_0\mathbb{1}_A + f\mathbb{1}_{\Omega\setminus A})d\mu\right\| &> \|\widetilde{G} + \mathbb{T}T\| - \varepsilon \\ &= 1 + \|T\| - \varepsilon = 1 + \|S\| - \varepsilon.\end{aligned}$$

Therefore, we can write

$$\begin{aligned}1 + \|S\| - \varepsilon &\leqslant \left\|\frac{1}{\mu(B)} \int_B (\widetilde{G} + \mathbb{T}T)(x_0\mathbb{1}_A + f\mathbb{1}_{\Omega\setminus A})d\mu\right\| \\ &= \left\|\frac{1}{\mu(B)} \int_B G(x_0)\mathbb{1}_A + \mathbb{T}T(x_0\mathbb{1}_A + f\mathbb{1}_{\Omega\setminus A})d\mu\right\| \\ &= \left\|G(x_0) + \mathbb{T}\frac{1}{\mu(B)} \int_B T(x_0\mathbb{1}_A + f\mathbb{1}_{\Omega\setminus A})d\mu\right\| \\ &= \left\|G(x_0) + \mathbb{T}S\left(\frac{1}{\mu(B)} \int_B x_0\mathbb{1}_A + f\mathbb{1}_{\Omega\setminus A}d\mu\right)\right\| = \|G(x_0) + \mathbb{T}S(x_0)\|.\end{aligned}$$

Thus, we get $\|G + \mathbb{T}S\| > 1 + \|S\| - \varepsilon$ and so G is a spear operator. □

We next deal with lushness for $L_\infty(\mu,X)$.

Proof (of Theorem 8.8.b). Assume first that G is lush. To prove that so is $\widetilde{G}$, we will check that Proposition 3.28.iii is satisfied. Let $f_0\in S_{L_\infty(\mu,X)}$, $g_0\in S_{L_\infty(\mu,Y)}$ and $\varepsilon>0$. By density, we can assume that f_0,g_0 can be written as

$$f_0=\sum_{A\in\pi}x_A\mathbb{1}_A,\qquad g_0=\sum_{A\in\pi}y_A\mathbb{1}_A$$

where $\pi\subset\Sigma^+_{\text{fin}}$ is a countable partition of Ω and $x_A\in B_X$, $y_A\in B_Y$ for each $A\in\pi$. Since $\|g_0\|_{L_\infty(\mu,Y)}=1$, we can assume without loss of generality that there is $A_0\in\pi$ with $\|y_{A_0}\|=1$. Using that G is lush, we can find $y_0^*\in S_{Y^*}$ such that

$$\operatorname{Re}y_0^*(y_{A_0})>1-\varepsilon \tag{8.6}$$

and elements $x_j\in\operatorname{gSlice}(S_X,G^*y_0^*,\varepsilon)$, $\theta_j\in\mathbb{T}$, $\lambda_j>0$ for $j=1,\dots,m$ ($m\in\mathbb{N}$) with $\sum_j\lambda_j=1$ satisfying that

$$\left\|x_{A_0}-\sum_{j=1}^m\lambda_j\theta_jx_j\right\|<\varepsilon. \tag{8.7}$$

Consider now $h_0^*:=y_0^*\mathbb{1}_{A_0}/\mu(A_0)\in\mathscr{A}$, and for each $j=1,\dots,m$ let

$$f_j:=\sum_{A\in\pi,A\neq A_0}\overline{\theta_j}x_A\mathbb{1}_A+x_j\mathbb{1}_{A_0}.$$

Then, by (8.6) we have that

$$\operatorname{Re}h_0^*(g_0)=\operatorname{Re}y_0^*(y_{A_0})>1-\varepsilon.$$

A similar argument shows that

$$f_j\in\operatorname{gSlice}(S_{L_\infty(\mu,X)},\widetilde{G}^*h^*,\varepsilon)$$

for every $j=1,\dots,m$, since $\operatorname{Re}G^*y_0^*(x_j)>1-\varepsilon$. Moreover, using (8.7) we immediately conclude that

$$\left\|f_0-\sum_{j=1}^m\lambda_j\theta_jf_j\right\|=\max\left\{\sup_{\substack{A\in\pi\\A\neq A_0}}\left\|x_A-\sum_{j=1}^m\lambda_jx_A\right\|,\left\|x_{A_0}-\sum_{j=1}^m\lambda_j\theta_jx_j\right\|\right\}<\varepsilon.$$

Let us see the converse: assume that $\widetilde{G}$ is lush, fix an element $B\in\Sigma^+_{\text{fin}}$, and let $x\in S_X$, $y\in S_Y$ and $\varepsilon>0$. Then, defining $f:=x\mathbb{1}_B\in S_{L_\infty(\mu,X)}$ and $g:=y\mathbb{1}_B\in S_{L_\infty(\mu,Y)}$, we can use the hypothesis to find $h^*\in\operatorname{Slice}(\mathscr{A},g,\varepsilon)$ such that

$$\operatorname{dist}\Big(f,\operatorname{aconv}\big(\operatorname{gSlice}(B_{L_\infty}(\mu,X),\widetilde{G}^*h^*,\varepsilon)\big)\Big)<\varepsilon. \tag{8.8}$$

Since we can write $h^* = y_0^* \mathbb{1}_A / \mu(A)$ for some $A \in \Sigma_{\text{fin}}^+$ and $y_0^* \in S_{Y^*}$, the condition $h^* \in \operatorname{Slice}(\mathscr{A}, g, \varepsilon)$ can be rewritten as

$$\operatorname{Re} y_0^*(y_0) \frac{\mu(A \cap B)}{\mu(A)} > 1 - \varepsilon. \tag{8.9}$$

By (8.8) we find elements $f_j \in \operatorname{gSlice}(B_{L_\infty(\mu,X)}, \widetilde{G}^* h^*, \varepsilon)$, $\theta_j \in \mathbb{T}$ and $\lambda_j > 0$ for $j = 1, \ldots, m$ satisfying $\sum_{j=1}^m \lambda_j = 1$ and

$$\left\| f - \sum_{j=1}^m \lambda_j \theta_j f_j \right\| < \varepsilon. \tag{8.10}$$

Next, for each $j = 1, \ldots, m$, we consider the element

$$x_j := \frac{1}{\mu(A)} \int_A f_j \, d\mu \in B_X$$

which clearly satisfies that

$$\operatorname{Re} G^* y_0^*(x_j) = \frac{1}{\mu(A)} \int_A \operatorname{Re} G^* y_0^*(f_j) \, d\mu = \operatorname{Re} \widetilde{G}^* h^*(f_j) > 1 - \varepsilon.$$

Moreover, making use of (8.9) we get that

$$\left\| x - \frac{1}{\mu(A)} \int_A f \, d\mu \right\| = \left\| x - \frac{\mu(A \cap B)}{\mu(A)} x \right\| < \varepsilon$$

and, combining this with (8.10), we conclude that

$$\begin{aligned} \left\| x - \sum_{j=1}^m \lambda_j \theta_j x_j \right\| &\leqslant \varepsilon + \left\| \frac{1}{\mu(A)} \int_A f \, d\mu - \sum_{j=1}^m \lambda_j \theta_j \frac{1}{\mu(A)} \int_A f_j \, d\mu \right\| \\ &\leqslant \varepsilon + \frac{1}{\mu(A)} \int_A \left\| f - \sum_{j=1}^m \lambda_j \theta_j f_j \right\| d\mu < 2\varepsilon, \end{aligned}$$

which finishes the proof. □

Proof (of Theorem 8.8.c). We start showing that when G has the aDP so does $\widetilde{G}$. Indeed, observe that in the first part of the proof of Theorem 8.8.a, if the operator T has rank one then the operator $S \in \mathcal{L}(X, Y)$ constructed there also has rank one.

To prove the reversed implication, fix $A_0 \in \Sigma$ which is an atom for μ and observe that we can identify

$$L_\infty((\Omega, \mu), X) \equiv X \oplus_\infty L_\infty((\Omega \setminus A_0, \mu), X)$$

and

$$L_\infty((\Omega,\mu),Y) \equiv Y \oplus_\infty L_\infty((\Omega\setminus A_0,\mu),Y).$$

Now, if for $g \in L_\infty((\Omega\setminus A_0,\mu),X)$ we write

$$\widehat{g}(t) = \begin{cases} 0 & \text{if } t \in A_0 \\ g(t) & \text{if } t \notin A_0 \end{cases} \qquad (t \in \Omega),$$

and we consider the operator $\widehat{G}\colon L_\infty((\Omega\setminus A_0,\mu),X) \longrightarrow L_\infty((\Omega\setminus A_0,\mu),Y)$ given by

$$\widehat{G}(g) = [\widetilde{G}(\widehat{g})]|_{\Omega\setminus A_0} \qquad \big(g \in L_\infty((\Omega\setminus A_0,\mu),X)\big),$$

then we can write

$$\widetilde{G}(x,g) = (Gx, \widehat{G}(g)) \qquad \big(x \in X, g \in L_\infty((\Omega\setminus A_0,\mu),X)\big).$$

Therefore, as $\widetilde{G}$ has the aDP, we may use Proposition 8.4 to deduce that G has the aDP. □

Proof (of Theorem 8.8.d). We prove first the sufficiency. We will use Theorem 3.6.iii to show that $\widetilde{G}$ has the aDP. So, fixed $f \in S_{L_\infty(\mu,Y)}$ and $\varepsilon > 0$, we write

$$\Delta_\varepsilon(f) = \left\{ g \in B_{L_\infty(\mu,X)} \colon \|\widetilde{G}(g) + \mathbb{T}f\| > 2-\varepsilon \right\}$$

and we have to show that $\overline{\text{conv}}(\Delta_\varepsilon(f)) = B_{L_\infty(\mu,X)}$. Using Lemma 2.2 in [100], we may find $y \in S_Y$ and $A \in \Sigma^+_{\text{fin}}$ such that

$$\|f - (y\mathbb{1}_A + f\mathbb{1}_{\Omega\setminus A})\| < \frac{\varepsilon}{2}.$$

As μ is atomless, for every $n \in \mathbb{N}$ we may and do pick pairwise disjoint sets $U_1,\dots,U_n$ with positive measure such that $U_i \subset A$ for every $i = 1,\dots,n$. Next, we use that $G(B_X)$ is a spear set to find $x \in B_X$ and $\theta \in \mathbb{T}$ such that $\|G(x)+\theta y\| > 2-\varepsilon/2$. Now, fixed $h \in B_{L_\infty(\mu,X)}$, we define $g_j = x\mathbb{1}_{U_j} + h\mathbb{1}_{K\setminus U_j} \in B_{L_\infty(\mu,X)}$ for $j = 1,\dots,n$. On the one hand, observe that for every $t \in U_j$ we get the following estimation

$$\|G(g_j(t)) + \theta f(t)\| \geqslant \|G(x)+\theta y\| - \|f(t)-y\| > 2-\varepsilon,$$

so $\|\widetilde{G}(g_j)+\theta f\| \geqslant 2-\varepsilon$ which implies that $g_j \in \Delta_\varepsilon(f)$. On the other hand, for $t \in U_k$ we have that

$$\left\| h(t) - \frac{1}{n}\sum_{j=1}^n g_j(t) \right\| = \left\| h(t) - \frac{n-1}{n}h(t) - \frac{1}{n}g_k(t) \right\| = \frac{1}{n}\|h(t)-g_k(t)\| \leqslant \frac{2}{n};$$

and, for $t \notin \bigcup_j U_j$, it follows that $h(t) - \frac{1}{n}\sum_{j=1}^n g_j(t) = 0$. This proves that $h \in \overline{\text{conv}}(\Delta_\varepsilon(f))$ and so $\widetilde{G}$ has the aDP.

Conversely, suppose now that $\widetilde{G}$ has the aDP. Fixed $\varepsilon > 0$ and a non-zero $y \in B_Y$, we take the constant function $f \in L_\infty(\mu, Y)$ given by $f \equiv \frac{y}{\|y\|}$ and we use Theorem 3.6.iii to find $g \in B_{L_\infty(\mu,X)}$ such that $\|\widetilde{G}(g) + \mathbb{T} f\| > 2 - \varepsilon$. So, there is $t_0 \in \Omega$ satisfying that $\left\| G(g(t_0)) + \mathbb{T}\frac{y}{\|y\|} \right\| > 2 - \varepsilon$ and, therefore,

$$\begin{aligned}\|G(g(t_0)) + \mathbb{T}y\| &\geqslant \left\| G(g(t_0)) + \mathbb{T}\frac{y}{\|y\|} \right\| - \left\| \frac{y}{\|y\|} - y \right\| \\ &> 2 - \varepsilon - (1 - \|y\|) = 1 + \|y\| - \varepsilon.\end{aligned}$$

This shows that $G(B_X)$ is a spear set, concluding thus the proof. □

Finally, we would like to work with spaces of integrable functions.

Theorem 8.11. *Let X, Y be Banach spaces, let (Ω, Σ, μ) be a σ-finite measure space, and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. Consider the norm-one composition operator $\widetilde{G}\colon L_1(\mu, X) \longrightarrow L_1(\mu, Y)$ given by $\widetilde{G}(f) = G \circ f$ for every $f \in L_1(\mu, X)$. Then:*

(a) $\widetilde{G}$ is a spear operator if and only if G is a spear operator.
(b) $\widetilde{G}$ is lush if and only if G is lush.
(c) If μ has an atom, then $\widetilde{G}$ has the aDP if and only if G has the aDP.
(d) If μ is atomless, then $\widetilde{G}$ has the aDP if and only if

$$B_X = \overline{\operatorname{aconv}}\{x \in B_X : \|Gx\| > 1 - \varepsilon\} \text{ for every } \varepsilon > 0.$$

Remark 8.12. The results in items (a), (c), and (d) of the above theorem were known for the case of the identity (see [97, 98]). The content of (b) is completely new even for the identity.

Corollary 8.13. *Let X be a Banach space and let (Ω, Σ, μ) be a σ-finite measure space. Then, $L_1(\mu, X)$ is lush if and only if X is lush.*

We claim that for the proof of Theorem 8.11 we can assume without loss of generality that (Ω, Σ, μ) is a probability space, as vector-valued L_1-spaces associated to σ-finite measures are (up to an isometric isomorphism) vector-valued L_1-spaces associated to probability measures (see [26, Proposition 1.6.1], for instance), with the same null-sets and the same atoms.

Now, in order to prove Theorem 8.11 for probability spaces, we need to introduce some notation. If (Ω, Σ, μ) is a probability space and X and Y are Banach spaces, the set

$$\mathscr{A} := \left\{ \sum_{A \in \pi} y_A^* \mathbb{1}_A : \pi \subset \Sigma^+ \text{ finite partition of } \Omega \ , y_A^* \in S_{Y^*} \right\} \subset S_{L_\infty(\mu, Y^*)}$$

satisfies that

$$B_{L_1(\mu,Y)^*} = \overline{\mathrm{conv}}^{w^*}(\mathscr{A}), \tag{8.11}$$

since $\mathscr{A}$ is rounded and it is clearly norming for the simple functions of $L_1(\mu,Y)$. On the other hand, we will write

$$\mathscr{B} := \left\{ x\,\frac{\mathbb{1}_B}{\mu(B)} : x \in S_X,\, B \in \Sigma^+ \right\}$$

which satisfies that

$$B_{L_1(\mu,X)} = \overline{\mathrm{conv}}(\mathscr{B}). \tag{8.12}$$

Indeed, it is enough to notice that every simple function f in $S_{L_1(\mu,X)}$ belongs to the convex hull of $\mathscr{B}$: such an f can be written as $f = \sum_{B\in\pi} x_B \mathbb{1}_B$, where $\pi \subset \Sigma^+$ is a finite family of pairwise disjoint sets of Ω and $x_B \in X \setminus \{0\}$ for each $B \in \pi$. Then

$$\|f\| = \sum_{B\in\pi} \|x_B\|\mu(B) = 1,$$

and hence

$$f = \sum_{B\in\pi} \|x_B\|\mu(B)\,\frac{x_B}{\|x_B\|}\,\frac{\mathbb{1}_B}{\mu(B)} \in \mathrm{conv}(\mathscr{B}).$$

Proof (of Theorem 8.11.a). Suppose that $\widetilde{G}$ is a spear operator. Fix $T \in \mathcal{L}(X,Y)$ and consider $\widetilde{T} \in \mathcal{L}\big(L_1(\mu,X), L_1(\mu,Y)\big)$ given by $\widetilde{T}(f) = T\circ f$, which obviously satisfies $\|\widetilde{T}\| = \|T\|$. Given $\varepsilon > 0$, we can find $x \in S_X$ and $B \in \Sigma^+$ such that

$$\left\|(\widetilde{G} + \mathbb{T}\widetilde{T})(x\tfrac{\mathbb{1}_B}{\mu(B)})\right\| > 1 + \|T\| - \varepsilon. \tag{8.13}$$

But notice that

$$\left\|\widetilde{G}(x\tfrac{\mathbb{1}_B}{\mu(B)}) + \mathbb{T}\widetilde{T}(x\tfrac{\mathbb{1}_B}{\mu(B)})\right\| = \left\|\big(G(x) + \mathbb{T}T(x)\big)\tfrac{\mathbb{1}_B}{\mu(B)}\right\| = \|G(x) + \mathbb{T}T(x)\|.$$

This, together with (8.13) and the arbitrariness of ε, tells us that G is a spear operator.

Assume now that G is a spear operator. Fixed $T \in \mathcal{L}\big(L_1(\mu,X), L_1(\mu,Y)\big)$ with $\|T\| = 1$ and $\varepsilon > 0$, we may find by (8.12) elements $x_0 \in S_X$ and $B \in \Sigma^+$ such that

$$\left\|T\left(x_0\tfrac{\mathbb{1}_B}{\mu(B)}\right)\right\| > 1 - \varepsilon.$$

Using now (8.11), there exists $f^* = \sum_{A\in\pi} y_A^* \mathbb{1}_A$, where π is a finite partition of Ω into sets of Σ^+ and $y_A^* \in S_{Y^*}$ for each $A \in \pi$, satisfying that

$$\mathrm{Re}\, f^*\left(T\left(x_0\tfrac{\mathbb{1}_B}{\mu(B)}\right)\right) = \mathrm{Re}\sum_{A\in\pi} y_A^*\left(\int_A T\left(x_0\tfrac{\mathbb{1}_B}{\mu(B)}\right)\, d\mu\right) > 1 - \varepsilon. \tag{8.14}$$

Then, we can write

$$T\left(x_0 \frac{\mathbb{1}_B}{\mu(B)}\right) = \sum_{\substack{A\in\pi \\ \mu(A\cap B)\neq 0}} \frac{\mu(B\cap A)}{\mu(B)} T\left(x_0 \frac{\mathbb{1}_{B\cap A}}{\mu(B\cap A)}\right)$$

so by a standard convexity argument we can assume that there is $A_0 \in \pi$ such that $A_0 \subset B$ and (8.14) is still satisfied. By the density of norm-attaining functionals, we can and do assume that every y_A^* is norm-ataining, so there is $y_{A_0} \in S_Y$ such that $y_{A_0}^*(y_{A_0}) = 1$. Define the operator $S\colon X \longrightarrow Y$ by

$$S(x) = \int_{A_0} T\left(x\frac{\mathbb{1}_B}{\mu(B)}\right) d\mu + \left[\sum_{A\in\pi\setminus\{A_0\}} y_A^*\left(\int_A T\left(x\frac{\mathbb{1}_B}{\mu(B)}\right) d\mu\right)\right] y_{A_0} \quad (x \in X).$$

It is easy to check that $\|S\| \leqslant 1$, and moreover $\|S\| > 1-\varepsilon$ since as a consequence of (8.14) we obtain that

$$\|S(x_0)\| \geqslant |y_{A_0}^*(Sx_0)| = \left|f^*\left(T\left(x_0\frac{\mathbb{1}_B}{\mu(B)}\right)\right)\right| > 1-\varepsilon.$$

By hypothesis, we can find $x_1 \in S_X$ and $\theta_1 \in \mathbb{T}$ such that $\|G(x_1)+\theta_1 S(x_1)\| > 2-\varepsilon$. Now,

$$\begin{aligned}
&\left\|\widetilde{G}\left(x_1\frac{\mathbb{1}_B}{\mu(B)}\right) + \theta_1 T\left(x_1\frac{\mathbb{1}_B}{\mu(B)}\right)\right\| \\
&\quad = \int_B \left\|G(x_1)\frac{\mathbb{1}_B}{\mu(B)} + \theta_1 T\left(x_1\frac{\mathbb{1}_B}{\mu(B)}\right)\right\| d\mu + \int_{\cup\pi\setminus B} \left\|T(x_1\frac{\mathbb{1}_B}{\mu(B)})\right\| d\mu \\
&\quad \geqslant \left\|G(x_1) + \theta_1 \int_B T(x_1\frac{\mathbb{1}_B}{\mu(B)})\, d\mu\right\| + \int_{A_0\setminus B} \left\|T(x_1\frac{\mathbb{1}_B}{\mu(B)})\right\| d\mu \\
&\qquad\qquad + \sum_{A\in\pi\setminus\{A_0\}} \int_A \left\|T(x_1\frac{\mathbb{1}_B}{\mu(B)})\right\| d\mu \\
&\quad \geqslant \left\|G(x_1) + \theta_1 \int_{A_0} T(x_1\frac{\mathbb{1}_B}{\mu(B)})\, d\mu\right\| + \sum_{A\in\pi\setminus\{A_0\}} \left\|y_A^*\left(\int_A T(x_1\frac{\mathbb{1}_B}{\mu(B)})\, d\mu\right) y_{A_0}\right\| \\
&\quad \geqslant \|G(x_1)+\theta_1 S(x_1)\| > 2-\varepsilon.
\end{aligned}$$

This shows that $\|\widetilde{G} + \mathbb{T}T\| > 2-\varepsilon$, finishing the proof. □

Our next aim is to deal with lushness. To this end, we will make use of the following immediate numerical result which we prove for the sake of completeness.

Lemma 8.14. *Let $\varepsilon > 0$, $\delta > 0$, and let $\lambda_i \geqslant 0$ for $i = 1,\dots,n$. Suppose that $\alpha_i, \beta_i \in \mathbb{R}$ are such that $\alpha_i \leqslant \beta_i$ for $i = 1,\dots,n$ and satisfy $(\sum_{i=1}^n \lambda_i\beta_i) - \varepsilon\delta \leqslant \sum_{i=1}^n \lambda_i\alpha_i$. Then,*

$$\sum\{\lambda_i\colon \alpha_i \leqslant \beta_i - \varepsilon\} < \delta.$$

In particular, if $\sum_{i=1}^n \lambda_i = 1$, then

$$\sum\{\lambda_i\colon \alpha_i > \beta_i - \varepsilon\} > 1-\delta.$$

Proof. Calling $I = \{1 \leqslant i \leqslant n \colon \alpha_i > \beta_i - \varepsilon\}$ it suffices to observe that

$$\left(\sum_{i=1}^{n} \lambda_i \beta_i\right) - \varepsilon\delta \leqslant \sum_{i=1}^{n} \lambda_i \alpha_i \leqslant \sum_{i\in I} \lambda_i \beta_i + \sum_{i\notin I} \lambda_i(\beta_i - \varepsilon) = \sum_{i=1}^{n} \lambda_i \beta_i - \varepsilon \sum_{i\notin I} \lambda_i$$

from where it easily follows that $\sum_{i\notin I} \lambda_i < \delta$. The last claim is clear. □

Proof (of Theorem 8.11.b). Assume that G is lush. To check that $\widetilde{G}$ is lush, we just have to show that Proposition 3.28.iii is satisfied. Fix $\varepsilon > 0$, $g_0 \in S_{L_1(\mu,Y)}$ and $f_0 \in \mathscr{B}$ of the form $f_0 = x_0 \mathbb{1}_B/\mu(B)$ for some $x_0 \in S_X$ and $B \in \Sigma^+$. By density, we can assume that

$$g_0 = \sum_{A\in\pi} y_A \frac{\mathbb{1}_A}{\mu(A)}$$

where $\pi \subset \Sigma^+$ is a finite partition of Ω and $y_A \in Y$ satisfy that $\sum_{A\in\pi} \|y_A\| = 1$. By Proposition 3.28.iii, the lushness of G lets us find for each $A \in \pi$ an element $y_A^* \in S_{Y^*}$ such that $\operatorname{Re} y_A^*(y_A) \geqslant (1-\varepsilon)\|y_A\|$ and

$$\operatorname{dist}\big(x_0, \operatorname{aconv}\big(\operatorname{gSlice}(S_X, G^* y_A^*, \varepsilon)\big)\big) < \varepsilon. \tag{8.15}$$

Let $h^* := \sum_{A\in\pi} y_A^* \mathbb{1}_A$, which satisfies that $h^* \in \operatorname{Slice}(S_{L_\infty(\mu,Y^*)}, g_0, \varepsilon)$ as

$$\operatorname{Re} h^*(g_0) = \sum_{A\in\pi} \operatorname{Re} y_A^*(y_A) > \sum_{A\in\pi} (1-\varepsilon)\|y_A\| = 1-\varepsilon.$$

Our aim is to prove now that

$$\operatorname{dist}\Big(f_0, \operatorname{aconv}\Big(\operatorname{gSlice}(B_{L_1(\mu,X)}, \widetilde{G}^* h^*, \varepsilon)\Big)\Big) < \varepsilon, \tag{8.16}$$

which will finish the proof. First notice that for each $A \in \pi$ with $\mu(B\cap A) \neq 0$ we have that

$$\operatorname{gSlice}(S_X, G^* y_A^*, \varepsilon) \frac{\mathbb{1}_{B\cap A}}{\mu(B\cap A)} \subset \operatorname{gSlice}(B_{L_1(\mu,X)}, \widetilde{G}^* h^*, \varepsilon), \tag{8.17}$$

since every $x_A \in \operatorname{gSlice}(S_X, G^* y_A^*, \varepsilon)$ satisfies

$$\operatorname{Re} \widetilde{G}^* h^* \left(x_A \tfrac{\mathbb{1}_{B\cap A}}{\mu(B\cap A)}\right) = \operatorname{Re} G^* y_A^*(x_A) > 1-\varepsilon.$$

In particular, if for each $A \in \pi$ we take an element $x_A \in \operatorname{aconv}\big(\operatorname{gSlice}(S_X, G^* y_A^*, \varepsilon)\big)$ satisfying $\|x_A - x_0\| < \varepsilon$, which exists by (8.15), then the inclusion in (8.17) yields that

$$f := \sum_{\substack{A\in\pi \\ \mu(A\cap B)\neq 0}} x_A \frac{\mathbb{1}_{B\cap A}}{\mu(B)}$$

$$= \sum_{\substack{A\in\pi \\ \mu(A\cap B)\neq 0}} \frac{\mu(B\cap A)}{\mu(B)} x_A \frac{\mathbb{1}_{B\cap A}}{\mu(B\cap A)} \in \operatorname{aconv}\big(\operatorname{gSlice}(B_{L_1(\mu,X)}, \widetilde{G}^* h^*, \varepsilon)\big).$$

Moreover, we have that

$$\|f - f_0\| = \left\| f - x_0 \frac{\mathbb{1}_B}{\mu(B)} \right\| = \sum_{\substack{A\in\pi \\ \mu(A\cap B)\neq 0}} \|x_A - x_0\| \frac{\mu(B\cap A)}{\mu(B)} < \varepsilon$$

which shows that (8.16) holds.

Let us see the converse: to check that G is lush, we will show that condition (iii) of Proposition 3.28 is satisfied. To do so, let $0<\varepsilon<1/8$, $x_0 \in S_X$ and $y_0 \in S_Y$. The mentioned condition applied to the lush operator $\widetilde{G}$ for ε, $x_0\mathbb{1}_\Omega \in S_{L_1(\mu,X)}$ and $y_0\mathbb{1}_\Omega \in S_{L_1(\mu,Y)}$ provides $n\in\mathbb{N}$, functions

$$g^* \in S(\mathscr{A}, y_0\mathbb{1}_\Omega, \varepsilon^3) \qquad \text{and} \qquad f_1,\dots,f_n \in \operatorname{gSlice}(S_{L_1(\mu,X)}, \widetilde{G}^* g^*, \varepsilon^3) \tag{8.18}$$

and scalars $\theta_1,\dots,\theta_n\in\mathbb{T}$, $\lambda_1,\dots,\lambda_n\in[0,1]$ with $\sum_{i=1}^n \lambda_i = 1$ satisfying that

$$\left\| x_0\mathbb{1}_\Omega - \sum_{i=1}^n \lambda_i\theta_i f_i \right\| < \varepsilon^3. \tag{8.19}$$

By density, we can assume that the functions f_i are simple and moreover that there is a finite partition $\{A_1,\dots,A_m\}\subset\Sigma^+$ of Ω such that

$$g^* = \sum_{j=1}^m y_j^* \mathbb{1}_{A_j} \qquad \text{and} \qquad f_i = \sum_{j=1}^m x_{i,j}\mathbb{1}_{A_j} \qquad (i=1,\dots,n)$$

where $y_j^* \in S_{Y^*}$ and $x_{i,j}\in X$ for every i,j. Then, conditions (8.18) and (8.19) can be rewritten as

$$1-\varepsilon^3 < \operatorname{Re} g^*(y_0\mathbb{1}_\Omega) = \sum_{j=1}^m \operatorname{Re} y_j^*(y_0)\mu(A_j), \tag{8.20}$$

$$\sum_{j=1}^m \mu(A_j) \left\| x_0 - \sum_{i=1}^n \lambda_i\theta_i x_{i,j} \right\| < \varepsilon^3, \tag{8.21}$$

and

$$1-\varepsilon^3 < \operatorname{Re}\widetilde{G}^* g^*(f_i) = \sum_{j=1}^{m} \operatorname{Re} G^* y_j^*(x_{i,j})\mu(A_j) \tag{8.22}$$
$$\leqslant \sum_{j=1}^{m} \|x_{i,j}\|\mu(A_j) = 1 \qquad (i=1,\dots,n).$$

Applying Lemma 8.14 to (8.20) and (8.21), we obtain respectively that

$$\sum\{\mu(A_j)\colon \operatorname{Re} y_j^*(y_0) > 1-\varepsilon\} > 1-\varepsilon^2 \tag{8.23}$$

and

$$\sum\left\{\mu(A_j)\colon \left\|x_0 - \sum_{i=1}^{n}\lambda_i\theta_i x_{i,j}\right\| < \varepsilon\right\} > 1-\varepsilon^2. \tag{8.24}$$

Using that $\|x_0\| = 1$, the last inequality yields in particular that

$$\sum\left\{\mu(A_j)\colon \sum_{i=1}^{n}\lambda_i\|x_{i,j}\| > 1-\varepsilon\right\} > 1-\varepsilon^2. \tag{8.25}$$

Combining the relations of (8.22) in a convex sum with the λ_i's as coefficients we obtain that

$$1-\varepsilon^3 < \sum_{j=1}^{m}\mu(A_j)\sum_{i=1}^{n}\lambda_i \operatorname{Re} G^* y_j^*(x_{i,j}) \leqslant \sum_{j=1}^{m}\mu(A_j)\sum_{i=1}^{n}\lambda_i\|x_{i,j}\| = 1. \tag{8.26}$$

Actually, from the right-hand equality of the previous expression we get that

$$1 = \sum_{i=1}^{n}\lambda_i \sum_{j=1}^{m}\|x_{i,j}\|\mu(A_j) = \sum_{j=1}^{m}\mu(A_j)\sum_{i=1}^{n}\lambda_i\|x_{i,j}\|$$
$$\geqslant (1+\varepsilon)\sum\left\{\mu(A_j)\colon \sum_{i=1}^{n}\lambda_i\|x_{i,j}\| > 1+\varepsilon\right\}$$
$$+(1-\varepsilon)\sum\left\{\mu(A_j)\colon 1-\varepsilon < \sum_{i=1}^{n}\lambda_i\|x_{i,j}\| \leqslant 1+\varepsilon\right\},$$

which, together with (8.25), implies that the number

$$\alpha := \sum\left\{\mu(A_j)\colon 1-\varepsilon < \sum_{i=1}^{n}\lambda_i\|x_{i,j}\| \leqslant 1+\varepsilon\right\}$$

satisfies the relation

$$(1+\varepsilon)(1-\varepsilon^2-\alpha)+(1-\varepsilon)\alpha \leqslant 1.$$

A simple computation shows that necessarily $1-\varepsilon-\varepsilon^2 \leqslant 2\alpha$, and so

$$\sum \left\{ \mu(A_j) \colon 1-\varepsilon < \sum_{i=1}^{n} \lambda_i \|x_{i,j}\| \leqslant 1+\varepsilon \right\} = \alpha \geqslant \frac{1}{2} - \varepsilon. \tag{8.27}$$

On the other hand, an application of Lemma 8.14 to the left-hand side part of (8.26) gives that

$$\sum \left\{ \mu(A_j) \colon \left(\sum_{i=1}^{n} \lambda_i \|x_{i,j}\| \right) - \varepsilon^2 < \sum_{i=1}^{n} \lambda_i \operatorname{Re} G^* y_j^*(x_{i,j}) \right\} > 1-\varepsilon. \tag{8.28}$$

Using that $\varepsilon < 1/8$ combined with (8.23), (8.24), (8.27), and (8.28), we deduce the existence of some $j_0 \in \{1,\dots,m\}$ satisfying simultaneaously

$$1-\varepsilon < \operatorname{Re} y_{j_0}^*(y_0),$$

$$\left\| x_0 - \sum_{i=1}^{n} \lambda_i \theta_i x_{i,j_0} \right\| < \varepsilon, \tag{8.29}$$

$$1-\varepsilon < \sum_{i=1}^{n} \lambda_i \|x_{i,j_0}\| \leqslant 1+\varepsilon, \tag{8.30}$$

and

$$\left(\sum_{i=1}^{n} \lambda_i \|x_{i,j_0}\| \right) - \varepsilon^2 < \sum_{i=1}^{n} \lambda_i \operatorname{Re} G^* y_{j_0}^*(x_{i,j_0}). \tag{8.31}$$

If we denote

$$I := \{1 \leqslant i \leqslant n \colon \operatorname{Re} G^* y_{j_0}^*(x_{i,j_0}) > \|x_{i,j_0}\|(1-\varepsilon)\},$$

then again we can apply Lemma 8.14 to (8.31) with

$$\beta_i = 1, \quad \alpha_i = \frac{\operatorname{Re} G^* y_{j_0}^*(x_{i,j_0})}{\|x_{i,j_0}\|}, \quad \text{and} \quad \left(\sum_{i=1}^{n} \lambda_i \|x_{i,j_0}\| \beta_i \right) - \varepsilon^2 < \sum_{i=1}^{n} \lambda_i \|x_{i,j_0}\| \alpha_i$$

to get that

$$\sum_{i \notin I} \lambda_i \|x_{i,j_0}\| < \varepsilon.$$

This, together with (8.30), yields that

$$1-2\varepsilon < \sum_{i \in I} \lambda_i \|x_{i,j_0}\| \leqslant 1+\varepsilon. \tag{8.32}$$

Consider now the elements

$$\tilde{x}_i := \frac{x_{i,j_0}}{\|x_{i,j_0}\|} \in S_X \qquad \text{and} \qquad \tilde{\lambda}_i := \frac{\lambda_i \|x_{i,j_0}\|}{\sum_{k \in I} \lambda_k \|x_{k,j_0}\|} \geqslant 0$$

which satisfy

$$\sum_{i\in I}\tilde{\lambda}_i = 1 \qquad \text{and} \qquad \tilde{x}_i \in \operatorname{gSlice}(S_X, G^* y^*_{j_0}, \varepsilon) \qquad \text{for each } i \in I.$$

Finally, using (8.29) and (8.32) we conclude that

$$\begin{aligned}\left\| x_0 - \sum_{i\in I}\tilde{\lambda}_i\theta_i\tilde{x}_i \right\| &\leqslant \varepsilon + \left\| \sum_{i=1}^{n}\lambda_i\theta_i x_{i,j_0} - \sum_{i\in I}\tilde{\lambda}_i\theta_i\tilde{x}_i \right\| \\ &\leqslant \varepsilon + \sum_{i\notin I}\lambda_i\|x_{i,j_0}\| + \sum_{i\in I}\left\|\lambda_i x_{i,j_0} - \tilde{\lambda}_i\tilde{x}_i\right\| \\ &\leqslant \varepsilon + \sum_{i\notin I}\lambda_i\|x_{i,j_0}\| + \left| 1 - \frac{1}{\sum_{k\in I}\lambda_k\|x_{k,j_0}\|}\right| \sum_{i\in I}\lambda_i\|x_{i,j_0}\| \\ &= \varepsilon + \sum_{i\notin I}\lambda_i\|x_{i,j_0}\| + \left| 1 - \sum_{i\in I}\lambda_i\|x_{i,j_0}\|\right| \leqslant 4\varepsilon\end{aligned}$$

which finishes the proof. □

Proof (of Theorem 8.11.c). Let us fix an atom $A_0 \in \Sigma^+$. Assume that $\widetilde{G}$ has the aDP. We will show that G satisfies condition (iii) of Theorem 3.6: Let $x_0 \in S_X$, $y_0 \in S_Y$ and $\varepsilon > 0$. By hypothesis, we have that

$$x_0\frac{\mathbb{1}_{A_0}}{\mu(A_0)} \in \overline{\operatorname{aconv}}\left(\left\{ f \in \mathscr{B} \colon \left\|\widetilde{G}(f) + y_0\frac{\mathbb{1}_{A_0}}{\mu(A_0)}\right\| > 2 - \varepsilon\,\mu(A_0)\right\}\right).$$

Then, for each $\eta \in (0,1)$ we can find a finite family $\mathcal{F} \subset \Sigma^+$, elements $x_B \in S_X$ satisfying

$$2 - \varepsilon\,\mu(A_0) < \left\| G(x_B)\frac{\mathbb{1}_B}{\mu(B)} + y_0\frac{\mathbb{1}_{A_0}}{\mu(A_0)}\right\| \qquad \text{for every } B \in \mathcal{F}; \tag{8.33}$$

and scalars $\lambda_B \in \mathbb{K}$ with $\sum_{B\in\mathcal{F}}|\lambda_B| = 1$ such that

$$\left\| x_0\frac{\mathbb{1}_{A_0}}{\mu(A_0)} - \sum_{B\in\mathcal{F}}\lambda_B x_B\frac{\mathbb{1}_B}{\mu(B)}\right\| < \eta. \tag{8.34}$$

But $\mu(A_0\cap B)$ is either 0 or $\mu(A_0)$ for each $B \in \mathcal{F}$ as A_0 is an atom. Then, if we just integrate in (8.34) over the atom A_0 we will get that

$$\left\| x_0 - \sum_{\substack{B\in\mathcal{F}\\ \mu(A_0\setminus B)=0}}\lambda_B x_B\frac{\mu(A_0)}{\mu(B)}\right\| < \eta. \tag{8.35}$$

In particular, this yields that

$$\alpha := \sum_{\substack{B\in\mathcal{F}\\ \mu(A_0\setminus B)=0}} |\lambda_B|\frac{\mu(A_0)}{\mu(B)} > 1-\eta. \tag{8.36}$$

Combining (8.35) and (8.36), we deduce that

$$\begin{aligned}\left\| x_0 - \sum_{\substack{B\in\mathcal{F}\\ \mu(A_0\setminus B)=0}} \frac{\lambda_B\mu(A_0)}{\alpha\mu(B)}x_B\right\| &\leqslant \eta + \left\| \sum_{\substack{B\in\mathcal{F}\\ \mu(A_0\setminus B)=0}} \left(\frac{1}{\alpha}-1\right)\lambda_B x_B\frac{\mu(A_0)}{\mu(B)}\right\| \\ &\leqslant \eta + \frac{1-\alpha}{\alpha} \leqslant \eta + \frac{\eta}{1-\eta}.\end{aligned} \tag{8.37}$$

Since $\eta\in(0,1)$ was arbitrary, we deduce from (8.37) that x_0 belongs to the closed absolute convex hull of the set of all $x\in S_X$ for which there is $B\in\Sigma^+$ satisfying $\mu(A_0\setminus B)=0$ and

$$\left\| G(x)\frac{\mathbb{1}_B}{\mu(B)} + y_0\frac{\mathbb{1}_{A_0}}{\mu(A_0)}\right\| > 2-\varepsilon\,\mu(A_0).$$

But then such elements x and B satisfy in particular that

$$\begin{aligned} 2-\varepsilon\,\mu(A_0) &< \|G(x)\|\frac{\mu(B\setminus A_0)}{\mu(B)} + \left\| G(x)\frac{\mu(A_0)}{\mu(B)} + y_0\right\| \\ &\leqslant \|G(x)\|\frac{\mu(B\setminus A_0)}{\mu(B)} + \|y_0\|\frac{\mu(B\setminus A_0)}{\mu(B)} + \frac{\mu(A_0)}{\mu(B)}\|G(x)+y_0\| \\ &\leqslant \frac{\mu(B\setminus A_0)}{\mu(B)}2 + \frac{\mu(A_0)}{\mu(B)}\|G(x)+y_0\|,\end{aligned}$$

and hence

$$2-\varepsilon \leqslant 2-\mu(B)\,\varepsilon \leqslant \|G(x)+y_0\|.$$

We then conclude that

$$x_0 \in \overline{\operatorname{aconv}}(\{x\in B_X\colon \|G(x)+y_0\| > 2-\varepsilon\}).$$

Let us prove now the converse of (c). We remark here that this implication does not use that μ has atoms. Assuming that G has the aDP, we will now check that $\widetilde{G}$ satisfies Theorem 3.6.ii. For this, it is enough to prove that given a simple function $g_0\in S_{L_1(\mu,Y)}$ of the form

$$g_0 = \sum_{A\in\pi} y_A\frac{\mathbb{1}_A}{\mu(A)},$$

where $\pi\subset\Sigma^+$ is a finite partition of Ω and $y_A\in Y$ ($A\in\pi$), we have that

$$\mathscr{B} \subset \overline{\operatorname{aconv}}(\{f\in\mathscr{B}\colon \|\widetilde{G}(f)+g_0\| > 2-\varepsilon\}).$$

Let $x_0\in S_X$ and $B\in\Sigma^+$. Then

$$x_0 \frac{\mathbb{1}_B}{\mu(B)} = \sum_{\substack{A\in\pi \\ \mu(A\cap B)\neq 0}} \frac{\mu(B\cap A_j)}{\mu(B)} x_0 \frac{\mathbb{1}_{B\cap A}}{\mu(B\cap A)},$$

and so in order to show that

$$x_0 \frac{\mathbb{1}_B}{\mu(B)} \in \overline{\text{aconv}}(\{f \in \mathscr{B}\colon \|\widetilde{G}(f)+g_0\| > 2-\varepsilon\})$$

we can assume without loss of generality, by using a standard convexity argument, that B is contained in some $A_0 \in \pi$. Since G has the aDP, using Theorem 3.6.iii, for each $\delta > 0$ there is a finite set

$$F \subset \{x \in S_X\colon \left\|G(x)+y_{A_0}\right\| > 1+\|y_{A_0}\| - \varepsilon\}$$

such that $\text{dist}\,(x_0, \text{aconv}\,(F)) < \delta$. In particular, this implies that

$$\text{dist}\left(x_0 \tfrac{\mathbb{1}_B}{\mu(B)}, \text{aconv}\left(\left\{x \tfrac{\mathbb{1}_B}{\mu(B)} : x \in F\right\}\right)\right) < \delta. \tag{8.38}$$

Finally, notice that each $x \in F$ satisfies that

$$\begin{aligned}
\left\|\widetilde{G}\left(x\tfrac{\mathbb{1}_B}{\mu(B)}\right)+g_0\right\| &= \int_B \left\|G(x)\tfrac{\mathbb{1}_B}{\mu(B)} + y_{A_0}\mathbb{1}_{A_0}\right\| d\mu + \int_{\cup\pi\setminus B} \|g_0\|\,d\mu \\
&= \|G(x)+y_{A_0}\mu(B)\| + \|y_{A_0}\|\mu(A_0\setminus B) + \sum_{A\in\pi\setminus\{A_0\}} \|y_A\|\mu(A) \\
&\geqslant 1+\|y_{A_0}\|\mu(B) - \varepsilon + \|y_{A_0}\|\mu(A_0\setminus B) + \sum_{A\in\pi\setminus\{A_0\}} \|y_A\|\mu(A) \\
&= 1+\|g_0\| - \varepsilon = 2-\varepsilon.
\end{aligned}$$

Therefore, (8.38) leads to

$$\text{dist}\left(x_0 \tfrac{\mathbb{1}_B}{\mu(B)}, \text{aconv}\,(\{f \in \mathscr{B}\colon \|\widetilde{G}(f)+g_0\| > 2-\varepsilon\})\right) < \delta$$

for arbitrary $\delta > 0$. □

Proof (of Theorem 8.11.d). Assuming hat μ has no atoms, we claim that given a simple function $g_0 \in S_{L_1(\mu,Y)}$, for every $\delta > 0$ we can write g_0 as

$$g_0 = \sum_{A\in\pi} y_A \frac{\mathbb{1}_A}{\mu(A)} \tag{8.39}$$

where $\pi \subset \Sigma^+$ is a finite partition of Ω and the coefficients $y_A \in \delta B_Y$ for each $A \in \pi$. Let us check this: of course, we can write g_0 as in (8.39) for a partition $\pi \subset \Sigma^+$ and elements $y_A \in Y$ with $\sum_{A\in\pi} \|y_A\| = 1$. But since μ has no atoms, we can find for each $A \in \pi$ a partition of A into elements $C \in \Sigma^+$ satisfying $\mu(C) \leqslant \delta\mu(A)$. If π' is the collection of all such subsets, then this is a finer partition than π and

$$g_0 = \sum_{A\in\pi} y_A \frac{\mathbb{1}_A}{\mu(A)} = \sum_{A\in\pi}\sum_{\substack{C\in\pi'\\ C\subset A}} \left(\frac{\mu(C)}{\mu(A)} y_A\right) \frac{\mathbb{1}_C}{\mu(C)}$$

and the proof of the claim is over.

Let us then prove now that if

$$B_X = \overline{\operatorname{aconv}}\{x \in B_X\colon \|Gx\| > 1-\varepsilon\} \tag{8.40}$$

for every $\varepsilon > 0$, then $\widetilde{G}$ has the aDP. By Theorem 3.6.iii, it is enough to show that given a simple function $g_0 \in S_{L_1(\mu,Y)}$ as in (8.39) and $\varepsilon > 0$, we have that

$$\mathscr{B} \subset \operatorname{aconv}\left\{f \in \mathscr{B}\colon \|\widetilde{G}(f) + g_0\| > 2-\varepsilon\right\} + \delta B_{L_1(\mu,X)}$$

for every $\delta > 0$. Let $x_0 \in S_X$, $B \in \Sigma^+$ and $0 < \delta < \varepsilon/3$. Let $\pi \subset \Sigma^+$ and $y_A \in \delta B_Y$ ($A \in \pi$) as in the claim above for the given g_0. We can moreover assume that every $A \in \pi$ is either contained in B or in $\Omega \setminus B$. Using (8.40), we can find $m \in \mathbb{N}$, $x_j \in B_X$ with $\|G(x_j)\| > 1-\delta$ and $\lambda_j \in \mathbb{K}$ ($j = 1,\ldots,m$) such that $\sum_{j=1}^m |\lambda_j| = 1$ and

$$\left\| x_0 - \sum_{j=1}^m \lambda_j x_j \right\| < \delta.$$

Then, it is easy to check that

$$\left\| x_0 \frac{\mathbb{1}_B}{\mu(B)} - \sum_{\substack{A\in\pi\\ A\subset B}} \sum_{j=1}^m \left(\frac{\mu(A)}{\mu(B)}\lambda_j\right) x_j \frac{\mathbb{1}_A}{\mu(A)} \right\| < \delta. \tag{8.41}$$

This shows that $x_0\mathbb{1}_B/\mu(B)$ is δ-approximated by an absolutely convex sum of elements of the form $x_j\mathbb{1}_A/\mu(A)$ for some $1 \leqslant j \leqslant m$ and $A \in \pi$. Finally, notice that every such element satisfies that

$$\begin{aligned}
\left\| \widetilde{G}\left(x_j \frac{\mathbb{1}_A}{\mu(A)}\right) + g_0 \right\| &= \left\| G(x_j)\frac{\mathbb{1}_A}{\mu(A)} + g_0 \right\| \\
&= \|G(x_j) + y_A\| + \sum_{A'\in\pi, A'\neq A} \|y_{A'}\| \\
&\geqslant \|G(x_j)\| - \delta + 1 - \delta > 2 - 3\delta.
\end{aligned} \tag{8.42}$$

Therefore,

$$x_0 \frac{\mathbb{1}_B}{\mu(B)} \in \operatorname{aconv}\left(\left\{f \in B_{L_1(\mu,X)}\colon \|\widetilde{G}(f) + g_0\| > 2-3\delta\right\}\right) + \delta B_{L_1(\mu,X)}.$$

Using that $0 < \delta < \varepsilon/3$ was arbitrary, we conclude the result.

Conversely, suppose that $\widetilde{G}$ has the aDP and let $\varepsilon > 0$, $x_0 \in S_X$ and $y_0 \in S_Y$. By Theorem 3.6.iii, we have that

$$x_0 \mathbb{1}_\Omega \in \overline{\operatorname{aconv}}\left(\left\{f \in \mathscr{B}\colon \left\|\widetilde{G}(f) + y_0 \mathbb{1}_\Omega\right\| > 2 - \varepsilon\right\}\right).$$

Therefore, given any $\delta > 0$ there is a finite set $\mathcal{F} \subset \Sigma^+$ and elements $x_A \in S_X$, $\lambda_A \in \mathbb{K}$ $(A \in \mathcal{F})$ such that $\sum_{A \in \mathcal{F}} |\lambda_A| = 1$ satisfying that

$$\left\|\widetilde{G}\left(x_A \frac{\mathbb{1}_A}{\mu(A)}\right) + y_0 \mathbb{1}_\Omega\right\| > 2 - \varepsilon \tag{8.43}$$

and

$$\left\|x_0 \mathbb{1}_\Omega - \sum_{A \in \mathcal{F}} \lambda_A x_A \frac{\mathbb{1}_A}{\mu(A)}\right\| < \delta. \tag{8.44}$$

It easily follows from (8.43) that

$$1 - \varepsilon < \left\|G(x_A) \frac{\mathbb{1}_A}{\mu(A)} + y_0 \mathbb{1}_\Omega\right\| - 1 \leqslant \|G(x_A)\|.$$

On the other hand, (8.44) yields that

$$\left\|x_0 - \sum_{A \in \mathcal{F}} \lambda_A x_A\right\| = \left\|\int_\Omega \left(x_0 \mathbb{1}_\Omega - \sum_{A \in \mathcal{F}} \lambda_A x_A \frac{\mathbb{1}_A}{\mu(A)}\right) d\mu\right\| < \delta.$$

Therefore,

$$\operatorname{dist}\left(x_0, \operatorname{aconv}\left\{x \in B_X : \|Gx\| > 1 - \varepsilon\right\}\right) < \delta,$$

and since $\delta > 0$ and $x_0 \in S_X$ were arbitrary, we conclude that (8.40) holds. □

8.4 Target operators, lushness and ultraproducts

Now, we will prove the stability of target operators and lush operators with respect to the operation of taking ultraproducts. These results extend Corollaries 4.4 and 4.5 of [21] about stability of lush spaces with respect to ultraproducts.

Let us recall the basic definitions, taken from [55]. Let $\mathscr{U}$ be a free ultrafilter on $\mathbb{N}$. The limit of a sequence with respect to the ultrafilter $\mathscr{U}$ is denoted by $\lim_{\mathscr{U}} a_n$, or $\lim_{n,\mathscr{U}} a_n$, if it is necessary to stress that the limit is taken with respect to the variable n. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces. We can consider the ℓ_∞-sum of the family, $[\oplus_{n\in\mathbb{N}} X_n]_{\ell_\infty}$, together with its closed subspace

$$N(\mathscr{U}) = \left\{(x_n)_{n\in\mathbb{N}} \in [\oplus_{n\in\mathbb{N}} X_n]_{\ell_\infty} : \lim_{\mathscr{U}} \|x_n\| = 0\right\}.$$

The quotient space $(X_n)_{\mathscr{U}} = [\oplus_{n\in\mathbb{N}} X_n]_{\ell_\infty} / N(\mathscr{U})$ is called the *ultraproduct* of the family $(X_n)_{n\in\mathbb{N}}$ relative to the ultrafilter $\mathscr{U}$. Let $(x_n)_{\mathscr{U}}$ stand for the element of $(X_n)_{\mathscr{U}}$ containing a given representative $(x_n) \in [\oplus_{n\in\mathbb{N}} X_n]_{\ell_\infty}$. It is easy to check that

$$\|(x_n)_{\mathscr{U}}\| = \lim_{\mathscr{U}} \|x_n\|.$$

Moreover, every $\tilde{x} \in (X_n)_{\mathscr{U}}$ can be represented as $\tilde{x} = (x_n)_{\mathscr{U}}$ in such a way that $\|x_n\| = \|\tilde{x}\|$ for all $n \in \mathbb{N}$.

If all the spaces X_n are equal to the same Banach space X, the ultraproduct of the family is called the $\mathscr{U}$*-ultrapower* of X. We denote this ultrapower by $X_{\mathscr{U}}$.

Let $(X_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$ be two sequences of Banach spaces and let $(T_n)_{n\in\mathbb{N}}$ be a norm-bounded sequence of operators where $T_n \in \mathcal{L}(X_n, Y_n)$ for every $n \in \mathbb{N}$. We denote $(T_n)_{\mathscr{U}}$ the operator that acts from $(X_n)_{\mathscr{U}}$ to $(Y_n)_{\mathscr{U}}$ as follows:

$$\big[(T_n)_{\mathscr{U}}\big](x_n)_{\mathscr{U}} = (T_n x_n)_{\mathscr{U}} \qquad \big((x_n)_{\mathscr{U}} \in (X_n)_{\mathscr{U}}\big).$$

Evidently,

$$\|(T_n)_{\mathscr{U}}\| = \lim_{\mathscr{U}} \|T_n\|.$$

Now, we state our main result about ultraproducts.

Theorem 8.15. *Let $\mathscr{U}$ be a free ultrafilter on $\mathbb{N}$, $(X_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$, $(Z_n)_{n\in\mathbb{N}}$ be sequences of Banach spaces and let $(G_n)_{n\in\mathbb{N}}$, $(T_n)_{n\in\mathbb{N}}$ be norm bounded sequences of operators such that $G_n \in S_{\mathcal{L}(X_n,Y_n)}$ and $T_n \in \mathcal{L}(X_n, Z_n)$ for every $n \in \mathbb{N}$. If each T_n is a target for the corresponding G_n for every $n \in \mathbb{N}$, then $T = (T_n)_{\mathscr{U}} \in \mathcal{L}((X_n)_{\mathscr{U}}, (Z_n)_{\mathscr{U}})$ is a target for $G = (G_n)_{\mathscr{U}} \in \mathcal{L}((X_n)_{\mathscr{U}}, (Y_n)_{\mathscr{U}})$.*

We need the following easy remark about the absolutely convex hull of a convex set. In fact, this idea already appeared implicitly in the proof of implication (i) $\Rightarrow$ (iii) of Corollary 2.8.

Proposition 8.16. *Let $F \subset B_X$ be a convex set. If X is a real space, then*

$$\operatorname{aconv}(F) = \{\lambda_1 x_1 - \lambda_2 x_2 \colon x_1, x_2 \in F, \lambda_1, \lambda_2 \geqslant 0, \lambda_1 + \lambda_2 = 1\}.$$

If X is a complex space, then for every $m \in \mathbb{N}$ and every $x \in \operatorname{aconv}(F)$, there are $\lambda_1, \dots, \lambda_m \geqslant 0$ with $\sum_{k=1}^m \lambda_k = 1$ and $x_1, \dots, x_m \in F$, such that

$$\left\| x - \sum_{k=1}^m \lambda_k \exp\left(\frac{2\pi i k}{m}\right) x_k \right\| \leqslant \frac{2\pi}{m}. \tag{8.45}$$

Proof. We demonstrate only the more complicated complex case. As $x \in \operatorname{aconv}(F)$ there are $\mu_j \in [0,\, 1]$, $j = 1, \dots, N$ with $\sum_{j=1}^N \mu_j = 1$, $\theta_j \in [0,\, 2\pi]$ and $y_j \in F$ satisfying

$$x = \sum_{j=1}^{N} \mu_j \exp(i\theta_j) y_j.$$

Taking into account that the points $\left\{\frac{2\pi k}{m} : k = 1, \ldots, m\right\}$ form a $\frac{2\pi}{m}$-net of $[0, 2\pi]$ we can represent the set of indices $\{1, \ldots, N\}$ as a disjoint union of sets A_k, $k = 1, \ldots, m$ in such a way that

$$\left|\theta_j - \frac{2\pi k}{m}\right| \leqslant \frac{2\pi}{m} \qquad \text{for every} \qquad j \in A_k.$$

Let us show that

$$\lambda_k = \sum_{j \in A_k} \mu_j, \quad \text{and} \quad x_k = \frac{1}{\lambda_k} \sum_{j \in A_k} \mu_j y_j \quad \text{if} \quad A_k \neq \emptyset$$

and

$$\lambda_k = 0, \quad \text{and arbitrary} \quad x_k \in F \quad \text{if} \quad A_k = \emptyset$$

fulfill the desired condition (8.45). Indeed, it is clear that $x_k \in F$ and $\sum_{k=1}^{m} \lambda_k = 1$. Now,

$$\begin{aligned}
\left\| x - \sum_{k=1}^{m} \lambda_k \exp\left(\frac{2\pi i k}{m}\right) x_k \right\| &= \left\| x - \sum_{\{k:\, A_k \neq \emptyset\}} \sum_{j \in A_k} \mu_j \exp\left(\frac{2\pi i k}{m}\right) y_j \right\| \\
&\leqslant \left\| x - \sum_{\{k:\, A_k \neq \emptyset\}} \sum_{j \in A_k} \mu_j \exp(i\theta_j) y_j \right\| + \frac{2\pi}{m} \\
&= \left\| x - \sum_{j=1}^{N} \mu_j \exp(i\theta_j) y_j \right\| + \frac{2\pi}{m} = \frac{2\pi}{m},
\end{aligned}$$

as desired. □

Proof (of Theorem 8.15). We demonstrate the theorem only for the more complicated complex case. Also, we may and do suppose that $\|T\| = \|T_n\| = 1$ for every $n \in \mathbb{N}$.

Let $x_0 = (x_{0,n})_{\mathscr{U}} \in B_{(X_n)_{\mathscr{U}}}$, $y = (y_n)_{\mathscr{U}} \in S_{(Y_n)_{\mathscr{U}}}$ and $\varepsilon > 0$ be fixed. Evidently, the "coordinates" $x_{0,n}$ can be selected in such a way that $x_{0,n} \in B_{X_n}$ and $y_n \in S_{Y_n}$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ applying ($\diamond$) in Definition 3.9 for $\varepsilon/2$, $x_{0,n} \in B_{X_n}$, and $y_n \in S_{Y_n}$ we obtain the corresponding $F_n \subset B_{X_n}$ satisfying

$$\begin{aligned}
&\operatorname{conv}(F_n) \subset \left\{ x \in B_{X_n} : \|G_n x + y_n\| > 2 - \frac{\varepsilon}{2} \right\} \quad \text{and} \\
&\operatorname{dist}\big(T_n x_{0,n}, T_n\big(\operatorname{aconv}(F_n)\big)\big) < \frac{\varepsilon}{2}.
\end{aligned} \tag{8.46}$$

Without loss of generality, we assume that F_n is convex, otherwise we just substitute F_n by its convex hull. Our choice means that there is $x_n \in \operatorname{aconv}(F_n)$ such that

$$\left\|T_n x_{0,n} - T_n x_n\right\| < \frac{\varepsilon}{2}.$$

Select $m \in \mathbb{N}$ such that $\frac{2\pi}{m} < \frac{\varepsilon}{2}$. Using Proposition 8.16, we can find for each $n \in \mathbb{N}$ corresponding $\lambda_{n,1} \ldots, \lambda_{n,m} \geqslant 0$, $\sum_{k=1}^{m} \lambda_{n,k} = 1$ and $x_{n,1}, \ldots, x_{n,m} \in F_n$ such that

$$\left\| x_n - \sum_{k=1}^{m} \lambda_{n,k} \exp\left(\frac{2\pi ik}{m}\right) x_{n,k} \right\| < \frac{\varepsilon}{2}$$

and, consequently,

$$\left\| T_n x_{0,n} - \sum_{k=1}^{m} \lambda_{n,k} \exp\left(\frac{2\pi ik}{m}\right) T_n x_{n,k} \right\| < \varepsilon. \tag{8.47}$$

For each $k = 1, \ldots, m$, denote $\lambda_k = \lim_{n,\mathscr{U}} \lambda_{n,k}$ and $\tilde{x}_k = (x_{n,k})_{\mathscr{U}} \in B_{(X_n)_{\mathscr{U}}}$. Also, denote $F = \{\tilde{x}_1, \ldots, \tilde{x}_m\}$. Since $x_{n,1}, \ldots, x_{n,m} \in F_n$, and F_n is convex, by (8.46) we have

$$\operatorname{conv}(F) \subset \left\{ x \in B_{(X_n)_{\mathscr{U}}} : \|Gx + (y_n)_{\mathscr{U}}\| \geq 2 - \frac{\varepsilon}{2} \right\}.$$

Also, since m is fixed, (8.47) implies

$$\left\| Tx_0 - \sum_{k=1}^{m} \lambda_k \, \exp\left(\frac{2\pi ik}{m}\right) T\tilde{x}_k \right\| =$$

$$\lim_{n,\mathscr{U}} \left\| T_n x_{0,n} - \sum_{k=1}^{m} \lambda_{n,k} \exp\left(\frac{2\pi ik}{m}\right) T_n x_{n,k} \right\| \leqslant \varepsilon,$$

that is

$$\operatorname{dist}\big(Tx_0, T\left(\operatorname{aconv}(F)\right)\big) \leqslant \varepsilon.$$

Consequently, F satisfies ($\diamond$) for ε, x_0, and y, i.e. T is a target for G by using this set F in Definition 3.9. □

In the case of ultrapowers, the converse result is also true.

Theorem 8.17. *Let $\mathscr{U}$ be a free ultrafilter on $\mathbb{N}$, let X, Y, Z be Banach spaces, and let $G \in S_{\mathcal{L}(X,Y)}$ and $T \in \mathcal{L}(X,Z)$ be operators. If $T_{\mathscr{U}} = (T,T,\ldots)_{\mathscr{U}} \in \mathcal{L}(X_{\mathscr{U}}, Z_{\mathscr{U}})$ is a target for $G_{\mathscr{U}} = (G,G,\ldots)_{\mathscr{U}} \in \mathcal{L}(X_{\mathscr{U}}, Y_{\mathscr{U}})$, then T is a target for G.*

Proof. For given $x_0 \in B_X$, $\varepsilon > 0$ and $y \in S_Y$, by ($\diamond$) in Definition 3.9 applied to

$$\tilde{x}_0 = (x_0, x_0, \ldots)_{\mathscr{U}} \in B_{X_{\mathscr{U}}} \qquad \text{and} \qquad \tilde{y} = (y, y, \ldots)_{\mathscr{U}} \in S_{Y_{\mathscr{U}}},$$

we can find a finite set $F = \{(x_{1,n})_{\mathscr{U}}, \ldots, (x_{m,n})_{\mathscr{U}}\} \subset B_{X_{\mathscr{U}}}$, $\lambda_1, \ldots, \lambda_m \geqslant 0$ with $\sum \lambda_k = 1$, and $\theta_1, \ldots, \theta_m \in \mathbb{T}$ such that

$$\operatorname{conv}(F) \subset \{\tilde{x} \in B_{X_{\mathscr{U}}} : \|G_{\mathscr{U}}\tilde{x} + \tilde{y}\| > 2 - \varepsilon/2\} \tag{8.48}$$

and

$$\lim_{n,\mathscr{U}} \left\| Tx_0 - \sum_{k=1}^{m} \lambda_k \theta_k T(x_{k,n}) \right\| < \varepsilon. \tag{8.49}$$

Write $F_n = \{x_{1,n}, \dots, x_{m,n}\} \subset B_X$ and denote by E the set of those $n \in \mathbb{N}$ for which

$$\operatorname{conv}(F_n) \subset \{x \in B_X : \|Gx + y\| > 2 - \varepsilon\}.$$

We claim that $E \in \mathscr{U}$. Indeed, if this is not so, then $\mathbb{N} \setminus E \in \mathscr{U}$. For every $n \in \mathbb{N} \setminus E$ choose $\mu_{1,n}, \dots, \mu_{m,n} \geqslant 0$ with $\sum_{k=1}^{m} \mu_{k,n} = 1$ such that

$$\left\| G\left(\sum_{k=1}^{m} \mu_{k,n} x_{k,n} \right) + y \right\| \leqslant 2 - \varepsilon.$$

Then, for $\mu_k = \lim_{n,\mathscr{U}} \mu_{k,n}$ we have

$$\left\| G_{\mathscr{U}} \left(\sum_{k=1}^{m} \mu_k (x_{k,n})_{\mathscr{U}} \right) + \tilde{y} \right\| \leqslant 2 - \varepsilon,$$

which contradicts (8.48).

Now, since $E \in \mathscr{U}$, according to (8.49) there is an $n_0 \in E$ such that

$$\left\| Tx_0 - \sum_{k=1}^{m} \lambda_k \theta_k T(x_{k,n_0}) \right\| < \varepsilon.$$

The corresponding F_{n_0} fulfills ($\diamond$) in Definition 3.9. □

Since lushness of an operator reduces to the fact that the identity operator is a target for it, we obtain the following two corollaries.

Corollary 8.18. *Let $\mathscr{U}$ be a free ultrafilter on $\mathbb{N}$, $(X_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$ be sequences of Banach spaces, $(G_n)_{n\in\mathbb{N}}$ be a sequence of lush operators where $G_n \in S_{\mathcal{L}(X_n,Y_n)}$ for every $n \in \mathbb{N}$. Then $G = (G_n)_{\mathscr{U}} \in \mathcal{L}((X_n)_{\mathscr{U}}, (Y_n)_{\mathscr{U}})$ is lush.*

Corollary 8.19. *Let $\mathscr{U}$ be a free ultrafilter on $\mathbb{N}$, X, Y be Banach spaces, $G \in S_{\mathcal{L}(X,Y)}$. If $G_{\mathscr{U}} = (G, G, \dots)_{\mathscr{U}} \in \mathcal{L}(X_{\mathscr{U}}, Y_{\mathscr{U}})$ is lush, then G is lush.*

Chapter 9

Open problems

Corresponding to *Spear sets and spear vectors*:

Problem 9.1. Let X be a complex Banach space. If $\operatorname{Spear}(X)$ is not compact, does X contain a copy of c_0 or ℓ_1?

Problem 9.2. If X is a complex smooth Banach space and $\operatorname{Spear}(X^*) \neq \emptyset$, can we deduce that $X \cong \mathbb{C}$?

Corresponding to *Lush operators*:

Problem 9.3. Are items (a) and (b) in Proposition 3.32 necessary for G to be lush? If there is a counterexample, notice that the domain must be non separable.

Corresponding to *Examples in classical Banach spaces*:

Problem 9.4. Is lush the dual of the Fourier transform on L_1? Is lush the Fourier-Stieltjes transform?

Problem 9.5. Is Proposition 4.2 always an equivalence?

Problem 9.6. Is there an intrinsic characterization of lush operators or of spear operators acting from an $L_1(\mu)$ space analogous to the one given in Theorem 4.18 for the aDP?

Corresponding to *Further results*:

Problem 9.7. Are spearness and lushness equivalent when the codomain space is SCD? The aDP and spearness are, see Remark 5.4.

Problem 9.8. Are the aDP and lushness equivalent when the image of the operator is Asplund? They are equivalent when the codomain is Asplund, see Proposition 5.3.

Problem 9.9. Can the results about rank-one operators be extended to finite-rank operators? They are Corollary 5.9 and Proposition 5.20.

Problem 9.10. If $G\colon X \longrightarrow Y$ is lush and Y is L-embedded, is G^* lush? This is true for spearness and the aDP (see Proposition 5.21).

Corresponding to *Isomorphic and isometric consequences*:

Problem 9.11. Is Theorem 6.1 valid in the complex case? That is, does the dual of the domain of a complex operator with infinite rank and the aDP always contain ℓ_1?

Problem 9.12. Let $G\colon X \longrightarrow Y$ be an operator with the aDP. Does $X = \mathbb{K}$ if X is strictly convex or smooth? Does $Y = \mathbb{K}$ if Y is strictly convex or smooth?

Problem 9.13. Does every spear operator attain its norm? This is true for lush operators, see Proposition 6.8, but it is not true for operators with the aDP, see Example 8.7.

Corresponding to *Stability results*:

Problem 9.14. Are there results about the relationship between spear and lush operators with quotients by the kernel of the operator analogous to the one given in Proposition 8.1 for the aDP?

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