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On the Bishop–Phelps–Bollobás theorem for operators and numerical radius

by

Sun Kwang Kim (Suwon), Han Ju Lee (Seoul) and
Miguel Martín (Granada)

Abstract. We study the Bishop–Phelps–Bollobás property for numerical radius (for short, BPBp-nu) of operators on ℓ_1-sums and ℓ_∞-sums of Banach spaces. More precisely, we introduce a property of Banach spaces, which we call strongly lush. We find that if X is strongly lush and $X \oplus_1 Y$ has the weak BPBp-nu, then (X, Y) has the Bishop–Phelps–Bollobás property (BPBp). On the other hand, if Y is strongly lush and $X \oplus_\infty Y$ has the weak BPBp-nu, then (X, Y) has the BPBp. Examples of strongly lush spaces are $C(K)$ spaces, $L_1(\mu)$ spaces, and finite-codimensional subspaces of $C[0, 1]$.

1. Introduction. Let X be a (real or complex) Banach space and X^* be its dual space. The unit sphere of X will be denoted by S_X and the closed unit ball by B_X. We write $\mathcal{L}(X)$ for the space of all bounded linear operators on X. The *numerical radius* of $T \in \mathcal{L}(X)$ is defined by

$$v(T) = \sup\{|x^*Tx| : (x, x^*) \in \Pi(X)\},$$

where $\Pi(X) := \{(x, x^*) \in S_X \times S_{X^*} : x^*(x) = 1\}$. It is clear that v is a seminorm on $\mathcal{L}(X)$ with $v(T) \leq \|T\|$ for every $T \in \mathcal{L}(X)$. We refer the reader to the classical monographs [10, 11] for background. An operator $T \in \mathcal{L}(X)$ *attains its numerical radius* if there exists $(x_0, x_0^*) \in \Pi(X)$ such that $v(T) = |x_0^* T x_0|$. A lot of attention has been paid to the study of the denseness of numerical radius attaining operators [1, 3, 6, 8, 14, 15, 16, 27].

Motivated by the work [4] of M. Acosta, R. Aron, D. García and M. Maestre on the Bishop–Phelps–Bollobás property for operators, A. Guirao and O. Kozhushkina [19] introduced the notion of Bishop–Phelps–Bollobás property for numerical radius, which is a quantitative way to study the denseness of numerical radius attaining operators.

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Definition 1.1 ([22]). A Banach space X is said to have the *weak Bishop–Phelps–Bollobás property for numerical radius* (for short, *weak BPBp-nu*) if for every $0 < \varepsilon < 1$, there exists $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(X)$ and $(x, x^*) \in \Pi(X)$ satisfy $v(T) = 1$ and $|x^*Tx| > 1 - \eta(\varepsilon)$, there exist $S \in \mathcal{L}(X)$ and $(y, y^*) \in \Pi(X)$ such that

$$v(S) = |y^*Sy|, \quad \|T - S\| < \varepsilon, \quad \|x - y\| < \varepsilon, \quad \|x^* - y^*\| < \varepsilon.$$

A pair (X, Y) of Banach spaces is said to have the *Bishop–Phelps–Bollobas property for numerical radius* (for short, BPBP-nu) if together with all requirements of Definition 1.1, also $v(S) = |y^*Sy| = 1$. From the definitions, it is clear that the BPBp-nu implies the weak BPBp-nu, while in [22] some conditions are given ensuring that the converse also holds.

Let X, Y be Banach spaces and denote by $\mathcal{L}(X, Y)$ the Banach space of all bounded linear operators from X to Y. We recall that $T \in \mathcal{L}(X, Y)$ is said to be *norm attaining* if there is $x \in B_X$ such that $\|T\| = \|Tx\|$. A pair (X, Y) is said to have the *Bishop–Phelps–Bollobás property for operators* (for short, *BPBp*) [4] if, given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x \in S_X$ satisfy $\|Tx\| > 1 - \eta(\varepsilon)$, there exist $S \in \mathcal{L}(X, Y)$ with $\|S\| = 1$ and $y \in S_X$ such that

$$\|Sy\| = 1, \quad \|T - S\| < \varepsilon, \quad \|x - y\| < \varepsilon.$$

It is shown in [19] that the real or complex spaces c_0 and ℓ_1 have the BPBp-nu. The result on ℓ_1 has been extended to the real space $L_1(\mathbb{R})$ by J. Falcó [18]. For the result on c_0, A. Avilés, A. J. Guirao and J. Rodríguez [7] gave sufficient conditions on a compact space K for the real space $C(K)$ to have the BPBp-nu, which, in particular, include all metrizable compact spaces. In [22] the BPBp-nu is studied for more general spaces. For instance, it is shown that finite-dimensional spaces and general $L_1(\mu)$ spaces have the BPBp-nu. It is also shown that $L_p(\mu)$ has the BPBp-nu for every measure μ when $1 < p < \infty$, $p \neq 2$. It has been shown very recently [23] that every real Hilbert space has the BPBp-nu. As for negative results, it is shown in [22] that every separable infinite-dimensional Banach space can be equivalently renormed to fail the BPBp-nu, even though for reflexive spaces (actually for spaces with the Radon–Nikodým property [6]) the set of numerical radius attaining operators is always dense. To get this result, it is shown in [22] that there is a relation between the BPBp-nu and the Bishop–Phelps–Bollobás property for operators. More precisely, if $L_1(\mu) \oplus_1 Y$ has the BPBp-nu, then $(L_1(\mu), Y)$ has the BPBp [22, Theorem 15].

In this paper, we generalize this fact as follows. Let X, Y be Banach spaces. If X is strongly lush (see the definition below) and $X \oplus_1 Y$ has the weak BPBp-nu, then (X, Y) has the BPBp. On the other hand, if Y is strongly lush and $X \oplus_\infty Y$ has the weak BPBp-nu, then (X, Y) has the

BPBp. It is also shown that none of the converses of these results holds. More precisely, there exist strongly lush spaces X and Y such that (X,Y) has the BPBp, but the set of numerical radius attaining operators is dense in neither $\mathcal{L}(X \oplus_1 Y)$ nor $\mathcal{L}(X \oplus_\infty Y)$.

We need some notation. Given a subset F of a Banach space X, we denote the absolutely closed convex hull of F by $\overline{\operatorname{aconv}}(F)$. For $C \subset X^*$, $\overline{\operatorname{aconv}}^{w^*}(C)$ denotes the absolutely weak-$*$ closed convex hull of C. We write $\mathrm{NA}(X)$ to denote the subset of those elements in X^* which attain their norm. Note that this set is dense by the classical Bishop–Phelps theorem [9]. Given $x^* \in \mathrm{NA}(X) \cap S_{X^*}$, we write $F(x^*)$ to denote the (non-empty) *face* generated by x^*, i.e. $F(x^*) = \{x \in B_X : x^*(x) = 1\}$.

Definition 1.2. We say that a Banach space X is *strongly lush* if there is a subset C of S_{X^*} such that $B_{X^*} = \overline{\operatorname{aconv}}^{w^*}(C)$ and $B_X = \overline{\operatorname{aconv}}(F(x^*))$ for every $x^* \in C$.

This definition appeared, without name, in some papers, including [21, Corollary 4.5] or [24, Proposition 2.1]. There are many examples of spaces with this property, the easiest ones being the almost-CL-spaces [26, §2]. We recall that a Banach space is an *almost-CL-space* if its unit ball is the absolutely closed convex hull of every maximal face. $L_1(\mu)$ spaces and their isometric preduals (in particular, $C(K)$ spaces), the disk algebra etc. are examples of almost-CL-spaces (see [17, 21, 26] and references therein for background).

Moreover, separable lush spaces are strongly lush. We recall that a Banach space X is *lush* [13] if given $x, y \in S_X$ and $\varepsilon > 0$, there is $y^* \in S_{X^*}$ such that $\operatorname{Re} y^*(y) > 1 - \varepsilon$ and the distance from x to

$$\overline{\operatorname{aconv}}(\{z \in B_X : \operatorname{Re} y^*(z) > 1 - \varepsilon\})$$

is less than ε. We refer to [12, 13, 21, 24] and references therein for background. Almost-CL-spaces are lush, but the converse is not true [13]. As commented before, separable lush spaces are strongly lush ([21, Corollary 4.5] for the real case, [24, Proposition 2.1] for the complex case). This implies, in particular, that finite-codimensional subspaces of $C[0,1]$ are strongly lush.

Let us also mention that there is a reformulation of strong lushness in terms of extreme points of the bidual ball: A Banach space X is strongly lush if and only if there exists a subset $C \subset S_{X^*}$ with $B_{X^*} = \overline{\operatorname{aconv}}^{w^*}(C)$ such that $|x^{**}(x^*)| = 1$ for every $x^* \in C$ and every extreme point x^{**} of $B_{X^{**}}$. Indeed, Milman's theorem shows that the necessity holds. The converse is shown by [5, Corollary 3.5].

2. The results. Let us present first the result for ℓ_1-sums, which generalizes [22, Theorem 15].

THEOREM 2.1. *Let X and Y be Banach spaces and suppose that X is strongly lush. If $X \oplus_1 Y$ has the weak BPBp-nu, then (X,Y) has the BPBp.*

Proof. Suppose that $X \oplus_1 Y$ has the weak BPBp-nu with a function η; we will show that (X,Y) has the BPBp with the function $\varepsilon \mapsto \eta\big(\frac{\varepsilon}{2+\varepsilon}\big)$. Fix $0<\varepsilon<1$ and let $\tilde\varepsilon = \varepsilon/(\varepsilon+2)$. Let $T \in \mathcal{L}(X,Y)$ with $\|T\|=1$ and $x_0 \in S_X$ satisfying $\|Tx_0\| > 1-\eta(\tilde\varepsilon)$. We pick $y_0^* \in S_{Y^*}$ such that

$$|y_0^*(Tx_0)| = \|Tx_0\| > 1-\eta(\tilde\varepsilon).$$

We consider the extension $\tilde T$ of T from $X \oplus_1 Y$ to $X \oplus_1 Y$ given by

$$\tilde T(x,y) = (0,Tx) \qquad ((x,y) \in X \oplus_1 Y).$$

We claim that $v(\tilde T) = \|\tilde T\| = 1$. Indeed, $v(\tilde T) \le \|\tilde T\| = \|T\| = 1$. On the other hand,

$$\begin{aligned} v(\tilde T) &= \sup\big\{|(x^*,y^*)\tilde T(x,y)| : \max\{\|x^*\|,\|y^*\|\}=1,\ \|x\|+\|y\|=1, \\ &\qquad\qquad x^*(x)+y^*(y)=1\big\} \\ &= \sup\big\{|(x^*,y^*)\tilde T(x,y)| : (x^*,y^*) \in B_{X^*}\times B_{Y^*},\ \|x\|+\|y\|=1, \\ &\qquad\qquad x^*(x)=\|x\|,\ y^*(y)=\|y\|\big\} \\ &\ge \sup\{|(x^*,y^*)\tilde T(x,0)| : x \in S_X,\ x^* \in S_{X^*},\ x^*(x)=1,\ y^* \in S_{Y^*}\} \\ &= \sup\{|y^*(Tx)| : y^* \in S_{Y^*},\ x \in S_X\} = \|T\|. \end{aligned}$$

Now, pick any $x_0^* \in S_{X^*}$ with $x_0^*(x_0)=1$ and observe that

$$((x_0,0),(x_0^*,y_0^*)) \in \Pi(X \oplus_1 Y)$$

and

$$|(x_0^*,y_0^*)\tilde T(x_0,0)| = |y_0^*(Tx_0)| = \|T(x_0)\| > 1-\eta(\tilde\varepsilon).$$

Since $X \oplus_1 Y$ has the weak BPBp-nu with the function η, there exist $(x_1,y_1) \in S_{X\oplus_1 Y}$, $(x_1^*,y_1^*) \in S_{X^*\oplus_\infty Y^*}$ and $S' \in \mathcal{L}(X \oplus_1 Y)$ satisfying

$$x_1^*(x_1)+y_1^*(y_1)=1, \qquad v(S') = |(x_1^*,y_1^*)S'(x_1,y_1)|$$

and

$$\|S'-\tilde T\| < \tilde\varepsilon, \qquad \|x_1-x_0\|+\|y_1\| < \varepsilon, \qquad \max\{\|x_1^*-x_0^*\|,\|y_1^*-y_0^*\|\} < \varepsilon.$$

So $|v(S')-1| < \tilde\varepsilon$ and $\big|\|S'\|-1\big| < \tilde\varepsilon$. Hence

$$\begin{aligned} \left\|\frac{S'}{v(S')} - \tilde T\right\| &\le \left\|\frac{S'}{v(S')} - S'\right\| + \|S'-\tilde T\| \\ &< \frac{\|S'\|\cdot|v(S')-1|}{v(S')} + \tilde\varepsilon \le \frac{(1+\tilde\varepsilon)\tilde\varepsilon}{1-\tilde\varepsilon} + \tilde\varepsilon = \varepsilon. \end{aligned}$$

Write $\tilde S = S'/v(S')$ and observe that

$$v(\tilde S) = 1 = \big|(x_1^*,y_1^*)\tilde S(x_1,y_1)\big|, \qquad \|\tilde S - \tilde T\| < \varepsilon.$$

It follows that $x_1^*(x_1) = \|x_1\|$ and $y_1^*(y_1) = \|y_1\|$.

We claim that $y_1 = 0$. Otherwise,

$$\left\| \tilde{S}\left(0, \frac{y_1}{\|y_1\|}\right) - \tilde{T}\left(0, \frac{y_1}{\|y_1\|}\right) \right\| = \left\| \tilde{S}\left(0, \frac{y_1}{\|y_1\|}\right) \right\| < \varepsilon.$$

If $x_1 \neq 0$, then

$$\begin{aligned} 1 &= |(x_1^*, y_1^*)\tilde{S}(x_1, y_1)| \\ &\leq \left|(x_1^*, y_1^*)\tilde{S}\left(\frac{x_1}{\|x_1\|}, 0\right)\right| \|x_1\| + \left|(x_1^*, y_1^*)\tilde{S}\left(0, \frac{y_1}{\|y_1\|}\right)\right| \|y_1\| \\ &\leq \|x_1\| + \varepsilon\|y_1\| < \|x_1\| + \|y_1\| = 1, \end{aligned}$$

a contradiction. The case $x_1 = 0$ is even easier.

By the claim, we have $x_1^*(x_1) = \|x_1\| = 1$. Next, write $\tilde{S}(x, y) = (\tilde{S}_1(x, y), \tilde{S}_2(x, y))$ and define $S_1 \in \mathcal{L}(X, X)$ and $S_2 \in \mathcal{L}(X, Y)$ by

$$S_1(x) = \tilde{S}_1(x, 0), \quad S_2(x) = \tilde{S}_2(x, 0) \quad (x \in X).$$

Observe that

$$\begin{aligned} 1 = v(\tilde{S}) &= |(x_1^*, y_1^*)\tilde{S}(x_1, 0)| = |x_1^*(S_1(x_1)) + y_1^*(S_2(x_1))| \\ &\leq \|S_1(x_1)\| + \|S_2(x_1)\| \leq \sup\{\|S_1(x)\| + \|S_2(x)\| : x \in B_X\} \\ &= \sup\{|x^*(S_1(x))| + |y^*(S_2(x))| : x \in B_X,\ x^* \in C,\ y^* \in S_{Y^*}\} \\ &= \sup\{|x^*(S_1(x)) + y^*(S_2(x))| : x \in B_X,\ x^* \in C,\ y^* \in S_{Y^*}\} \end{aligned}$$

where we have used the fact that $\overline{\mathrm{aconv}}^{w^*}(C) = B_{X^*}$. For $x^* \in C$, we have $B_X = \overline{\mathrm{aconv}}(F(x^*))$ and the function $x \mapsto |x^*(S_1(x)) + y^*(S_2(x))|$ is convex, so we may continue the previous chain of inequalities as follows:

$$\begin{aligned} &= \sup\{|x^*(S_1(x)) + y^*(S_2(x))| : x^* \in C,\ x \in F(x^*),\ y^* \in S_{Y^*}\} \\ &= \sup\{|(x^*, y^*)(\tilde{S}(x, 0))| : x^* \in C,\ x \in F(x^*),\ y^* \in S_{Y^*}\} \\ &\leq \sup\{|(x^*, y^*)(\tilde{S}(x, y))| : ((x, y), (x^*, y^*)) \in \Pi(X \oplus_1 Y)\} = v(\tilde{S}) = 1. \end{aligned}$$

We conclude that

$$\begin{aligned} \sup\{\|S_1(x)\| + \|S_2(x)\| : x \in B_X\} &= \|S_1(x_1)\| + \|S_2(x_1)\| \\ &= |x_1^*(S_1(x_1)) + y_1^*(S_2(x_1))| = 1, \end{aligned}$$

and it follows in particular that there exists $\omega \in S_{\mathbb{K}}$ such that

$$\|S_1(x_1)\| = \omega x_1^*(S_1(x_1)) \quad \text{and} \quad \|S_2(x_1)\| = \omega y_1^*(S_2(x_1)).$$

We now claim that $S_2(x_1) \neq 0$. Indeed, for all $x \in S_X$,

$$\varepsilon > \|\tilde{S} - \tilde{T}\| \geq \|S_1 x\| + \|S_2 x - T x\|.$$

So $\|S_1\| \leq \varepsilon$ and $\|S_2 - T\| < \varepsilon$. If $S_2(x_1) = 0$, then

$$1 = \|S_1(x_1)\| + \|S_2(x_1)\| = \|S_1(x_1)\| \leq \|S_1\| < \varepsilon,$$

a contradiction.

Finally, define $R \in \mathcal{L}(X,Y)$ by

$$R(x) = S_2(x) + \omega \frac{S_2(x_1)}{\|S_2(x_1)\|} x_1^* S_1(x) \qquad \big(x \in X\big).$$

Observe that

$$\begin{aligned}|y_1^* R(x_1)| &= \left| y_1^* S_2(x_1) + \omega \frac{y_1^* S_2(x_1)}{\|S_2(x_1)\|} x_1^* S_1(x_1) \right| \\ &= |x_1^*(S_1(x_1)) + y_1^*(S_2(x_1))| = 1\end{aligned}$$

and $\|R(x)\| \le \|S_2(x)\| + \|S_1(x)\| \le 1$. Therefore, $\|R\| = 1 = \|R(x_1)\|$ and

$$\|R - T\| \le \|S_2 - T\| + \|S_1\| \le \|\tilde{S} - \tilde{T}\| < \varepsilon.$$

Notice also that $\|x_1 - x_0\| < \varepsilon$. This completes the proof. ∎

As mentioned in the introduction, almost-CL-spaces and separable lush spaces are strongly lush. Therefore, we have the following corollary.

Corollary 2.2. *Let X be an almost-CL-space or a separable lush space and let Y be a Banach space. If $X \oplus_1 Y$ has the weak BPBp-nu, then (X,Y) has the BPBp.*

Concerning ℓ_∞-sums, we have the following result in which a condition has to be imposed on the range space instead of on the domain space.

Theorem 2.3. *Let X and Y be Banach spaces and suppose that Y is strongly lush. If $X \oplus_\infty Y$ has the weak BPBp-nu, then (X,Y) has the BPBp.*

Proof. Suppose that $X \oplus_\infty Y$ has the BPBp-nu with a function η; we will show that (X,Y) has the BPBp with the function $\varepsilon \mapsto \eta\big(\frac{\varepsilon}{4+\varepsilon}\big)$. Fix $0 < \varepsilon < 1$ and let $\tilde{\varepsilon} = \varepsilon/(4+\varepsilon)$. Let $T \in \mathcal{L}(X,Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ satisfying $\|Tx_0\| > 1 - \eta(\tilde{\varepsilon})$. Then, by the Bishop–Phelps theorem, there exists $y_0^* \in S_{Y^*} \cap \mathrm{NA}(Y)$ such that

$$|y_0^*(Tx_0)| = \|Tx_0\| > 1 - \eta(\tilde{\varepsilon}).$$

We pick $y_0 \in S_Y$ such that $y_0^*(y_0) = 1$. Now, we consider the extension $\tilde{T}$ of T from $X \oplus_\infty Y$ to $X \oplus_\infty Y$ defined by $\tilde{T}(x,y) = (0, Tx)$ for every $(x,y) \in X \oplus_\infty Y$. Clearly, $v(\tilde{T}) \le \|\tilde{T}\| = \|T\| = 1$ and, on the other hand,

$$\begin{aligned}v(\tilde{T}) &\ge \sup\big\{|(x^*,y^*)\tilde{T}(x,y)| : (x,y) \in S_X \times S_Y,\ \|x^*\| + \|y^*\| = 1, \\ &\qquad\qquad x^*(x) = \|x^*\|,\ y^*(y) = \|y^*\|\big\} \\ &\ge \sup\{|y^*(Tx)| : y^* \in S_{Y^*} \cap \mathrm{NA}(Y),\ x \in S_X\} = \|T\| = 1.\end{aligned}$$

So $v(\tilde{T}) = \|\tilde{T}\| = 1$. Since $|(0,y_0^*)\tilde{T}(x_0,y_0)| = |y_0^* T x_0| > 1 - \eta(\tilde{\varepsilon})$ and $X \oplus_\infty Y$ has the weak BPBp-nu with the function η, there exist $S' \in \mathcal{L}(X \oplus_\infty Y)$, $(x_1,y_1) \in S_{X \oplus_\infty Y}$ and $(x_1^*, y_1^*) \in S_{X^* \oplus_1 Y^*}$ such that

$$x_1^*(x_1) + y_1^*(y_1) = 1, \qquad v(S') = |(x_1^*, y_1^*) S'(x_1, y_1)|$$

and

$$\|\tilde{T}-S'\| < \tilde{\varepsilon}, \quad \max\{\|x_1-x_0\|, \|y_0-y_1\|\} < \varepsilon/2, \quad \|x_1^*\|+\|y_0^*-y_1^*\| < \varepsilon/2.$$

So $|v(S')-1| < \tilde{\varepsilon}$ and $\big|\|S'\|-1\big| < \tilde{\varepsilon}$. Hence

$$\begin{aligned}\left\|\frac{S'}{v(S')}-\tilde{T}\right\| &\le \left\|\frac{S'}{v(S')}-S'\right\| + \|S'-\tilde{T}\| < \frac{\|S'\|\cdot|v(S')-1|}{v(S')}+\tilde{\varepsilon}\\ &\le \frac{(1+\tilde{\varepsilon})\tilde{\varepsilon}}{1-\tilde{\varepsilon}}+\tilde{\varepsilon} = \varepsilon/2.\end{aligned}$$

Now, for $\tilde{S} = S'/v(S')$ we have

$$v(\tilde{S}) = 1 = |(x_1^*, y_1^*)\tilde{S}(x_1, y_1)|, \quad \|\tilde{S}-\tilde{T}\| < \varepsilon/2.$$

Observe that

$$x_1^*(x_1) = \|x_1^*\|, \quad y_1^*(y_1) = \|y_1^*\|, \quad \|x_1^*\|+\|y_1^*\| = 1.$$

We claim that $x_1^* = 0$. Otherwise,

$$\begin{aligned}\left\|\left(\frac{x_1^*}{\|x_1^*\|}, 0\right)\tilde{S}(x_1, y_1)\right\| &= \left\|\left(\frac{x_1^*}{\|x_1^*\|}, 0\right)\tilde{S}(x_1, y_1) - \left(\frac{x_1^*}{\|x_1^*\|}, 0\right)\tilde{T}(x_1, y_1)\right\|\\ &\le \|\tilde{S}-\tilde{T}\| < \varepsilon.\end{aligned}$$

Hence, if $y_1^* \neq 0$, then

$$\begin{aligned}1 &= |(x_1^*, y_1^*)\tilde{S}(x_1, y_1)|\\ &\le \left|\left(\frac{x_1^*}{\|x_1^*\|}, 0\right)\tilde{S}(x_1, y_1)\right|\|x_1^*\| + \left|\left(0, \frac{y_1^*}{\|y_1^*\|}\right)\tilde{S}(x_1, y_1)\right|\|y_1^*\|\\ &\le \varepsilon\|x_1^*\| + \|y_1^*\| < \|x_1^*\| + \|y_1^*\| = 1,\end{aligned}$$

a contradiction. The case $y_1^* = 0$ is similar.

By the claim, we get $y_1^*(y_1) = \|y_1\| = 1$. Write $\tilde{T}(x, y) = (0, \tilde{T}_2(x, y))$ and $\tilde{S}(x, y) = (\tilde{S}_1(x, y), \tilde{S}_2(x, y))$ for every $(x, y) \in X \oplus_\infty Y$.

We claim that $\|\tilde{S}_2\| = 1 = \|\tilde{S}_2(x_1, y_1)\|$. Indeed,

$$\begin{aligned}1 = v(\tilde{S}) = |(0, y_1^*)\tilde{S}(x_1, y_1)| &= |y_1^*(\tilde{S}_2(x_1, y_1))| \le \|\tilde{S}_2(x_1, y_1)\|\\ &\le \sup\{\|\tilde{S}_2(x, y)\| : x \in B_X,\, y \in B_Y\}\\ &= \sup\{|y^*\tilde{S}_2(x, y)| : x \in B_X,\, y \in B_Y,\, y^* \in C\}\end{aligned}$$

where we have used the fact that $\overline{\mathrm{aconv}}^{w^*}(C) = B_{X^*}$. For $y^* \in C$, the function $y \mapsto |y^*\tilde{S}_2(x, y)|$ is convex and $B_Y = \overline{\mathrm{aconv}}(F(y^*))$, so we may continue the previous chain of inequalities as follows:

$$\begin{aligned}&= \sup\{|y^*\tilde{S}_2(x, y)| : x \in B_X,\, y^* \in C,\, y \in F(y^*)\}\\ &= \sup\{|(0, y^*)\tilde{S}(x, y)| : x \in B_X,\, y^* \in C,\, y \in F(y^*)\}\\ &\le v(\tilde{S}) = 1,\end{aligned}$$

which proves the claim.

As $\|x_0\| = 1$ and $\|x_0 - x_1\| < \varepsilon/2$, it follows that $\|x_1\| > 1 - \varepsilon/2$ (so, in particular, $x_1 \neq 0$), and $\bar{x}_1 = x_1/\|x_1\|$ satisfies

$$\|\bar{x}_1 - x_0\| < \varepsilon.$$

Next, we claim that $\|\tilde{S}_2(\bar{x}_1, y_1)\| = 1$. Otherwise,

$$\begin{aligned}\|\tilde{S}_2(x_1, y_1)\| &\leq \|x_1\|\|S_2(\bar{x}_1, y_1)\| + (1 - \|x_1\|)\|S_2(0, y_1)\| \\ &< \|x_1\| + (1 - \|x_1\|) = 1,\end{aligned}$$

a contradiction.

Finally, choose $x_2^* \in S_{X^*}$ with $x_2^*(\bar{x}_1) = 1$ and define $R \in \mathcal{L}(X, Y)$ by

$$R(x) = \tilde{S}_2(x, x_2^*(x)y_1) \quad (x \in X).$$

We clearly have $\|R\| \leq \|\tilde{S}_2\| \leq 1$ and

$$\|R(\bar{x}_1)\| = \|\tilde{S}_2(\bar{x}_1, y_1)\| = 1.$$

So it is enough to show that $\|R-T\| < \varepsilon$. Note that for $x \in B_X$ and $y \in B_Y$,

$$\|\tilde{S}_2(x, y) - Tx\| = \|\tilde{S}_2(x, y) - \tilde{T}_2(x, y)\| \leq \|\tilde{S}_2 - \tilde{T}_2\| \leq \|\tilde{S} - \tilde{T}\| < \varepsilon/2.$$

In particular, for all $x \in B_X$,

$$\|Rx - Tx\| = \|\tilde{S}_2(x, x_2(x)y_1) - Tx\| < \varepsilon/2.$$

This completes the proof. ∎

As for the ℓ_1-sum, we obtain the following consequence.

Corollary 2.4. *Let Y be an almost-CL-space or a separable lush space and let X be a Banach space. If $X \oplus_\infty Y$ has the weak BPBp-nu, then (X, Y) has the BPBp.*

The proofs of Theorems 2.1 and 2.3 can be easily adapted to get analogous results for norm and numerical radius attaining operators:

Remark 2.5. *Let X and Y be Banach spaces.*

(a) *Suppose that X is strongly lush and the set of numerical radius attaining operators is dense in $\mathcal{L}(X \oplus_1 Y)$. Then the set of norm attaining operators is dense in $\mathcal{L}(X, Y)$.*
(b) *Suppose that Y is strongly lush and that the set of numerical radius attaining operators is dense in $\mathcal{L}(X \oplus_\infty Y)$. Then the set of norm attaining operators is dense in $\mathcal{L}(X, Y)$.*

R. Payá [27] showed that there exists a strictly convex space X isomorphic to c_0 such that the set of numerical radius attaining operators is not dense in $\mathcal{L}(X \oplus_\infty c_0)$. Remark 2.5 allows us to give a similar example, with an easy proof.

Example 2.6. Let Y be any strictly convex space containing a copy of c_0. Then the set of numerical radius attaining operators is not dense in $\mathcal{L}(c_0 \oplus_1 Y)$. Indeed, otherwise Remark 2.5 implies that the set of norm attaining operators is dense in $\mathcal{L}(c_0, Y)$ (since c_0 is an almost-CL-space). However, this is not the case, as was shown by J. Lindenstrauss [25, Proposition 4].

As a final remark, we show that none of the converses to Theorem 2.1 and Theorem 2.3 (or even Corollaries 2.2 and 2.4) holds.

Remark 2.7. *There exist almost-CL-spaces X and Y such that (X, Y) has the BPBp, but the set of numerical radius attaining operators is dense in neither $\mathcal{L}(X \oplus_1 Y)$ nor $\mathcal{L}(X \oplus_\infty Y)$.*

Indeed, J. Johnson and J. Wolfe [20] proved in 1982 that there is a compact metric space S such that the set of norm attaining operators is not dense in $\mathcal{L}(L_1[0,1], C(S))$. The proof was given for real spaces, but it is not difficult to check that it is also valid in the complex case. Now, let X and Y be the complex spaces $C(S)$ and $L_1[0,1]$, respectively. Then X and Y are almost-CL-spaces, and M. Acosta has recently shown [2] that (X, Y) has the BPBp. However, the set of numerical radius attaining operators is dense in neither $\mathcal{L}(X \oplus_1 Y)$ nor $\mathcal{L}(X \oplus_\infty Y)$. Otherwise, Remark 2.5 would imply that the set of norm attaining operators is dense in $\mathcal{L}(Y, X)$, which is not the case due to the above mentioned result of J. Johnson and J. Wolfe.

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References

[1] M. D. Acosta, *Every real Banach space can be renormed to satisfy the denseness of numerical radius attaining operators*, Israel J. Math. 81 (1993), 273–280.

[2] M. D. Acosta, *The Bishop–Phelps–Bollobás property for operators on $C(K)$*, Banach J. Math. Anal. 10 (2016), 307–319.

[3] M. D. Acosta, F. J. Aguirre and R. Payá, *A new sufficient condition for the denseness of norm-attaining operators*, Rocky Mountain J. Math. 26 (1996), 407–418.

[4] M. D. Acosta, R. M. Aron, D. García and M. Maestre, *The Bishop–Phelps–Bollobás Theorem for operators*, J. Funct. Anal. 254 (2008), 2780–2799.

[5] M. D. Acosta, J. Becerra Guerrero and A. Rodríguez-Palacios, *Weakly open sets in the unit ball of the projective tensor product of Banach spaces*, J. Math. Anal. Appl. 383 (2011), 461–473.

[6] M. D. Acosta and R. Payá, *Numerical radius attaining operators and the Radon–Nikodým property*, Bull. London Math. Soc. 25 (1993), 67–73.

[7] A. Avilés, A. J. Guirao and J. Rodríguez, *On the Bishop–Phelps–Bollobás property for numerical radius in $C(K)$-spaces*, J. Math. Anal. Appl. 419 (2014), 395–421.

[8] I. D. Berg and B. Sims, *Denseness of operators which attain their numerical radius*, J. Austral. Math. Soc. Ser. A 36 (1984), 130–133.

[9] E. Bishop and R. R. Phelps, *A proof that every Banach space is subreflexive*, Bull. Amer. Math. Soc. 67 (1961), 97–98.

[10] F. F. Bonsall and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*, London Math. Soc. Lecture Note Ser. 2, Cambridge Univ. Press, 1971.

[11] F. F. Bonsall and J. Duncan, *Numerical Ranges II*, London Math. Soc. Lecture Note Ser. 10, Cambridge Univ. Press, 1973.

[12] K. Boyko, V. Kadets, M. Martín and J. Merí, *Properties of lush spaces and applications to Banach spaces with numerical index 1*, Studia Math. 190 (2009), 117–133.

[13] K. Boyko, V. Kadets, M. Martín and D. Werner, *Numerical index of Banach spaces and duality*, Math. Proc. Cambridge Philos. Soc.

[14] C. S. Cardassi, *Density of numerical radius attaining operators on some reflexive spaces*, Bull. Austral. Math. Soc. 31 (1985), 1–3.

[15] C. S. Cardassi, *Numerical radius attaining operators*, in: Banach Spaces (Columbia, MO, 1984), Lecture Notes in Math. 1166, Springer, Berlin, 1985, 11–14.

[16] C. S. Cardassi, *Numerical radius-attaining operators on $C(K)$*, Proc. Amer. Math. Soc. 95 (1985), 537–543.

[17] L.-X. Cheng and M. Li, *Extreme points, exposed points, differentiability points in CL-spaces*, Proc. Amer. Math. Soc. 136 (2008), 2445–2451.

[18] J. Falcó, *The Bishop–Phelps–Bollobás property for numerical radius on L_1*, J. Math. Anal. Appl. 414 (2014), 125–133.

[19] A. J. Guirao and O. Kozhushkina, *The Bishop–Phelps–Bollobás property for numerical radius in $\ell_1(\mathbb{C})$*, Studia Math. 218 (2013), 41–54.

[20] J. Johnson and J. Wolfe, *Norm attaining operators and simultaneously continuous retractions*, Proc. Amer. Math. Soc. 86 (1982), 609–612.

[21] V. Kadets, M. Martín, J. Merí and R. Payá, *Convexity and smoothness of Banach spaces with numerical index one*, Illinois J. Math. 53 (2009), 163–182.

[22] S. K. Kim, H. J. Lee and M. Martín, *On the Bishop–Phelps–Bollobás property for numerical radius*, Abstr. Appl. Anal. 2014, art. ID 479208, 15 pp.

[23] S. K. Kim, H. J. Lee, M. Martín and J. Merí, *On a second numerical index for Banach spaces*, arXiv:1604.06198 (2016).

[24] H. J. Lee and M. Martín, *Polynomial numerical indices of Banach spaces with 1-unconditional bases*, Linear Algebra Appl. 437 (2012), 2001–2008.

[25] J. Lindenstrauss, *On operators which attain their norm*, Israel J. Math. 1 (1963), 139–148.

[26] M. Martín and R. Payá, *On CL-spaces and almost-CL-spaces*, Ark. Mat. 42 (2004), 107–118.

[27] R. Payá, *A counterexample on numerical radius attaining operators*, Israel J. Math. 79 (1992), 83–101.

Sun Kwang Kim
Department of Mathematics
Kyonggi University
Suwon 443-760, Republic of Korea
ORCID: 0000-0002-9402-2002
E-mail: sunkwang@kgu.ac.kr

Han Ju Lee (corresponding author)
Department of Mathematics Education
Dongguk University – Seoul
04620 Seoul, Republic of Korea
ORCID: 0000-0001-9523-2987
E-mail: hanjulee@dongguk.edu

Miguel Martín
Departamento de Análisis Matemático
Facultad de Ciencias
Universidad de Granada
E-18071 Granada, Spain
ORCID: 0000-0003-4502-798X
E-mail: mmartins@ugr.es