

# THE VERSION FOR COMPACT OPERATORS OF LINDENSTRAUSS PROPERTIES A AND B

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ABSTRACT. It has been very recently discovered that there are compact linear operators between Banach spaces which cannot be approximated by norm-attaining operators. The aim of this note is to discuss those examples and also sufficient conditions to ensure that compact linear operators can be approximated by norm attaining operators. To do so, we introduce the analogous for compact operators of Lindenstrauss properties A and B.

## 1. INTRODUCTION

The study of norm-attaining operators started with a celebrated paper by J. Lindenstrauss of 1963 [25]. There, he provided examples of pairs of Banach spaces such that there are (bounded linear) operators between them which cannot be approximated by norm-attaining operators. Also, sufficient conditions on the domain space or on the range space providing the density of norm-attaining operators were given. We recall that an operator  $T$  between two Banach spaces  $X$  and  $Y$  is said to *attain its norm* whenever there is  $x \in X$  with  $\|x\| = 1$  such that  $\|T\| = \|T(x)\|$  (that is, the supremum defining the operator norm is actually a maximum).

Very recently, it has been shown that there exist compact linear operators between Banach spaces which cannot be approximated by norm-attaining operators [28], solving a question open since the 1970s. We recall that an operator between Banach spaces is *compact* if it carries bounded sets into relatively compact sets or, equivalently, if the closure of the image of the unit ball is compact. After the cited result of [28], it makes more sense to discuss sufficient conditions on the domain or the range space to ensure that every compact linear operator between them can be approximated by norm attaining operators. This is the objective of the present paper.

Prior to discuss results about the density of norm-attaining compact operators, let us make some remarks about the existence of compact operators which do not attain the norm. First, it is clear that if the domain of a linear operator is a finite-dimensional space, then the image by it of the unit ball is actually compact and, therefore, the operator automatically attains its norm. This argument extends to infinite-dimensional reflexive spaces. Indeed, if  $X$  is a reflexive space, every compact operator from  $X$  into a Banach space  $Y$  is completely continuous (i.e. it maps weakly

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convergent sequences into norm convergent sequences, see [13, Problem 30 in p. 515] for instance) and so the weak (sequential) compactness of the unit ball of  $X$  gives easily the result. We refer to [11, Theorem 6 in p. 16] for a discussion on when all bounded linear operators between reflexive spaces attain their norm. On the other hand, for every non-reflexive Banach space  $X$  there is a continuous linear functional on  $X$  which does not attain its norm (James' theorem, see [11, Theorem 2 in p. 7] for instance). The multiplication of such functional by a fix non-zero vector of a Banach space  $Y$  clearly produces a rank-one (hence compact) operator from  $X$  into  $Y$  which does not attain its norm.

Let us present a brief account on classical results about norm-attaining operators. The expository paper [3] contains a more detailed relate that can be used for reference and background. We need some notation first. Given two (real or complex) Banach spaces  $X$  and  $Y$ , we write  $L(X, Y)$  to denote the Banach space of all bounded linear operators from  $X$  into  $Y$ , endowed with the operator norm. By  $K(X, Y)$  and  $F(X, Y)$  we denote the subspaces of  $L(X, Y)$  of compact operators and finite-rank operators, respectively. We write  $X^*$  for the (topological) dual of  $X$ ,  $B_X$  for its closed unit ball and  $S_X$  for the unit sphere. The set of norm-attaining operators from  $X$  into  $Y$  is denoted by  $NA(X, Y)$ . The study on norm-attaining operators started as a negative answer by J. Lindenstrauss [25] to the question of whether it is possible to extend to the vector valued case the classical Bishop-Phelps theorem of 1961 [6] stating that the set of norm-attaining functionals is always dense in the dual of a Banach space. As the question has a negative answer in general, J. Lindenstrauss introduced two properties to study norm-attaining operators: a Banach space  $X$  (resp.  $Y$ ) has Lindenstrauss *property A* (resp. *property B*) if  $NA(X, Z)$  is dense in  $L(X, Z)$  (resp.  $NA(Z, Y)$  is dense in  $L(Z, Y)$ ) for every Banach space  $Z$ . It is shown in [25], for instance, that  $c_0$ ,  $C[0, 1]$  and  $L_1[0, 1]$  fail property A. Examples of spaces having property A (including reflexive spaces and  $\ell_1$ ) and of spaces having property B (including  $c_0$ ,  $\ell_\infty$  and every finite-dimensional space whose unit ball is a polyhedron) are also shown in this paper. There are many extensions of Lindenstrauss results from which we will comment only a representative sample. With respect to property A, J. Bourgain showed in 1977 that every Banach space with the Radon-Nikodým property have property A and that, conversely, if a Banach space  $X$  has property A in every equivalent norm, then it has the Radon-Nikodým property (this direction needs a refinement due to R. Huff, 1980). W. Schachermayer (1983) and B. Godun and S. Troyanski (1993) showed that “almost” every Banach space can be equivalently renormed to have property A. With respect to property B, J. Partington proved that every Banach space can be renormed to have property B (1982) and W. Schachermayer showed that  $C[0, 1]$  fails the property (1983). W. Gowers showed in 1990 that  $\ell_p$  does not have property B for  $1 < p < \infty$ , a result extended by M. Acosta (1999) to all infinite-dimensional strictly convex Banach spaces and to infinite-dimensional  $L_1(\mu)$  spaces. With respect to pairs of classical Banach spaces not covered by the above results, J. Johnson and J. Wolfe (1979) proved that, in the real case,  $NA(C(K), C(S))$  is dense in  $L(C(K), C(S))$  for all compact spaces  $K$  and  $S$ , and C. Finet and R. Payá (1998) showed the same result for the pair  $(L_1[0, 1], L_\infty[0, 1])$ . Concerning the study of norm-attaining compact operators, J. Diestel and J. Uhl (1976) [12] showed that norm-attaining finite-rank operators from  $L_1(\mu)$  into any Banach space are dense in the space of all compact operators. This study was continued by J. Johnson and J. Wolfe [23] (1979), who proved the same result when the domain space is a  $C(K)$  space or the range space is an  $L_1$ -space (only real case) or a predual of an  $L_1$ -space. In 2013, B. Cascales, A. Guirao, and V. Kadets [9, Theorem 3.6] showed that for every uniform algebra (in particular, the disk algebra  $A(\mathbb{D})$ ), the set of norm-attaining compact operators arriving to the algebra is dense in the set of all compact operators.

Our objective here is to discuss positive and negative result about density of norm-attaining compact operators. As this question is too general, and imitating what Lindenstrauss did in 1963, we introduce the following two properties.

**Definition 1.1.** Let  $X, Y$  be Banach spaces.

- (a)  $X$  is said to satisfy *property  $A^k$*  if  $K(X, Z) \cap NA(X, Z)$  is dense in  $K(X, Z)$  for every Banach space  $Z$ .
- (b)  $Y$  is said to satisfy *property  $B^k$*  if  $K(Z, Y) \cap NA(Z, Y)$  is dense in  $K(Z, Y)$  for every Banach space  $Z$ .

It is not clear whether, in general, property A implies property  $A^k$  or property B implies property  $B^k$ .

**Question 1.2.** Does Lindenstrauss property A imply property  $A^k$ ?

**Question 1.3.** Does Lindenstrauss property B imply property  $B^k$ ?

We do not know the answer to these two questions, but all sufficient conditions for Lindenstrauss properties A and B listed below also implies, respectively, properties  $A^k$  and  $B^k$ . This is so because the usual way of establishing the density of norm-attaining operators is by proving that every operator can be approximated by compact perturbations of it attaining the norm.

Besides of these examples, there are results which are specific for compact operators and they depend on some stronger forms of the approximation property. Let us recall the basic principles of this concept. We refer to the classical book [26] for background and to [8] for a more updated account. A Banach space  $X$  has the (Grothendieck) *approximation property* if for every compact set  $K$  and every  $\varepsilon > 0$ , there is  $R \in F(X, X)$  such that  $\|x - R(x)\| < \varepsilon$  for all  $x \in K$ . Useful and classical results about the approximation theory are the following.

**Proposition 1.4.** Let  $X, Y$  be Banach spaces.

- a)  $Y$  has the approximation property if and only if  $\overline{F(Z, Y)} = K(Z, Y)$  for every Banach space  $Z$ .
- b)  $X^*$  has the approximation property if and only if  $\overline{F(X, Z)} = K(X, Z)$  for every Banach space  $Z$ .
- c) If  $X^*$  has the approximation property, then so does  $X$ .
- d) (Enflo) There exist Banach spaces without the approximation property. Actually, there are closed subspaces of  $c_0$  failing the approximation property.

What is the relation between the approximation property and norm-attaining compact operators? On the one hand, the negative examples given in [28] exploit the failure of the approximation property of some subspaces of  $c_0$  together with an easy extension to its subspaces of a geometrical property proved by Lindenstrauss for  $c_0$  (see section 2 for details). On the other hand, the most of the positive results for properties  $A^k$  and  $B^k$  which are not related to Lindenstrauss properties A and B use some strong form of the approximation property. Actually, all positive results in this line try to give a partial answer to one of the following two open questions.

**Question 1.5.** Does the approximation property imply property  $B^k$ ?

**Question 1.6.** Has property  $A^k$  every Banach space whose dual has the approximation property?

Actually, by Proposition 1.4.a, Question 1.5 is equivalent to the following one, which is considered one of the most important open question in the theory of norm-attaining operators.

**Question 1.7.** Does every finite-dimensional Banach space have Lindenstrauss property B?

Surprising, this question is open even for the two-dimensional Euclidean space.

In section 3 we collect all results about property  $A^k$ . We start by listing the main known examples of Banach spaces with Lindenstrauss property A since all of them have property  $A^k$ . Next, we show a result of [23] that an stronger version of the approximation property of the dual of a Banach space (which we may call  $w^*$  metric  $\pi$  property) implies property  $A^k$ , show that most of the examples known in the literature are proved in this way, and also present some new examples. In particular, we get the following (non-exhaustive) list of examples of Banach spaces with property  $A^k$ :  $L_1(\mu)$  spaces,  $C_0(L)$  spaces, preduals of  $\ell_1$ , and closed subspaces of  $c_0$  with monotone basis.

The results for property  $B^k$  appear in section 4. Again, we start with a list of known examples of Banach spaces with Lindenstrauss property B which also have property  $B^k$ . Next, as we do not know whether the approximation property implies property  $B^k$  or, equivalently, whether finite-dimensionality implies property B (Questions 1.5 and 1.7), we provide two sufficient conditions. The first one is very simple: suppose that a Banach space  $Y$  has the approximation property and every finite-dimensional subspace of it is contained in another subspace of  $Y$  having property  $B^k$ . Then, the whole space  $Y$  has property  $B^k$ . The second sufficient condition deals with the existence of a bounded net of projections converging to the identity in the strong operator topology and such that their ranges have property  $B^k$ . These two ideas lead to a list including most of the known examples of Banach spaces with property  $B^k$ : preduals of  $L_1(\mu)$  spaces (in particular,  $C_0(L)$  spaces), real  $L_1(\mu)$  spaces, and polyhedral Banach spaces with the approximation property (in particular, subspaces of  $c_0$  with the approximation property, both in the real and in the complex case). Besides of these examples, uniform algebras have been very recently proved to have property  $B^k$ .

We would like to finish this introduction with some comments about the complex case of Bishop-Phelps theorem and its relation to Question 1.7 for the two dimensional real Hilbert space. First, let us comment that there is a complex version of the Bishop-Phelps theorem, easily deductible from the real case, which states that for every complex Banach space  $X$ , complex-linear norm-attaining functionals from  $X$  into  $\mathbb{C}$  are dense in  $X^* = L(X, \mathbb{C})$  (see [32] or [33, §2]). But this does not imply that  $\mathbb{C}$  viewed as the real two-dimensional Hilbert space have Lindenstrauss property B, as it does not allow to work with operators which are not complex-linear. On the other hand, V. Lomonosov showed in 2000 [27] that there is a complex Banach space  $X$  and a (non-complex symmetric) closed convex bounded subset  $C$  of  $X$  such that there is no element in  $X^*$  attaining the supremum of its modulus on  $C$ .

## 2. NEGATIVE EXAMPLES

Our goal here is to present the recent results in [28] providing examples of compact operators which cannot be approximated by norm-attaining operators. The key idea is to combine the approximation property with the following simple geometric idea. Let  $X$  and  $Y$  be Banach spaces and let  $T \in L(X, Y)$  with  $\|T\| = 1$ . Suppose that  $T$  attains its norm at a point  $x_0 \in S_X$  which is not an extreme point of  $B_X$ , let  $z \in X$  be such that  $\|x_0 \pm z\| \leq 1$  and observe that  $\|Tx_0 \pm Tz\| \leq 1$ . If  $Tx_0 \in S_Y$  is an extreme point of  $B_Y$ , then  $Tz = 0$ . Summarizing:

$$\|x_0 \pm z\| \leq 1 \text{ and } Tx_0 \text{ is an extreme point of } B_Y \implies Tz = 0. \quad (1)$$

Two remarks are pertinent. First, the most quantity of vectors  $z$ 's we may use in the above equation, the most information we get about  $T$ . Second, to get that  $Tx_0 \in S_Y$  is an extreme point,

the easiest way is to require that all points in  $S_Y$  are extreme points of  $B_Y$ , that is, that  $Y$  is *strictly convex*. Observe that geometrically, this argument means that an operator into a strictly

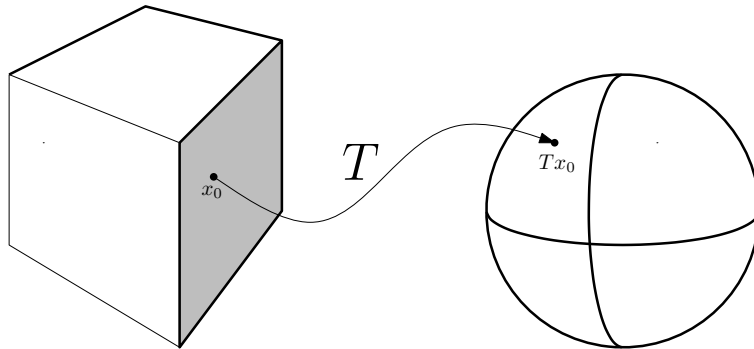


FIGURE 1. An operator attaining its norm into a strictly convex space

convex Banach space which attains its norm in the interior of a face of the unit sphere carries the whole sphere to the same point (see Figure 1).

Next, suppose that for every  $x_0 \in S_X$ , the set of  $z$ 's working in (1) generate a finite-codimensional subspace. Then  $T$  has finite-rank. We have proved the key ingredient for all the examples.

**Lemma 2.1** (Geometrical key lemma, Lindenstrauss). *Let  $X, Y$  be Banach spaces. Suppose that for every  $x_0 \in S_X$  the closed linear span of the set of those  $z \in X$  such that  $\|x_0 \pm z\| \leq 1$  is finite-codimensional and that  $Y$  is strictly convex. Then,  $NA(X, Y) \subseteq F(X, Y)$ .*

This result was used by J. Lindenstrauss [25] to give a direct proof of the fact that  $c_0$  fails property A (this result is deductible from earlier parts in the same paper). How was this done? Just observing that for every  $x_0 \in S_{c_0}$ , there is  $N \in \mathbb{N}$  such that  $\|x_0 \pm \frac{1}{2}e_n\| \leq 1$  for every  $n \geq N$  and that there is a strictly convex Banach space  $Y$  isomorphic to  $c_0$ . The isomorphism from  $c_0$  into  $Y$  is not compact and so it cannot be approximated by norm-attaining operators by Lemma 2.1. The argument also shows that strictly convex renorming of  $c_0$  do not have property B. Similar arguments like the above ones, replacing  $c_0$  by suitable Banach spaces, have been used by T. Gowers [18] to show that  $\ell_p$  fails Lindenstrauss property B, and by M. Acosta [1, 2] to show that no infinite-dimensional strictly convex Banach space nor infinite-dimensional  $L_1(\mu)$  space has Lindenstrauss property B.

But let us return to compact operators. Instead of looking for a non-compact operator which cannot be approximated by norm-attaining operators using Lemma 2.1, the idea in [28] is to use domain spaces without the approximation property. The key idea there is that Lindenstrauss' argument for  $c_0$  extends to all its closed subspaces.

**Lemma 2.2** ([28]). *Let  $X$  be a closed subspace of  $c_0$ . Then, for every  $x_0 \in S_X$  the closed linear span of the set of those  $z \in X$  such that  $\|x_0 \pm z\| \leq 1$  generates a finite-codimensional subspace.*

*Proof.* Fix  $x_0 \in S_X$ . As  $x_0 \in c_0$ , there is  $N \in \mathbb{N}$  such that  $|x_0(n)| < 1/2$  for every  $n \geq N$ . Now, consider the finite-codimensional subspace of  $X$  given by

$$Z = \{z \in X : z(i) = 0 \text{ for } 1 \leq i \leq N\},$$

and observe that for every  $z \in Z$  with  $\|z\| \leq 1/2$ , we have  $\|x_0 \pm z\| \leq 1$ . □

Therefore, combining this result with Lemma 2.1, we get the following.

**Corollary 2.3** ([28]). *Let  $X$  be a closed subspace of  $c_0$  and let  $Y$  be a strictly convex Banach space. Then,  $NA(X, Y) \subseteq F(X, Y)$ .*

What is next? We just have to recall the approximation property and use Proposition 1.4. Pick a closed subspace  $X$  of  $c_0$  without the approximation property. Then,  $X^*$  also fails the approximation property, so there is a Banach space  $Y$  and  $T \in K(X, Y)$  which is not in the closure of  $F(X, Y)$ . Considering  $Y$  as the range of  $T$ , which is separable, we may suppose that  $Y$  is strictly convex (using an equivalent renorming, see [10, §II.2]). Now, Corollary 2.3 gives that  $T$  cannot be approximated by norm-attaining operators. We have proved.

**Fact 2.4** ([28]). *There exist compact operators between Banach spaces which cannot be approximated by norm-attaining operators.*

Actually, we have proved the following result.

**Proposition 2.5** ([28]). *Every closed subspace of  $c_0$  whose dual does not have the approximation property fails property  $A^k$ .*

An specially interesting example can be given using an space constructed by W. Johnson and G. Schechtman [22, Corollary JS, p. 127] which is a closed subspace of  $c_0$  with Schauder basis whose dual fails the approximation property.

**Example 2.6** ([28]). There exist a subspace of  $c_0$  with Schauder basis failing property  $A^k$ .

Compare this result with Corollary 3.13.

Next we would like to produce more examples without property  $A^k$ . We say that the norm of a Banach space  $X$  *locally depends upon finitely many coordinates* if for every  $x \in X$ , there exist  $\varepsilon > 0$ , a finite subset  $\{f_1, f_2, \dots, f_n\}$  of  $X^*$  and a continuous function  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $\|y\| = \varphi(f_1(y), f_2(y), \dots, f_n(y))$  for every  $y \in X$  such that  $\|x - y\| < \varepsilon$ . We refer to [16] and references therein for background. Closed subspaces of  $c_0$  have this property [16, Proposition III.3] and, conversely, every infinite-dimensional Banach space whose norm locally depends upon finitely many coordinates contains an isomorphic copy of  $c_0$  [16, Corollary IV.5]. It is easy to extend the proof of Lemma 2.2 to this case and then use Lemma 2.1 to get the following extension of Proposition 2.5.

**Proposition 2.7.** *Let  $X$  be a Banach space whose norm locally depends upon finitely many coordinates and fails the approximation property. Then  $X$  does not have property  $A^k$ .*

*Proof.* Fix  $x_0 \in S_X$ . For  $\varepsilon = 1/2$ , consider  $\{f_1, f_2, \dots, f_n\} \subset X^*$  and  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  given by the hypothesis. Let  $Z = \bigcap_{i=1}^n \ker f_i$ , which is finite-codimensional. For  $z \in Z$  with  $\|z\| < 1/2$ , we have that

$$\|x_0 \pm z\| = \varphi(f_1(x_0 \pm z), f_2(x_0 \pm z), \dots, f_n(x_0 \pm z)) = \varphi(f_1(x_0), f_2(x_0), \dots, f_n(x_0)) = 1.$$

Now, we may repeat the proof of Proposition 2.5 using the above fact instead of Lemma 2.2.  $\square$

We now deal with the range space. Let  $Y$  be a strictly convex Banach space without the approximation property. By a result of A. Grothendieck [21, Theorem 18.3.2], there is a closed subspace  $X$  of  $c_0$  such that  $F(X, Y)$  is not dense in  $K(X, Y)$ . This, together with Corollary 2.3 show that  $Y$  fails property  $B^k$ .

**Proposition 2.8** ([28]). *Every strictly convex Banach space without the approximation property fails property  $B^k$ .*

The same kind of arguments can be applied to subspaces of complex  $L_1(\mu)$  spaces without the approximation property. Indeed, every subspace  $Y$  of the complex  $L_1(\mu)$  space is *complex strictly convex* (see [20, Proposition 3.2.3]) and this means that for every  $y \in Y$  with  $\|y\| = 1$  and  $z \in Y$ , the condition  $\|y + \theta z\| \leq 1$  for every  $\theta \in \mathbb{C}$  with  $|\theta| = 1$  implies  $z = 0$ . An obvious adaption of the proof of Corollary 2.3 and the proof of the above proposition provide the following result.

**Proposition 2.9** ([28]). *Every closed subspace of the complex space  $L_1(\mu)$  without the approximation property fails property  $B^k$ .*

It is even possible to produce a Banach space  $Z$  and a compact endomorphism of  $Z$  which cannot be approximated by norm-attaining operators.

**Example 2.10** ([28]). There exists a Banach space  $Z$  and a compact operator from  $Z$  into itself which cannot be approximated by norm-attaining operators.

This is an immediate consequence of the following lemma, which is proved in [28, Theorem 8].

**Lemma 2.11** ([28]). *Let  $X, Y$  be Banach spaces and let  $Z = X \oplus_\infty Y$ . If  $NA(Z, Z) \cap K(Z, Z)$  is dense in  $K(Z, Z)$  then  $NA(X, Y) \cap K(X, Y)$  is dense in  $K(X, Y)$ .*

We finish the section about negative examples with an easy consequence of Lemma 2.3 about Lindenstrauss property A which we do not know whether it was previously known.

**Proposition 2.12.** *No infinite-dimensional closed subspace of  $c_0$  satisfies Lindenstrauss property A.*

*Proof.* Let  $X$  be a infinite-dimensional closed subspace of  $c_0$  and let  $Y$  a strictly convex renorming of  $c_0$ . By Corollary 2.3,  $NA(X, Y) \subset F(X, Y)$ , but as  $X$  is infinite-dimensional, the inclusion from  $X$  into  $Y$  is non-compact and, therefore, it cannot be approximated by norm-attaining operators.  $\square$

Let us observe that this result solves in the negative Question 13 of [28] as it is written there. As we have done here, it can be solved using arguments from that paper. But, actually, there is an errata in the statement of this question and the exact question that the author wanted to propose is about property  $A^k$  (see Question 3.14).

### 3. POSITIVE RESULTS ON DOMAIN SPACES

As we commented in the introduction, every compact operator whose domain is reflexive attains its norm. In particular, reflexive spaces have property  $A^k$ . To get more examples, we first recall that even it is not known whether Lindenstrauss property A implies property  $A^k$  (Question 1.2), the usual way to prove property A for a Banach space  $X$  is by showing that every operator from  $X$  can be approximated by compact perturbations of it attaining the norm. Therefore, the known examples of spaces with property A actually have property  $A^k$ . The main examples of this kind are spaces with the Radon-Nikodým property (J. Bourgain 1977 [7]) and those with property  $\alpha$  (W. Schachermayer 1983 [34]). Let us start with the Radon-Nikodým property, which does not need presentation as it is one of the classical properties studied in geometry of Banach spaces, but let us just recall that reflexive spaces and  $\ell_1$  have it. Bourgain's result is much deeper than what we are going to present here and the paper also contains a kind of converse result. The paper [7] is consider one of the cornerstone results in the theory of norm-attaining operators and connect

this theory with the geometric concept of dentability. The proof given by Bourgain is based in a variational principle introduced in the same paper and approximates every operator by nuclear (hence compact) perturbations of it attaining the norm. The variational principle was extended to the non-linear case in 1978 by C. Stegall [36] getting rank-one perturbations. We are not going to present the proof here; we refer the reader to the 1986 paper [37] of C. Stegall for a simpler proof and applications.

**Theorem 3.1** (Bourgain). *The Radon-Nikodým property implies property  $A^k$ .*

Let us introduce the definition of property  $\alpha$ , probably less known. A Banach space  $X$  has *property  $\alpha$*  if there are two sets  $\{x_i : i \in I\} \subset S_X$ ,  $\{x_i^* : i \in I\} \subset S_{X^*}$  and a constant  $0 \leq \rho < 1$  such that the following conditions hold:

- (i)  $x_i^*(x_i) = 1, \forall i \in I$ .
- (ii)  $|x_i^*(x_j)| \leq \rho < 1$  if  $i, j \in I, i \neq j$ .
- (iii)  $B_X$  is the absolutely closed convex hull of  $\{x_i : i \in I\}$  or, equivalently, for every  $x^* \in X^*$ ,  $\|x^*\| = \sup_{i \in I} |x^*(x_i)|$ .

This property was introduced by W. Schachermayer [34] as an strengthening of a property used by J. Lindenstrauss in the seminal paper [25]. We refer to [29] and references therein for more information and background. The prototype of Banach space with property  $\alpha$  is  $\ell_1$ .

**Proposition 3.2** (Schachermayer). *Property  $\alpha$  implies property  $A^k$ .*

*Proof.* Let  $X$  be a Banach space with property  $\alpha$  with constant  $\rho \in [0, 1)$  and let  $Y$  be a Banach space. Fix  $T \in K(X, Y)$ ,  $T \neq 0$ , and  $\varepsilon > 0$ . We find  $i \in I$  such that

$$\|Tx_i\| > \frac{\|T\|(1 + \varepsilon\rho)}{1 + \varepsilon}$$

and define  $S \in K(X, Y)$  by

$$Sx = Tx + \varepsilon x_i^*(x)Tx_i \quad (x \in X).$$

Then  $\|Sx_i\| > \|T\|(1 + \varepsilon\rho)$ , while  $\|Sx_j\| \leq \|T\|(1 + \varepsilon\rho)$  for every  $j \neq i$ . This gives that  $S \in NA(X, Y)$  and it is clear that  $\|T - S\| \leq \varepsilon\|T\|$ .  $\square$

The main utility of property  $\alpha$  is that many Banach spaces can be renormed with property  $\alpha$  (B. Godun and S. Troyanski 1983 [17], previous results by W. Schachermayer [34]), so we obtain that property  $A^k$  is isomorphically innocuous in most cases.

**Corollary 3.3.** *Every Banach space  $X$  with a biorthogonal system whose cardinality is equal to the density character of  $X$  can be equivalently renormed to have property  $A^k$ . In particular, this happens if  $X$  is separable.*

Let us comment that property  $A^k$  for a Banach space  $X$  does not imply that for every Banach space  $Y$ , norm-attaining finite-rank operators from  $X$  into  $Y$  are dense in  $K(X, Y)$ . Indeed, by the above, all reflexive spaces have property  $A^k$ , while there are reflexive spaces whose duals fail the approximation property (even subspaces of  $\ell_p$  for  $p \neq 2$ ).

Let us pass to discuss on results which are specific of property  $A^k$  and do not follow from property A. All results we know of this kind follow from the same general principle: an stronger version of the approximation property of the dual, and so give partial answers to Question 1.6. The argument appeared in the 1979 paper by J. Johnson and J. Wolfe [23, Lemma 3.1]. The (easy) proof of it appeared in [28, Proposition 11].



**Proposition 3.4** (Johnson-Wolfe). *Let  $X$  be a Banach space. Suppose there is a net  $(P_\alpha)$  of finite-rank contractive projections on  $X$  such that  $(P_\alpha^*)$  converges to  $\text{Id}$  in the strong operator topology (i.e. for every  $x^* \in X^*$ ,  $(P_\alpha^*x^*) \rightarrow x^*$  in norm). Then  $X$  has property  $A^k$ .*

*Sketch of the proof.* Let  $Y$  be a Banach space and consider  $T \in K(X, Y)$ . First, all operators  $TP_\alpha$  attain their norm since  $TP_\alpha(B_X) = T(B_{P_\alpha(X)})$  and  $B_{P_\alpha(X)}$  is compact. By using the compactness of  $T$ , it can be easily proved that  $(P_\alpha^*T^*) \rightarrow T^*$ , so  $(TP_\alpha) \rightarrow T$ .  $\square$

This result was used in the cited paper [23] to get examples of spaces with property  $A^k$ . In [23, Proposition 3.2] it is shown that real  $C(K)$  spaces ensure the property given in Proposition 3.4, but the proof easily extends to real or complex  $C_0(L)$  spaces.

**Example 3.5** (Johnson-Wolfe). For every locally compact Hausdorff space  $L$ , the space  $C_0(L)$  has property  $A^k$ .

In 1976, J. Diestel and J. Uhl [12] showed that  $L_1(\mu)$  spaces have property  $A^k$ .

**Example 3.6** (Diestel-Uhl). For every positive measure  $\mu$ , the space  $L_1(\mu)$  has property  $A^k$ .

If the measure is finite, the above result also follows from Proposition 3.4. The general case can be obtained from the finite measure case using the following lemma, which is just the immediate adaptation to compact operators of [31, Lemma 2].

**Lemma 3.7.** *Let  $\{X_i : i \in I\}$  be a non-empty family of Banach spaces and let  $X$  denote the  $\ell_1$ -sum of the family. Then  $X$  has property  $A^k$  if and only if  $X_i$  does for every  $i \in I$ .*

Compare Examples 3.5 and 3.6 with the result by W. Schachermayer that  $NA(L_1[0, 1], C[0, 1])$  is not dense in  $L(L_1[0, 1], C[0, 1])$  [35].

More examples of spaces with property  $A^k$  can be deduced from Proposition 3.4. The first set is the family of preduals of  $\ell_1$ .

**Corollary 3.8.** *Let  $X$  be a Banach space such that  $X^*$  is isometrically isomorphic to  $\ell_1$ . Then  $X$  has property  $A^k$ .*

*Proof.* Let  $(x_n^*)_{n \in \mathbb{N}}$  be a Schauder basis of  $X^*$  isometrically equivalent to the usual  $\ell_1$ -basis and for every  $n \in \mathbb{N}$ , let  $Y_n$  the linear span of  $\{x_1^*, \dots, x_n^*\}$ . In the proof of [15, Corollary 4.1], a sequence of  $w^*$ -continuous contractive projections  $Q_n : X^* \rightarrow X^*$  with  $Q_n(X^*) = Y_n$  is constructed. The  $w^*$ -continuity of  $Q_n$  provides us with a sequence of finite-rank contractive projections on  $X$  satisfying the hypothesis of Proposition 3.4. Let us note that the results in [15] are given in the real case, but in the case we are using the proofs work in the complex case as well.  $\square$

We do not know whether the corollary above extends to isometric preduals of arbitrary  $L_1(\mu)$  spaces.

**Question 3.9.** Do all preduals of  $L_1(\mu)$  spaces have property  $A^k$ ?

Proposition 3.4 also applied to spaces with a shrinking monotone Schauder basis. Recall that a Schauder basis of a Banach space  $X$  is said to be *shrinking* if its sequence of coordinate functionals is a Schauder basis of  $X^*$ .

**Corollary 3.10.** *Every Banach space with a shrinking monotone Schauder basis has property  $A^k$ .*

It is well-known that an unconditional Schauder basis of a Banach space is shrinking if the space does not contain  $\ell_1$  (see [5, Theorem 3.3.1] for instance), so the following particular case appears.

**Corollary 3.11.** *Let  $X$  be a Banach space with unconditional monotone Schauder basis which does not contain  $\ell_1$ . Then  $X$  has property  $A^k$ .*

For the class of  $M$ -embedded spaces, this last result can be improved removing the unconditionality condition on the basis, by using the 1988 result of G. Godefroy and P. Saphar that Schauder bases in  $M$ -embedded spaces with basis constant less than 2 are shrinking (see [19, Corollary III.3.10], for instant). We recall that a Banach space  $X$  is said to be  $M$ -embedded if  $X^\perp$  is the kernel of an  $L_1$ -projection in  $X^*$  (i.e.  $X^* = X^\perp \oplus Z$  for some  $Z$  and  $\|x^\perp + z\| = \|x^\perp\| + \|z\|$  for every  $x^\perp \in X^\perp$  and  $z \in Z$ ). We refer the reader to [19] for background.

**Corollary 3.12.** *Every  $M$ -embedded space with monotone Schauder basis has property  $A^k$ .*

As  $c_0$  is an  $M$ -embedded space [19, Examples III.1.4] and  $M$ -embeddedness passes to closed subspaces [19, Theorem III.1.6], we get the following interesting particular case

**Corollary 3.13** ([28, Corollary 12]). *Every closed subspace of  $c_0$  with monotone Schauder basis has property  $A^k$ .*

Compare this result with the example of a closed subspace of  $c_0$  with Schauder basis failing property  $A^k$  (Example 2.6). It is an interesting question whether Corollary 3.13 extends to every closed subspace of  $c_0$  with the metric approximation property.

**Question 3.14.** Does every closed subspace of  $c_0$  with the metric approximation property have property  $A^k$ ?

We finish this section with an easy observation about Questions 1.2 and 1.6: we do not know whether either property A or the approximation property of the dual is sufficient to get property  $A^k$ , but the two properties together are.

**Proposition 3.15.** *Let  $X$  be a Banach space having Lindenstrauss property A and such that  $X^*$  has the approximation property. Then,  $X$  has property  $A^k$ .*

*Proof.* As  $X^*$  has the approximation property, it is enough to show that every finite-rank operator starting from  $X$  can be approximated by finite-rank norm-attaining operators (see Proposition 1.4.b). Fix a finite-rank operator  $T : X \rightarrow Y$  and write  $Z = T(X)$  which is finite-dimensional. As  $X$  has property A, we have that  $NA(X, Z) = NA(X, Z) \cap K(X, Z)$  is dense in  $L(X, Z) = K(X, Z)$ , so we may find a sequence  $T_n \in NA(X, Z)$  converging to  $T$  (viewed as an operator from  $X$  into  $Z$ ). It is now enough to consider the operators  $T_n$  as elements of  $NA(X, Y) \cap K(X, Y)$ .  $\square$

#### 4. POSITIVE RESULTS ON RANGE SPACES

The first example of a Banach space with property  $B^k$  is the base field by the classical Bishop-Phelps theorem [6]. This implies that every rank-one operator can be approximated by norm-attaining rank-one operators. It is not known whether this extends in general to finite-rank operators (Question 1.7 or even to rank-two operators).

To get more positive examples, and analogously to what is done in the previous section, we start by recalling that it is not known whether Lindenstrauss property B implies property  $B^k$  (Question 1.3) but, nevertheless, the usual way to prove property B for a Banach space  $X$

is by showing that every operator arriving to  $Y$  can be approximated by compact perturbations of it attaining the norm. This is what happens with property  $\beta$ , which is the main example of this kind. A Banach space  $Y$  has *property  $\beta$*  (J. Lindenstrauss 1963 [25]) if there are two sets  $\{y_i : i \in I\} \subset S_Y$ ,  $\{y_i^* : i \in I\} \subset S_{Y^*}$  and a constant  $0 \leq \rho < 1$  such that the following conditions hold:

- (i)  $y_i^*(y_i) = 1, \forall i \in I$ .
- (ii)  $|y_i^*(y_j)| \leq \rho < 1$  if  $i, j \in I, i \neq j$ .
- (iii) For every  $y \in Y$ ,  $\|y\| = \sup_{i \in I} |y_i^*(y)|$  or, equivalently,  $B_{Y^*}$  is the absolutely weakly\*-closed convex hull of  $\{y_i^* : i \in I\}$ .

We refer to [29] and references therein for more information and background. Examples of Banach spaces with property  $\beta$  are closed subspace of  $\ell_\infty(I)$  containing the canonical copy of  $c_0(I)$  and real finite-dimensional Banach spaces whose unit ball is a polyhedrum (actually, these are the only real finite-dimensional spaces with property  $\beta$ ).

**Proposition 4.1** (Lindenstrauss). *Property  $\beta$  implies property  $B^k$ .*

*Proof.* Let  $Y$  be a Banach space with property  $\beta$  with constant  $\rho \in [0, 1)$  and let  $X$  be a Banach space. Fix  $T \in K(X, Y)$  with  $\|T\| = 1$  and  $\varepsilon > 0$ , and consider  $\delta > 0$  such that

$$\left(1 + \frac{\varepsilon}{2}\right)(1 - \delta) > 1 + \rho\left(\frac{\varepsilon}{2} + \delta\right).$$

Next, we find  $i \in I$  such that

$$\|T^*y_i^*\| > 1 - \delta$$

and apply Bishop-Phelps theorem [6] to get  $x_0^* \in X^*$  attaining its norm such that

$$\|x_0^*\| = \|T^*y_i^*\| \quad \text{and} \quad \|T^*y_i^* - x_0^*\| < \delta.$$

Define  $S \in K(X, Y)$  by

$$Sx = Tx + \left[\left(1 + \frac{\varepsilon}{2}\right)x_0^*(x) - y_i^*(Tx)\right] \quad (x \in X).$$

Then, it is immediate to check that  $\|T - S\| < \frac{\varepsilon}{2} + \delta < \varepsilon$ . Now, we have that

$$S^*y_i^* = \left(1 + \frac{\varepsilon}{2}\right)x_0^*, \quad \|S^*y_i^*\| \geq \left(1 + \frac{\varepsilon}{2}\right)(1 - \delta),$$

and for  $j \neq i$ , we have

$$\|S^*y_j^*\| \leq 1 + \rho\left(\frac{\varepsilon}{2} + \delta\right).$$

This shows that  $S^*$  attains its norm at  $y_i^* \in S_{Y^*}$ . As  $S^*y_i^* = \left(1 + \frac{\varepsilon}{2}\right)x_0^* \in X^*$  also attains its norm, it follows that  $S \in NA(X, Y)$ . □

R. Partington proved in 1982 [30] that every Banach space can be renormed with property  $\beta$ , so we obtain that property  $B^k$  is isomorphically innocuous.

**Corollary 4.2.** *Every Banach space can be equivalently renormed to have property  $B^k$ .*

Let us comment that property  $B^k$  for a Banach space  $Y$  does not imply that for every Banach space  $X$ , norm-attaining finite-rank operators from  $X$  into  $Y$  are dense in  $K(X, Y)$ . Indeed, by the above, there are many Banach spaces with property  $\beta$  (and so  $B^k$ ) but without the approximation property.

Let us comment that the knowledge about property B is more unsatisfactory than the one about property A. Besides of property  $\beta$ , the only sufficient condition we know for property B is the so-called property quasi- $\beta$ , introduced in 1996 by M. Acosta, F. Agirre and R. Payá [4]. We are not going to give the definition of this property here (it is a weakening of property  $\beta$ ), but let us just comment that there are even examples of finite-dimensional spaces with property quasi- $\beta$  which do not have property  $\beta$ . Again, the proof of the fact that property quasi- $\beta$  implies property B can be adapted to the compact case.

**Proposition 4.3** (Acosta-Aguirre-Payá). *Property quasi- $\beta$  implies property  $B^k$ .*

Let us pass to discuss on results which are specific of property  $B^k$  and are not related to property B. Most results we know of this kind follows from two general principles. The first one is the following, whose proof is straightforward.

**Proposition 4.4.** *Let  $Y$  be a Banach space with the approximation property. Suppose that for every finite-dimensional subspace  $W$  of  $Y$ , there is a closed subspace  $Z$  having Lindenstrauss property B such that  $W \leq Z \leq Y$ . Then  $Y$  has property  $B^k$ .*

This result applied to those Banach space with the approximation property satisfying that all its finite-dimensional subspaces have Lindenstrauss property B. This is the case of the so-called polyhedral spaces. We recall that a *real* Banach space is said to be *polyhedral* if the unit balls of all of its finite-dimensional subspaces are polyhedra (i.e. the convex hull of finitely many points). A typical example of polyhedral space is  $c_0$  and hence, so are its closed subspaces. We refer to [14] for background on polyhedral spaces. Real finite-dimensional polyhedral spaces clearly fulfil property  $\beta$  [25].

**Corollary 4.5.** *A polyhedral Banach space with the approximation property has property  $B^k$ .*

To deal with the complex case, we observe that polyhedrality is equivalent to the fact that the norm of each finite-dimensional subspace can be calculated as the maximum of the absolute value of finitely many functionals, and this implies property  $\beta$  also in the complex case. With this idea, the result above can be extended to the complex case.

**Proposition 4.6.** *Let  $Y$  be a complex Banach space with the approximation property such that for every finite-dimensional subspace, the norm of the subspace can be calculated as the maximum of the modulus of finitely many functionals. Then  $Y$  has property  $B^k$ .*

It is easy to see that closed subspaces of  $c_0$  satisfy this condition (see [16, Proposition III.3]).

**Example 4.7.** Closed subspaces of real or complex  $c_0$  with the approximation property have property  $B^k$ .

The main limitation of Proposition 4.4 is that we only know few examples of finite-dimensional Banach spaces with property B. If we use property  $\beta$  (equivalent here to polyhedrality), what we actually are requiring in that proposition is that all finite-dimensional subspaces have property B. To deal with more examples, we present the second general principle to get property  $B^k$ , which appeared in [23, Lemma 3.4].

**Proposition 4.8** (Johnson-Wolfe). *A Banach space  $Y$  has property  $B^k$  provided that there is a net of projections  $\{Q_\lambda\}$  in  $Y$  such that  $\sup_\lambda \|Q_\lambda\| < \infty$  and converging to  $\text{Id}_Y$  in the strong operator topology (i.e.  $Q_\lambda(y) \rightarrow y$  in norm for every  $y \in Y$ ) and such that  $Q_\lambda(Y)$  has property  $B^k$  for every  $\lambda$ .*

*Proof.* Let  $X$  be a Banach space and fix  $T \in K(X, Y)$ . The compactness of  $T$  allows to show that  $Q_\lambda T$  converges in norm to  $T$ . Therefore, it is enough to prove that each  $Q_\lambda T$  can be approximated by norm-attaining compact operators, but this is immediate since  $Q_\lambda T$  arrives to the space  $Q_\lambda(Y)$  which has property  $B^k$ .  $\square$

Observe that if in the proposition above the projections have finite-dimensional range, what we are requiring is an stronger form of the approximation property.

This result was used in [23] to show that (real or complex) isometric preduals of  $L_1(\mu)$  spaces have property  $B^k$ . Indeed, by a classical result of A. Lazar and J. Lindenstrauss, the projections  $Q_\lambda$  can be chosen to have  $\|Q_\lambda\| = 1$  and  $Q_\lambda(Y) \equiv \ell_\infty^{n_\lambda}$  (see [24, Chapter 7] for instance).

**Corollary 4.9** (Johnson-Wolfe). *Every predual of a real or complex  $L_1(\mu)$  space has property  $B^k$ .*

In particular,

**Example 4.10** (Johnson-Wolfe). Real or complex  $C_0(L)$  spaces have property  $B^k$ .

Proposition 4.8 also applied to *real*  $L_1(\mu)$  spaces. Indeed, it is enough to consider conditional expectations to finite collections of subsets of positive and finite measure, and use the density of simple functions in  $L_1(\mu)$ . Doing that, we get a net  $(Q_\lambda)$  of norm-one projections converging to the identity in the strong operator topology such that  $Q_\lambda(L_1(\mu)) \equiv \ell_1^{n_\lambda}$ . In the real case,  $\ell_1^n$  is polyhedral, so it has property  $\beta$ ; in the complex case,  $\ell_1^n$  does not have property  $\beta$  and it is not known whether it has property B.

**Example 4.11** (Johnson-Wolfe). For every positive measure  $\mu$ , the *real* space  $L_1(\mu)$  has property  $B^k$ .

We do not know whether the above result extends to the complex case.

The last class of spaces with property  $B^k$  that we would like to present is the one of uniform algebras. The result recently appeared in [9, R2 in p. 380] and the proof is completely different from the previous ones in this chapter, as the authors do not use any kind of approximation property, but a nice complex version of Urysohn lemma constructed in the same paper. We recall that a *uniform algebra* is a closed subalgebra of a complex  $C(K)$  space that separates the points of  $K$ .

**Proposition 4.12** (Cascales-Guirao-Kadets). *Every uniform algebra has property  $B^k$ .*

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## REFERENCES

- [1] M. D. ACOSTA, Denseness of norm-attaining operators into strictly convex spaces, *Proc. Royal Soc. Edinburgh* **129A** (1999), 1107–1114.
- [2] M. D. ACOSTA, Norm-attaining operators into  $L_1(\mu)$ , in: Function spaces, *Contemporary Math.* **232** (1999), 1–11.

- [3] M. D. ACOSTA, Denseness of norm attaining mappings, *Rev. R. Acad. Cien. Serie A. Mat.* **100** (2006), 9–30.
- [4] M. D. ACOSTA, F. J. AGUIRRE, AND R. PAYÁ, A new sufficient condition for the denseness of norm attaining operators, *Rocky Mountain J. Math.* **26** (1996), 407–418.
- [5] ALBIAC F., KALTON N., *Topics in Banach Space Theory*, Graduate Texts in Mathematics **233**, Springer, New York, 2006.
- [6] E. BISHOP AND R. R. PHELPS, A proof that every Banach space is subreflexive, *Bull. Amer. Math. Soc.* **67** (1961), 97–98.
- [7] J. BOURGAIN, On dentability and the Bishop-Phelps property, *Israel J. Math.* **28** (1977), 265271.
- [8] P. G. CASAZZA, Approximation properties, *Handbook of the geometry of Banach spaces*, Vol. I, 271–316, North-Holland, Amsterdam, 2001.
- [9] B. CASCALES, A. J. GUIRAO, AND V. KADETS, A Bishop-Phelps-Bollobás type theorem for uniform algebras, *Adv. Math.* **240** (2013), 370–382.
- [10] R. DEVILLE, G. GODEFROY, AND V. ZIZLER, *Smoothness and Renormings in Banach spaces*, Pitman Monographs 64, New York 1993.
- [11] J. DIESTEL, *Geometry of Banach Spaces: Selected Topics*, Springer-Verlag, New York, 1975.
- [12] J. DIESTEL AND J. J. UHL, The Radon-Nikodým theorem for Banach space valued measures, *Rocky Mountain J. Math.* **6** (1976), 1–46.
- [13] N. DUNFORD AND J. SCHWARTZ, *Linear Operators. Part I: General Theory*, Interscience, New York, 1957.
- [14] V. P. FONF, J. LINDENSTRAUSS, R. R. PHELPS, Infinite dimensional convexity. In: *Handbook of the geometry of Banach spaces*, Vol. I, pp. 599–670, North-Holland, Amsterdam, 2001.
- [15] I. GASPARIS, On contractively complemented subspaces of separable  $L_1$ -preduals, *Israel J. Math.* **128** (2002), 77–92.
- [16] G. GODEFROY, The Banach space  $c_0$ , *Extracta Math.* **16** (2001), 1–25.
- [17] B. V. GODUN, S. L. TROYANSKI, *Renorming Banach spaces with fundamental biorthogonal system*, *Contemporary Mathematics* **144** (1993), 119–126.
- [18] W. T. GOWERS, Symmetric block bases of sequences with large average growth, *Israel J. Math.* **69** (1990), 129–149.
- [19] P. HARMAND, D. WERNER, AND D. WERNER, *M-ideals in Banach spaces and Banach algebras*, Lecture Notes in Math. **1547**, Springer-Verlag, Berlin 1993.
- [20] V. I. ISTRATESCU, *Strict convexity and complex strict convexity*, Lecture Notes in Pure and Applied Mathematics, vol. **89**, Marcel Dekker, New York, 1984.
- [21] H. JARCHOW, *Locally convex spaces*, Teubner, Stuttgart, 1981.
- [22] W. B. JOHNSON AND T. OIKHBERG, Separable lifting property and extensions of local reflexivity, *Illinois J. Math.* **45** (2001), 123–137.
- [23] J. JOHNSON AND J. WOLFE, Norm attaining operators, *Studia Math.* **65** (1979), 7–19.
- [24] H. E. LACEY, *The isometric theory of classical Banach spaces*, Springer-Verlag, Berlin 1972.
- [25] J. LINDENSTRAUSS, On operators which attain their norm, *Israel J. Math.* **1** (1963), 139–148.
- [26] J. LINDENSTRAUSS AND L. TZAFRIRI, *Classical Banach Spaces I: Sequence Spaces*, Springer-Verlag, Berlin 1977.
- [27] V. LOMONOSOV, A counterexample to the Bishop-Phelps theorem in complex spaces, *Israel J. Math.* **115** (2000), 25–28.
- [28] M. MARTÍN, Norm-attaining compact operators, *J. Funct. Anal.* **267** (2014), 1585–1592.
- [29] J. P. MORENO, Geometry of Banach spaces with  $(\alpha, \epsilon)$ -property or  $(\beta, \epsilon)$ -property, *Rocky Mount. J. Math.* **27** (1997), 241–256.
- [30] J. R. PARTINGTON, Norm attaining operators, *Israel J. Math.* **43** (1982), 273–276.
- [31] R. PAYÁ AND Y. SALEH, Norm attaining operators from  $L_1(\mu)$  into  $L_\infty(\nu)$ , *Arch. Math.* **75** (2000) 380–388.
- [32] R. R. PHELPS, The Bishop-Phelps Theorem in Complex Spaces: An Open Problem. In: *Function spaces* (Edwardsville, IL, 1990), *Lecture Notes in Pure and Appl. Math.*, **136**, Dekker, New York, 1992, pp. 337–340.
- [33] R. R. PHELPS, The Bishop-Phelps theorem. In: *Ten mathematical essays on approximation in analysis and topology*, Elsevier B. V., Amsterdam, 2005, pp. 235244.
- [34] W. SCHACHERMAYER, Norm attaining operators and renormings of Banach spaces, *Israel J. Math.* **44** (1983), 201–212.
- [35] W. SCHACHERMAYER, Norm attaining operators on some classical Banach spaces, *Pacific J. Math.* **105** (1983), 427–438.
- [36] C. STEGALL, Optimization of functions on certain subsets of Banach spaces, *Math. Ann.* **236** (1978), 171–176.
- [37] C. STEGALL, Optimization and differentiation in Banach spaces, *Linear Alg. Appl.* **84** (1986), 191–211.