THE BISHOP-PHELPS-BOLLOBÁS THEOREM FOR OPERATORS ON $L_1(\mu)$

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Dedicated to the memory of Robert R. Phelps

ABSTRACT. In this paper we show that the Bishop-Phelps-Bollobás theorem holds for $\mathcal{L}(L_1(\mu), L_1(\nu))$ for all measures μ and ν and also holds for $\mathcal{L}(L_1(\mu), L_\infty(\nu))$ for every arbitrary measure μ and every localizable measure ν. Finally, we show that the Bishop-Phelps-Bollobás theorem holds for two classes of bounded linear operators from a real $L_1(\mu)$ into a real $C(K)$ if μ is a finite measure and K is a compact Hausdorff space. In particular, one of the classes includes all Bochner representable operators and all weakly compact operators.

1. INTRODUCTION

The celebrated Bishop-Phelps theorem of 1961 [12] states that for a Banach space X, every element in its dual space X^* can be approximated by ones that attain their norms. Since then, there has been an extensive research to extend this result to bounded linear operators between Banach spaces [14, 27, 33, 34, 37] and non-linear mappings [2, 7, 11, 16, 17, 30]. On the other hand, Bollobás [13], motivated by problems arising in the theory of numerical ranges, sharpened the Bishop-Phelps theorem in 1970, and got what is nowadays called the Bishop-Phelps-Bollobás theorem. Previously to presenting this result, let us introduce some notations. Given a (real or complex) Banach space X, we write B_X for the unit ball, S_X for its unit sphere, and X^* for the topological dual space of X. If Y is another Banach space, we write $\mathcal{L}(X,Y)$ to denote the space of all bounded linear operators from X into Y.

Theorem 1.1 (Bishop-Phelps-Bollobás theorem)**.** *Let X be a Banach space. If $x \in S_X$ and $x^* \in S_{X^*}$ satisfy $|x^*(x) - 1| < \varepsilon^2/4$, then there exist $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|x^* - y^*\| < \varepsilon$ and $\|x - y\| < \varepsilon$.*

In 2008, Acosta, Aron, García and Maestre [3] introduced the Bishop-Phelps-Bollobás property to study extensions of the theorem above to operators between Banach spaces.

Definition 1.2. Let X and Y be Banach spaces. The pair (X,Y) is said to have the *Bishop-Phelps-Bollobás property* (*BPBp*) if for every $0 < \varepsilon < 1$, there is $\eta(\varepsilon) > 0$ such that for every $T \in \mathcal{L}(X,Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ satisfying $\|T(x_0)\| > 1 - \eta(\varepsilon)$, there exist $y_0 \in S_X$ and $S \in \mathcal{L}(X,Y)$ with $\|S\| = 1$ satisfying the following conditions:

$$\|Sy_0\| = 1, \qquad \|y_0 - x_0\| < \varepsilon, \quad \text{and} \quad \|S - T\| < \varepsilon.$$

In this case, we also say that the Bishop-Phelps-Bollobás theorem holds for $\mathcal{L}(X,Y)$.

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This property has been studied by many authors. See for instance [5, 8, 10, 15, 18, 19, 28, 29]. Observe that the BPBp of a pair (X, Y) implies obviously that the set of norm attaining operators is dense in $\mathcal{L}(X, Y)$. However, its converse is false, as shown by the pair (X, Y) where X is the 2-dimensional L_1-space and Y is a strictly, but not uniformly convex space (see [3] or [10]). Let us also comment that the Bishop-Phelps-Bollobás theorem states that the pair $(X, \mathbb{K})$ has the Bishop-Phelps-Bollobás property for every Banach space X ($\mathbb{K}$ is the base scalar field $\mathbb{R}$ or $\mathbb{C}$).

In this paper we first deal with the problem of when the pair $(L_p(\mu), L_q(\nu))$ has the BPBp. Let us start with a presentation of both already known results and our new results. Iwanik [25] showed in 1979 that the set of norm-attaining operators from $L_1(\mu)$ to $L_1(\nu)$ is dense in the space $\mathcal{L}(L_1(\mu), L_1(\nu))$ for arbitrary measures μ and ν. Our first main result in this paper is that the pair $\mathcal{L}(L_1(\mu), L_1(\nu))$ has the BPBp. This is the content of section 3.

On the other hand, Aron et al. [9] showed that if μ is a σ-finite measure, then the pair $(L_1(\mu), L_\infty[0, 1])$ has the BPBp, improving a result of Finet and Payá [24] about the denseness of norm-attaining operators. We generalize this result in section 4 showing that $(L_1(\mu), L_\infty(\nu))$ has the BPBp for every measure μ and every localizable measure ν. This is also a strengthening of a result of Payá and Saleh [35] which stated only the denseness of norm-attaining operators.

One of the tools used to prove the results above is the fact that one can reduce the proofs to some particular measures. We develop this idea in section 2, where, as its first easy application, we extend to arbitrary measures μ the result in [18] that $(L_1(\mu), L_p(\nu))$ has the BPBp for σ-finite measures μ.

The following result summarizes all what is known about the BPBp for the pair $(L_p(\mu), L_q(\nu))$.

Corollary 1.3. *The pair $(L_p(\mu), L_q(\nu))$ has the BPBp*

(1) *for all measures μ and ν if $p = 1$ and $1 \leqslant q < \infty$.*
(2) *for any measure μ and any localizable measure ν if $p = 1$, $q = \infty$.*
(3) *for all measures μ and ν if $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$.*
(4) *for all measures μ and ν if $p = \infty$, $q = \infty$, in the real case.*

(1) and (2) follows from the results of this paper (Corollary 2.3, Theorem 3.1 and Theorem 4.1). Since $L_p(\mu)$ is uniformly convex when $1 < p < \infty$, (3) follows from [5, 29] in the σ-finite case, generalized here to arbitrary measures μ (Corollary 2.3). Finally, (4) follows from [4], because every L_∞ space is isometrically isomorphic to a $C(K)$ space.

As far as we know, the cases $(L_\infty(\mu), L_q(\nu))$ for $1 \leqslant q < \infty$ and the complex case of (4) remain open.

Let μ be a finite measure. Since any L_∞ space is isometrically isomorphic to $C(K)$ for some compact Hausdorff space K, it is natural to ask when $(L_1(\mu), C(K))$ has the BPBp. Schachermayer [38] showed that the set of all norm-attaining operators is not dense in $\mathcal{L}(L_1[0, 1], C[0, 1])$. Hence, $(L_1[0, 1], C[0, 1])$ cannot have the BPBp. On the other hand, Johnson and Wolfe [27] proved that if X is a Banach space and if either Y or Y^* is a $L_1(\mu)$ space, then every compact operator from X into Y can be approximated by norm-attaining finite-rank operators. They also showed that every weakly compact operator from $L_1(\mu)$ into $C(K)$ can be approximated by norm-attaining weakly compact ones. In this direction, Acosta et al. have shown that $(L_1(\mu), Y)$ has the BPBp for representable operators (in particular, for weakly compact operators) if (ℓ_1, Y) has the BPBp, and this is the case of $Y = C(K)$ [6].

On the other hand, Iwanik [26] studied two classes of bounded linear operators from a real $L_1(\mu)$ space to a real $C(K)$ space such that every element of each class can be approximated by norm-attaining elements, and showed that one of the classes strictly contains all Bochner representable operators and all weakly compact operators. In section 5, we deal with Bishop-Phelps-Bollobás versions of these Iwanik's results. In particular, we show that for every $0 < \varepsilon < 1$, there is $\eta(\varepsilon) > 0$ such that if $T \in \mathcal{L}(L_1(\mu), C(K))$ with $\|T\| = 1$ is Bochner representable (resp. weakly compact) and $f_0 \in S_{L_1(\mu)}$ satisfy $\|Tf_0\| > 1 - \eta(\varepsilon)$, then there is a Bochner representable (resp. weakly compact) operator $S \in \mathcal{L}(L_1(\mu), C(K))$ and $f \in S_{L_1(\mu)}$ such that $\|Sf\| = \|S\| = 1$, $\|S - T\| < \varepsilon$ and $\|f - f_0\| < \varepsilon$.

Let us finally comment that the proofs presented in sections 3 and 4 are written for the complex case. Their corresponding proofs for the real case are easily obtained, even easier, from the ones presented there.

2. Some preliminary results

We start with some terminologies and known facts about $L_1(\mu)$. Suppose that (Ω, Σ, μ) is an arbitrary measure space and put $X = L_1(\mu)$. Suppose G is a countable subset of X. Since the closed linear span $[G]$ of G is separable, we may assume that $[G]$ is the closed linear span of a countable set $\{\chi_{E_n}\}$ of characteristic functions of measurable subsets with finite positive measure. Let $E = \bigcup_n E_n$ and $Z = \{f\chi_E : f \in X\}$. Then, $Z = L_1(\mu|_E)$, where $\mu|_E$ is restriction of the measure μ to the σ-algebra $\Sigma|_E = \{E \cap A : A \in \Sigma\}$. Since $\mu|_E$ is σ-finite, Z is isometrically (lattice) isomorphic to $L_1(m)$ for some positive finite Borel regular measure m defined on a compact Hausdorff space by the Kakutani representation theorem (see [32, Theorem 9, § 14] for a reference). This space Z is called the *band* generated by G, and the *canonical band projection* $P : X \longrightarrow Z$, defined by $P(f) := f\chi_E$ for $f \in X$, satisfies $\|f\| = \|Pf\| + \|(\mathrm{Id} - P)f\|$ for all $f \in X$. For more details, we refer the reader to the classical books [32, 39].

Next, we state the following equivalent formulation of the BPBp from [10] which helps to better understand the property and will be useful for our preliminary results. Given a pair (X, Y) of Banach spaces, let

$$\Pi(X, Y) = \{(x, T) \in X \times \mathcal{L}(X, Y) : \|T\| = \|x\| = \|Tx\| = 1\}$$

and define, for $0 < \varepsilon < 1$,

$$\eta(X, Y)(\varepsilon) = \inf\{1 - \|Tx\| : x \in S_X,\, T \in \mathcal{L}(X, Y),\, \|T\| = 1,\, \operatorname{dist}\big((x, T), \Pi(X, Y)\big) \geqslant \varepsilon\},$$

where $\operatorname{dist}\big((x, T), \Pi(X, Y)\big) = \inf\{\max\{\|x - y\|, \|T - S\|\} : (y, S) \in \Pi(X, Y)\}$. Equivalently, for every $\varepsilon \in (0, 1)$, $\eta(X, Y)(\varepsilon)$ is the supremum of those $\xi \geqslant 0$ such that whenever $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x \in S_X$ satisfy $\|Tx\| \geqslant 1 - \xi$, then there exists $(y, S) \in \Pi(X, Y)$ with $\|T - S\| \leqslant \varepsilon$ and $\|x - y\| \leqslant \varepsilon$. It is clear that (X, Y) has the BPBp if and only if $\eta(X, Y)(\varepsilon) > 0$ for all $0 < \varepsilon < 1$.

Our first preliminary result deals with operators acting on an $L_1(\mu)$ space and shows that the proof of some results can be reduced to the case when μ is a positive finite Borel regular measure defined on a compact Hausdorff space.

Proposition 2.1. *Let Y be a Banach space. Suppose that there is a function $\eta : (0, 1) \longrightarrow (0, \infty)$ such that*

$$\eta\big(L_1(m), Y\big)(\varepsilon) \geqslant \eta(\varepsilon) > 0 \qquad (0 < \varepsilon < 1)$$

for every positive finite Borel regular measure m defined on a compact Hausdorff space. Then, for every measure μ, the pair $(L_1(\mu), Y)$ has the BPBp with $\eta\big(L_1(\mu), Y\big) \geqslant \eta$.

Moreover, if $Y = L_1(\nu)$ for an arbitrary measure ν, then it is enough to show that

$$\eta\big(L_1(m_1), L_1(m_2)\big)(\varepsilon) \geqslant \eta(\varepsilon) > 0 \qquad (0 < \varepsilon < 1)$$

for all positive finite Borel regular measures m_1 and m_2 defined on Hausdorff compact spaces in order to get that $(L_1(\mu), L_1(\nu))$ has the BPBp with $\eta\big(L_1(\mu), L_1(\nu)\big) \geqslant \eta$.

Proof. Let $0 < \varepsilon < 1$. Suppose that $T \in \mathcal{L}(L_1(\mu), Y)$ is a norm-one operator and $f_0 \in S_X$ satisfy that $\|Tf_0\| > 1 - \eta(\varepsilon)$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in X such that $\|f_n\| \leqslant 1$ for all n and $\lim_{n\to\infty} \|Tf_n\| = \|T\| = 1$. The band X_1 generated by $\{f_n : n \geqslant 0\}$ is isometric to $L_1(J, m)$ for a finite positive Borel regular measure m defined on a compact Hausdorff space J by the Kakutani representation theorem. Let T_1 be the restriction of T to X_1. Then $\|T_1\| = 1$ and $\|T_1 f_0\| > 1 - \eta(\varepsilon)$. By the assumption, there exist a norm-one operator $S_1 : X_1 \longrightarrow Y$ and $g \in S_{X_1}$ such that $\|S_1 g\| = 1$, $\|T_1 - S_1\| < \varepsilon$ and $\|f - g\| < \varepsilon$. Let P denote the canonical band projection from $L_1(\mu)$ onto X_1. Then $S := S_1 P + T(\mathrm{Id} - P)$ is a norm-one operator from $L_1(\mu)$ to Y, g can be viewed as a norm-one element in $S_{L_1(\mu)}$ (just extending by 0), $\|Sg\| = 1$, $\|S - T\| < \varepsilon$ and $\|f - g\| < \varepsilon$. This completes the proof of the first part of the proposition.

In the case when $Y = L_1(\nu)$, we observe that the image $T(X_1)$ is also contained in a band Y_1 of $L_1(\nu)$ which, again, is isometric to $L_1(m_2)$ for a finite positive Borel regular measure m_2 on a compact Hausdorff space J_2. Now, we work with the restriction of T to X_1 with values in Y_1, we follow the proof of the first part and finally we consider the operator S as an operator with values in $L_1(\nu)$ (just composing with the formal inclusion of Y_1 into $L_1(\nu)$). □

Since for every positive finite Borel regular measure m defined on a compact Hausdorff space, $L_1(m)$ is isometric to $L_1(\mu)$ for a probability measure μ, we get the following.

Corollary 2.2. *Let Y be a Banach space. Suppose that there is a strictly positive function $\eta : (0,1) \longrightarrow (0,\infty)$ such that $\eta\big(L_1(\mu_1), Y\big) \geqslant \eta$ for every probability measure μ_1. Then $(L_1(\mu), Y)$ has the BPBp for every measure μ, with $\eta\big(L_1(\mu), Y\big) \geqslant \eta$.*

Let us give the first application of the above results. For a σ-finite measure μ_1, it is shown in [18] that $(L_1(\mu_1), Y)$ has the BPBp if Y has the Radon-Nikodým property and (ℓ_1, Y) has the BPBp. By following the proof of [18, Theorem 2.2], we conclude that there is a strictly positive function $\eta_Y : (0,1) \longrightarrow (0,\infty)$ such that $\eta(L_1(\mu_1), Y) \geqslant \eta_Y$ for every probability measure μ_1. Therefore, the corollary above provides the same result without the assumption of σ-finiteness. We also recall that $L_q(\nu)$ is uniformly convex for all $1 < q < \infty$ and for all measures ν, so it has the Radon-Nikodým property and $(\ell_1, L_q(\nu))$ has the BPBp [3]. Hence we get the following.

Corollary 2.3. *Let μ be an arbitrary measure. If Y is a Banach space with the Radon-Nikodým property and such that (ℓ_1, Y) has the BPBp, then the pair $(L_1(\mu), Y)$ has the BPBp. In particular, $(L_1(\mu), L_q(\nu))$ has the BPBp for all $1 < q < \infty$ and all arbitrary measures ν.*

We now deal with operators with values on an ℓ_∞-sum of Banach spaces, giving the following result from [10] which we will use in section 4. Given a family $\{Y_j : j \in J\}$ of Banach spaces, we denote by $\Big[\bigoplus_{j\in J} Y_j\Big]_{\ell_\infty}$ the ℓ_∞-sum of the family.

Proposition 2.4 ([10])**.** *Let X be a Banach space and let $\{Y_j : j \in J\}$ be a family of Banach spaces and let $Y = \Big[\bigoplus_{j\in J} Y_j\Big]_{\ell_\infty}$ denote their ℓ_∞-sum. If $\inf\limits_{j\in J} \eta(X, Y_j)(\varepsilon) > 0$ for all $0 < \varepsilon < 1$, then (X, Y) has the BPBp with*

$$\eta(X, Y) = \inf_{j\in J} \eta(X, Y_j).$$

We will use this result for operators with values in $L_\infty(\nu)$. To present the result, we fist recall that given a localizable measure ν, we have the following representation

$$L_\infty(\nu) \equiv \Big[\bigoplus\nolimits_{j\in J} Y_j\Big]_{\ell_\infty}, \tag{1}$$

where each space Y_j is either 1-dimensional or of the form $L_\infty([0,1]^\Lambda)$ for some finite or infinite set Λ and $[0,1]^\Lambda$ is endowed with the product measure of the Lebesgue measures. For its background, see [35] and references therein. With this in mind, the following corollary follows from the proposition above.

Corollary 2.5. *Let X be a Banach space. Suppose that there is a strictly positive function $\eta : (0,1) \longrightarrow (0,\infty)$ such that*

$$\eta\big(X, L_\infty([0,1]^\Lambda)\big)(\varepsilon) \geqslant \eta(\varepsilon) \qquad (0 < \varepsilon < 1)$$

for every finite or infinite set Λ. Then the pair $(X, L_\infty(\nu))$ has the BPBp for every localizable measure ν with

$$\eta\big(X, L_\infty(\nu)\big)(\varepsilon) \geqslant \min\{\eta(\varepsilon), \varepsilon^2/2\} \qquad (0 < \varepsilon < 1).$$

The proof is just an application of Proposition 2.4, the representation formula given in (1) and the Bishop-Phelps-Bollobás theorem (Theorem 1.1).

Let us comment that the analogue of Proposition 2.4 is false for ℓ_1-sums in the domain space (see [10]), so Proposition 2.1 cannot be derived directly from the decomposition of $L_1(\mu)$ spaces analogous to (1).

Before finishing this section, we state the following lemma of [3] which we will frequently use afterwards.

Lemma 2.6 ([3, Lemma 3.3]). *Let $\{c_n\}$ be a sequence of complex numbers with $|c_n| \leqslant 1$ for every n, and let $\eta > 0$ be such that for a convex series $\sum \alpha_n$, $\operatorname{Re} \sum_{n=1}^{\infty} \alpha_n c_n > 1 - \eta$. Then for every $0 < r < 1$, the set $A := \{i \in \mathbb{N} : \operatorname{Re} c_i > r\}$, satisfies the estimate*

$$\sum_{i \in A} \alpha_i \geqslant 1 - \frac{\eta}{1-r}.$$

3. The Bishop-Phelps-Bollobás property of $(L_1(\mu), L_1(\nu))$

Our goal in this section is to prove the following result.

Theorem 3.1. *Let μ and ν be arbitrary measures. Then the pair $(L_1(\mu), L_1(\nu))$ has the BPBp. Moreover, there exists a strictly positive function $\eta : (0,1) \longrightarrow (0,\infty)$ such that*

$$\eta\big(L_1(\mu), L_1(\nu)\big)(\varepsilon) \geqslant \eta(\varepsilon) \qquad (0 < \varepsilon < 1).$$

By Proposition 2.1, it is enough to get the result for finite regular positive Borel measures defined on compact Hausdorff spaces. Therefore, Theorem 3.1 follows directly from the next result.

Theorem 3.2. *Let m_1 and m_2 be finite regular positive Borel measures on compact Hausdorff spaces J_1 and J_2, respectively. Let $0 < \varepsilon < 1$ and suppose that $T \in \mathcal{L}(L_1(m_1), L_1(m_2))$ with $\|T\| = 1$ and $f_0 \in S_{L_1(m_1)}$ satisfy $\|Tf_0\| > 1 - \frac{\varepsilon^{18}}{5^3 2^{27}}$. Then there are $S \in S_{L(L_1(m_1),L_1(m_2))}$ and $g \in S_{L_1(m_1)}$ such that*

$$\|Sg\| = 1, \quad \|f - g\| < 4\varepsilon \quad \text{and} \quad \|T - S\| < 4\sqrt{\varepsilon}.$$

Prior to presenting the proof of this theorem, we have to recall the following representation result for operators from $L_1(m_1)$ into $L_1(m_2)$. As we announced in the introduction, we deal with the complex spaces only. The proof of the real case is easier than the one given by us for the complex case.

Let m_1 and m_2 be finite regular positive Borel measures on compact Hausdorff spaces J_1 and J_2, respectively. For a complex-valued Borel measure μ on the product space $J_1 \times J_2$, we define their marginal measures μ^i on J_i $(i = 1, 2)$ as follows:

$$\mu^1(A) = \mu(A \times J_2) \quad \text{and} \quad \mu^2(B) = \mu(J_1 \times B),$$

where A and B are Borel measurable subsets of J_1 and J_2, respectively.

Let $M(m_1, m_2)$ be the complex Banach lattice consisting of all complex-valued Borel measures μ on the product space $J_1 \times J_2$ such that each $|\mu|^i$ is absolutely continuous with respect to m_i for $i = 1, 2$ with the norm

$$\left\| \frac{d|\mu|^1}{dm_1} \right\|_{\infty}.$$

It is clear that to each $\mu \in M(m_1, m_2)$ there corresponds a unique bounded linear operator $T_\mu \in L(L_1(m_1), L_1(m_2))$ defined by

$$\langle T_\mu(f), g \rangle = \int_{J_1 \times J_2} f(x) g(y)\, d\mu(x, y),$$

where $f \in L_1(m_1)$ and $g \in L_\infty(m_2)$. Iwanik [25] showed that the mapping $\mu \longmapsto T_\mu$ is a surjective lattice isomorphism and

$$\|T_\mu\| = \left\| \frac{d|\mu|^1}{dm_1} \right\|_{\infty}.$$

Even though he showed this for the real case, it can be easily generalized to the complex case. For details, see [25, Theorem 1] and [39, IV Theorem 1.5 (ii), Corollary 2].

Since the proof of Theorem 3.2 is complicated, we divide it into the following two lemmas.

Lemma 3.3. *Let $0 < \varepsilon < 1$. Suppose that T_μ is an element of $\mathcal{L}(L_1(m_1), L_1(m_2))$ with $\|T_\mu\| = 1$ for some $\mu \in M(m_1, m_2)$ and that $f_0 \in S_{L_1(m_1)}$ is a nonnegative simple function such that $\|T_\mu f_0\| > 1 - \frac{\varepsilon^3}{2^6}$. Then there are a norm-one bounded linear operator T_ν for some $\nu \in M(m_1, m_2)$ and a nonnegative simple function f_1 in $S_{L_1(m_1)}$ such that*

$$\|T_\mu - T_\nu\| < \varepsilon, \quad \|f_1 - f_0\| < 3\varepsilon$$

and we have, for all $x \in \operatorname{supp}(f_1)$,

$$\frac{d|\nu|^1}{dm_1}(x) = 1.$$

Proof. As $\|T_\mu\| = 1$, we have that $\dfrac{d|\mu|^1}{dm_1}(x) \leqslant 1$ almost everywhere. Let $f_0 = \sum_{j=1}^n \alpha_j \frac{\chi_{B_j}}{m_1(B_j)}$, where $\{B_j\}_{j=1}^n$ are mutually disjoint Borel subsets of J_1, $\alpha_j \geqslant 0$ and $m_1(B_j) > 0$ for all $1 \leqslant j \leqslant n$, and $\sum_{j=1}^n \alpha_j = 1$. Let $D = \{x \in J_1 : \frac{d|\mu|^1}{dm_1}(x) > 1 - \frac{\varepsilon}{8}\}$. It is clear that $m_1(D) > 0$. Since $\|T_\mu f_0\| > 1 - \frac{\varepsilon^3}{2^6}$, there is $g_0 \in S_{L_\infty(m_2)}$ such that

$$\operatorname{Re} \langle T_\mu f_0, g_0 \rangle > 1 - \frac{\varepsilon^3}{2^6}.$$

Let

$$J = \left\{ j \in \{1, \dots, n\} \,:\, \frac{1}{m_1(B_j)} \int_{B_j} \frac{d|\mu|^1}{dm_1}(x)\, dm_1(x) > 1 - \frac{\varepsilon^2}{2^6} \right\}.$$

Then we have

$$\sum_{j \in J} \alpha_j \geqslant 1 - \varepsilon > 0.$$

Indeed, since

$$\begin{aligned} 1 - \frac{\varepsilon^3}{2^6} &< \operatorname{Re} \langle T_\mu f_0, g_0 \rangle = \operatorname{Re} \int_{J_1 \times J_2} f_0(x) g_0(y)\, d\mu(x, y) \\ &\leqslant \int_{J_1 \times J_2} |f_0(x)|\, d|\mu|(x, y) = \int_{J_1} f_0(x)\, d|\mu|^1(x) \\ &= \sum_{j=1}^n \alpha_j \frac{1}{m_1(B_j)} \int_{B_j} d|\mu|^1(x) = \sum_{j=1}^n \alpha_j \frac{1}{m_1(B_j)} \int_{B_j} \frac{d|\mu|^1}{dm_1}(x)\, dm_1(x), \end{aligned}$$

we have $\sum_{j \in J} \alpha_j \geqslant 1 - \varepsilon > 0$ by Lemma 2.6. Note also that for each $j \in J$,

$$\begin{aligned} 1 - \frac{\varepsilon^2}{2^6} &< \frac{1}{m_1(B_j)} \int_{B_j} \frac{d|\mu|^1}{dm_1}(x)\, dm_1(x) \\ &= \frac{1}{m_1(B_j)} \int_{B_j \cap D} \frac{d|\mu|^1}{dm_1}(x)\, dm_1(x) + \frac{1}{m_1(B_j)} \int_{B_j \setminus D} \frac{d|\mu|^1}{dm_1}(x)\, dm_1(x) \\ &\leqslant \frac{m_1(B_j \cap D)}{m_1(B_j)} + \left(1 - \frac{\varepsilon}{8}\right) \frac{m_1(B_j \setminus D)}{m_1(B_j)} \\ &= 1 - \frac{\varepsilon}{8} \frac{m_1(B_j \setminus D)}{m_1(B_j)}. \end{aligned}$$

Hence we deduce that, for all $j \in J$,

$$\frac{m_1(B_j \setminus D)}{m_1(B_j)} \leqslant \frac{\varepsilon}{8}.$$

Let $\tilde{B_j} = B_j \cap D$ and $\beta_j = \frac{\alpha_j}{\sum_{j \in J} \alpha_j}$ for all $j \in J$ and define

$$f_1 = \sum_{j \in J} \beta_j \frac{\chi_{\tilde{B_j}}}{m_1(\tilde{B_j})}.$$

It is clear that f_1 is a nonnegative element in $S_{L_1(m_1)}$ and

$$\begin{aligned}
\|f_0 - f_1\| &\leqslant \left\| \sum_{j\in J} \alpha_j \frac{\chi_{B_j}}{m_1(B_j)} - \sum_{j\in J} \beta_j \frac{\chi_{\tilde{B}_j}}{m_1(\tilde{B}_j)} \right\| + \left\| \sum_{j\in\{1,\dots,n\}\setminus J} \alpha_j \frac{\chi_{B_j}}{m_1(B_j)} \right\| \\
&\leqslant \left\| \sum_{j\in J} \alpha_j \left(\frac{\chi_{B_j}}{m_1(B_j)} - \frac{\chi_{\tilde{B}_j}}{m_1(\tilde{B}_j)} \right) \right\| + \left\| \sum_{j\in J} (\alpha_j - \beta_j) \frac{\chi_{\tilde{B}_j}}{m_1(\tilde{B}_j)} \right\| + \sum_{j\in\{1,\dots,n\}\setminus J} \alpha_j \\
&< \left\| \sum_{j\in J} \alpha_j \left(\frac{\chi_{B_j}}{m_1(B_j)} - \frac{\chi_{\tilde{B}_j}}{m_1(\tilde{B}_j)} \right) \right\| + \sum_{j\in J} |\alpha_j - \beta_j| + \varepsilon \\
&= \left\| \sum_{j\in J} \alpha_j \left(\frac{\chi_{B_j}}{m_1(B_j)} - \frac{\chi_{\tilde{B}_j}}{m_1(\tilde{B}_j)} \right) \right\| + 1 - \sum_{j\in J} \alpha_j + \varepsilon \\
&\leqslant 2 \sum_{j\in J} \alpha_j \frac{m_1(B_j \setminus D)}{m_1(B_j)} + 2\varepsilon \leqslant \frac{\varepsilon}{4} + 2\varepsilon < 3\varepsilon.
\end{aligned}$$

Define

$$d\nu(x,y) = \sum_{j\in J} \chi_{\tilde{B}_j}(x) \left(\frac{d|\mu|^1}{dm_1}(x) \right)^{-1} d\mu(x,y) + \chi_{(J_1\setminus\tilde{B})}\, d\mu(x,y),$$

where $\tilde{B} = \bigcup_{j\in J} \tilde{B}_j$. It is clear that $\frac{d|\nu|^1}{dm_1}(x) = 1$ on $\tilde{B}$ and $\frac{d|\nu|^1}{dm_1}(x) \leqslant 1$ elsewhere. Note also that for all $x \in J_1$,

$$\begin{aligned}
\frac{d|\nu-\mu|^1}{dm_1}(x) &= \sum_{j\in J} \chi_{\tilde{B}_j}(x) \left(\left(\frac{d|\mu|^1}{dm_1}(x) \right)^{-1} - 1 \right) \frac{d|\mu|^1}{dm_1} \\
&\leqslant \frac{1}{1-\varepsilon/8} - 1 = \frac{\varepsilon}{8-\varepsilon} < \varepsilon.
\end{aligned}$$

Hence T_ν is a norm-one operator such that $\|T_\mu - T_\nu\| < \varepsilon$, $\|f_1 - f_0\| < 3\varepsilon$ and $\frac{d|\nu|^1}{dm_1}(x) = 1$ for all $x \in \operatorname{supp}(f_1)$. □

Lemma 3.4. *Let $0 < \varepsilon < 1$. Suppose that T_ν is a norm-one operator in $\mathcal{L}(L_1(m_1), L_1(m_2))$ and that f is a nonnegative norm-one simple function in $S_{L_1(m_1)}$ satisfying $\|T_\nu f\| > 1 - \frac{\varepsilon^6}{2^7}$ and $\frac{d|\nu|^1}{dm_1}(x) = 1$ for all x in the support of f. Then there are a nonnegative simple function $\tilde{f}$ in $S_{L_1(m_1)}$ and a norm-one operator $T_{\tilde{\nu}}$ in $\mathcal{L}(L_1(m_1), L_1(m_2))$ such that*

$$\|T_{\tilde{\nu}} \tilde{f}\| = 1, \quad \|T_\nu - T_{\tilde{\nu}}\| < 3\sqrt{\varepsilon} \quad \text{and} \quad \|f - \tilde{f}\| < 3\varepsilon.$$

Proof. Let $f = \sum_{j=1}^n \beta_j \frac{\chi_{B_j}}{m_1(B_j)}$, where $\{B_j\}_{j=1}^n$ are mutually disjoint Borel subsets of J_1, $\beta_j \geqslant 0$ and $m_1(B_j) > 0$ for all $1 \leqslant j \leqslant n$, and $\sum_{j=1}^n \beta_j = 1$. Since $\|T_\nu f\| > 1 - \frac{\varepsilon^6}{2^7}$, there is $g \in S_{L_\infty(m_2)}$ such that

$$1 - \frac{\varepsilon^6}{2^7} < \operatorname{Re} \langle T_\nu f, g \rangle = \sum_{j=1}^n \beta_j \operatorname{Re} \int_{J_1\times J_2} \frac{\chi_{B_j}(x)}{m_1(B_j)} (g(y))\, d\nu(x,y).$$

Let $J = \left\{ j \in \{1,\dots,n\} : \operatorname{Re} \int_{J_1\times J_2} \frac{\chi_{B_j}(x)}{m_1(B_j)} g(y)\, d\nu(x,y) > 1 - \frac{\varepsilon^3}{2^6} \right\}$. From Lemma 2.6 it follows that

$$\sum_{j\in J} \beta_j > 1 - \frac{\varepsilon^3}{2}.$$

Let $f_1 = \sum_{j\in J} \tilde{\beta}_j \frac{\chi_{B_j}}{m_1(B_j)}$, where $\tilde{\beta}_j = \beta_j/(\sum_{j\in J}\beta_j)$ for all $j \in J$. Then

$$\|f_1 - f\| \leqslant \left\| \sum_{j\in J} (\tilde{\beta}_j - \beta_j) \frac{\chi_{B_j}}{m_1(B_j)} \right\| + \sum_{j\in J} \beta_j \leqslant \varepsilon^3 < \varepsilon.$$

Note that there is a Borel measurable function h on $J_1 \times J_2$ such that $d\nu(x,y) = h(x,y)\, d|\nu|(x,y)$ and $|h(x,y)| = 1$ for all $(x,y) \in J_1 \times J_2$. Let

$$C = \left\{ (x,y) : |g(y)h(x,y) - 1| < \frac{\sqrt{\varepsilon}}{2^{3/2}} \right\}.$$

Define two measures ν_f and ν_c as follows:

$$\nu_f(A) = \nu(A \setminus C) \quad \text{and} \quad \nu_c(A) = \nu(A \cap C)$$

for every Borel subset A of $J_1 \times J_2$. It is clear that

$$d\nu = d\nu_f + d\nu_c, \quad d|\nu_f| = \bar{h} d\nu_f, \quad d|\nu_c| = \bar{h} d\nu_c, \quad \text{and} \quad d|\nu| = d|\nu_f| + d|\nu_c|.$$

Since $\frac{d|\nu|^1}{dm_1}(x) = 1$ for all $x \in \bigcup_{j=1}^n B_j$, we have $|\nu|^1(B_j) = m_1(B_j)$ for all $1 \leqslant j \leqslant n$ and

$$1 = \frac{d|\nu|^1}{dm_1}(x) = \frac{d|\nu_f|^1}{dm_1}(x) + \frac{d|\nu_c|^1}{dm_1}(x) \ \text{ for all } x \in B = \bigcup_{j=1}^n B_j.$$

We claim that $\frac{|\nu_f|^1(B_j)}{m_1(B_j)} \leqslant \frac{\varepsilon^2}{2^2}$ for all $j \in J$. Indeed, if $|g(y)h(x,y) - 1| \geqslant \frac{\sqrt{\varepsilon}}{2^{3/2}}$, then $\operatorname{Re}(g(y)h(x,y)) \leqslant 1 - \frac{\varepsilon}{2^4}$. So we have

$$\begin{aligned}
1 - \frac{\varepsilon^3}{2^6} &\leqslant \frac{1}{m_1(B_j)} \operatorname{Re} \int_{J_1\times J_2} \chi_{B_j(x)} g(y)\, d\nu(x,y) \\
&= \frac{1}{m_1(B_j)} \int_{J_1\times J_2} \chi_{B_j(x)} \operatorname{Re}\big(g(y)h(x,y)\big)\, d|\nu|(x,y) \\
&= \frac{1}{m_1(B_j)} \int_{J_1\times J_2} \chi_{B_j(x)} \operatorname{Re}\big(g(y)h(x,y)\big)\, d|\nu_f|(x,y) \\
&\quad + \frac{1}{m_1(B_j)} \int_{J_1\times J_2} \chi_{B_j(x)} \operatorname{Re}\big(g(y)h(x,y)\big)\, d|\nu_c|(x,y) \\
&\leqslant \frac{1}{m_1(B_j)} \Big((1 - \frac{\varepsilon}{2^4}) |\nu_f|^1(B_j) + |\nu_c|^1(B_j) \Big) \\
&= 1 - \frac{\varepsilon}{2^4} \frac{|\nu_f|^1(B_j)}{m_1(B_j)}.
\end{aligned}$$

This proves our claim.

We also claim that for each $j \in J$, there exists a Borel subset $\tilde{B}_j$ of B_j such that

$$\Big(1 - \frac{\varepsilon}{2}\Big) m_1(B_j) \leqslant m_1(\tilde{B}_j) \leqslant m_1(B_j)$$

and

$$\frac{d|\nu_f|^1}{dm_1}(x) \leqslant \frac{\varepsilon}{2}$$

for all $x \in \tilde{B}_j$. Indeed, set $\tilde{B}_j = B_j \cap \Big\{ x \in J_1 : \frac{d|\nu_f|^1}{dm_1}(x) \leqslant \frac{\varepsilon}{2} \Big\}$. Then

$$\int_{B_j \setminus \tilde{B}_j} \frac{\varepsilon}{2}\, dm_1(x) \leqslant \int_{B_j} \frac{d|\nu_f|^1}{dm_1}(x)\, dm_1(x) = |\nu_f^1|(B_j) \leqslant \frac{\varepsilon^2}{2^2} m_1(B_j).$$

This shows that $m_1(B_j \setminus \tilde{B}_j) \leqslant \frac{\varepsilon}{2} m_1(B_j)$. This proves our second claim.

Now, we define $\tilde{g}$ by $\tilde{g}(y) = \frac{g(y)}{|g(y)|}$ if $g(y) \neq 0$ and $\tilde{g}(y) = 1$ if $g(y) = 0$, and we write $\tilde{f} = \sum_{j\in J} \tilde{\beta}_j \frac{\chi_{\tilde{B}_j}}{m_1(\tilde{B}_j)}$. Finally, we define the measure

$$d\tilde{\nu}(x,y) = \sum_{j\in J} \chi_{\tilde{B}_j}(x)\overline{\tilde{g}(y)h(x,y)}d\nu_c(x,y)\left(\frac{d|\nu_c|^1}{dm_1}(x)\right)^{-1} + \chi_{J_1\setminus\tilde{B}}(x)d\nu(x,y),$$

where $\tilde{B} = \bigcup_{j\in J}\tilde{B}_j$. It is easy to see that $\frac{d|\tilde{\nu}|^1}{dm_1}(x) = 1$ on $\tilde{B}$ and $\frac{d|\tilde{\nu}|^1}{dm_1}(x) \leqslant 1$ elsewhere. Note that

$$\begin{aligned} d(\tilde{\nu}-\nu)(x,y) = &\sum_{j\in J}\chi_{\tilde{B}_j}(x)\left[\overline{\tilde{g}(y)h(x,y)}\left(\frac{d|\nu_c|^1}{dm_1}(x)\right)^{-1} - 1\right]d\nu_c(x,y) \\ &- \sum_{j\in J}\chi_{\tilde{B}_j}(x)d\nu_f(x,y). \end{aligned}$$

If $(x,y) \in C$, then $|g(y)| \geqslant 1 - \frac{\sqrt{\varepsilon}}{2^{3/2}} \geqslant 1 - \frac{1}{2^{3/2}}$ and

$$\begin{aligned} \left|\overline{\tilde{g}(y)h(x,y)} - 1\right| &= \left|\frac{g(y)}{|g(y)|}h(x,y) - 1\right| \\ &\leqslant \frac{|g(y)h(x,y)-1|}{|g(y)|} + \frac{\big|1-|g(y)|\big|}{|g(y)|} \\ &\leqslant 2\frac{|g(y)h(x,y)-1|}{|g(y)|} \leqslant 2\frac{\sqrt{\varepsilon}}{2^{3/2}}\frac{2^{3/2}}{2^{3/2}-1} \leqslant 2\sqrt{\varepsilon}. \end{aligned}$$

Hence, for all $(x,y) \in C$ we have

$$\begin{aligned} \left|\overline{\tilde{g}(y)h(x,y)}\left(\frac{d|\nu_c|^1}{dm_1}(x)\right)^{-1} - 1\right| &\leqslant \left|\overline{\tilde{g}(y)h(x,y)} - 1\right|\left(\frac{d|\nu_c|^1}{dm_1}(x)\right)^{-1} + \left|\left(\frac{d|\nu_c|^1}{dm_1}(x)\right)^{-1} - 1\right| \\ &\leqslant 2\sqrt{\varepsilon}\left(\frac{d|\nu_c|^1}{dm_1}(x)\right)^{-1} + \left|\left(\frac{d|\nu_c|^1}{dm_1}(x)\right)^{-1} - 1\right|. \end{aligned}$$

So, we have for all $x \in J_1$,

$$\begin{aligned} \frac{d|\tilde{\nu}-\nu|^1}{dm_1}(x) &\leqslant \sum_{j\in J}\chi_{\tilde{B}_j}(x)\left[2\sqrt{\varepsilon}\left(\frac{d|\nu_c|^1}{dm_1}(x)\right)^{-1} + \left|\left(\frac{d|\nu_c|^1}{dm_1}(x)\right)^{-1} - 1\right|\right]\frac{d|\nu_c|^1}{dm_1}(x) \\ &\quad + \sum_{j\in J}\chi_{\tilde{B}_j}(x)\frac{d|\nu_f|^1}{dm_1}(x) \\ &\leqslant \sum_{j\in J}\chi_{\tilde{B}_j}(x)\left(2\sqrt{\varepsilon} + \left(1 - \frac{d|\nu_c|^1}{dm_1}(x)\right)\right) + \sum_{j\in J}\chi_{\tilde{B}_j}(x)\left(\frac{d|\nu_f|^1}{dm_1}(x)\right) \\ &\leqslant 2\sqrt{\varepsilon} + \varepsilon < 3\sqrt{\varepsilon}. \end{aligned}$$

This gives that $\|T_\nu - T_{\tilde{\nu}}\| < 3\sqrt{\varepsilon}$. Note also that, for all $j \in J$,

$$\begin{aligned} \left\langle T_{\tilde{\nu}}\frac{\chi_{\tilde{B}_j}}{m_1(\tilde{B}_j)}, \tilde{g}\right\rangle &= \int_{J_1\times J_2}\frac{\chi_{\tilde{B}_j}(x)}{m_1(\tilde{B}_j)}\tilde{g}(y)\,d\tilde{\nu}(x,y) \\ &= \int_{J_1\times J_2}\frac{\chi_{\tilde{B}_j}(x)}{m_1(\tilde{B}_j)}\overline{h(x,y)}\left(\frac{d|\nu_c|^1}{dm_1}(x)\right)^{-1}\,d\nu_c(x,y) \\ &= \int_{J_1}\frac{\chi_{\tilde{B}_j}(x)}{m_1(\tilde{B}_j)}\left(\frac{d|\nu_c|^1}{dm_1}(x)\right)^{-1}\,d|\nu_c|^1(x) \\ &= \int_{J_1}\frac{\chi_{\tilde{B}_j}(x)}{m_1(\tilde{B}_j)}\,dm_1(x) = 1. \end{aligned}$$

Hence we get $\left\langle T_{\tilde{\nu}}\tilde{f}, \tilde{g}\right\rangle = 1$, which implies that $\|T_{\tilde{\nu}}\tilde{f}\| = \|T_{\tilde{\nu}}\| = 1$. Finally,

$$\begin{aligned}
\|\tilde{f} - f\| &\leqslant \|\tilde{f} - f_1\| + \|f_1 - f\| \\
&= \left\| \sum_{j\in J} \tilde{\beta}_j \frac{\chi_{\tilde{B}_j}}{m_1(\tilde{B}_j)} - \sum_{j\in J} \tilde{\beta}_j \frac{\chi_{B_j}}{m_1(B_j)} \right\| + \varepsilon \\
&\leqslant \sum_{j\in J} \tilde{\beta}_j \left(\left\| \frac{\chi_{\tilde{B}_j}}{m_1(\tilde{B}_j)} - \frac{\chi_{B_j}}{m_1(\tilde{B}_j)} \right\| + \left\| \frac{\chi_{B_j}}{m_1(\tilde{B}_j)} - \frac{\chi_{B_j}}{m_1(B_j)} \right\| \right) + \varepsilon \\
&= 2\sum_{j\in J} \tilde{\beta}_j \frac{m_1(B_j \setminus \tilde{B}_j)}{m_1(\tilde{B}_j)} + \varepsilon \\
&\leqslant 2\sum_{j\in J} \tilde{\beta}_j \frac{\frac{\varepsilon}{2} m_1(B_j)}{m_1(\tilde{B}_j)} + \varepsilon \leqslant \frac{\varepsilon}{1-\varepsilon/2} + \varepsilon < 3\varepsilon.
\end{aligned}$$

□

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. Let $0 < \varepsilon < 1$. Suppose that T is a norm-one element in $\mathcal{L}(L_1(m_1), L_1(m_2))$ and there is $f \in S_{L_1(m_1)}$ such that $\|Tf\| > 1 - \frac{\varepsilon^{18}}{5^3 2^{27}}$. Then there is an isometric isomorphism ψ from $L_1(m_1)$ onto itself such that $\psi(f) = |f|$. Using $T \circ \psi^{-1}$ instead of T, we may assume that f is nonnegative. Since simple functions are dense in $L_1(m_1)$, we can choose a nonnegative simple function $f_0 \in S_{L_1(m_1)}$ arbitrarily close to f so that

$$\|Tf_0\| > 1 - \frac{\varepsilon^{18}}{5^3 2^{27}} = 1 - \frac{\varepsilon_1^3}{2^6},$$

where $\varepsilon_1 = \frac{\varepsilon^6}{5\cdot 2^7}$. By Lemma 3.3, there exist a norm-one bounded linear operator T_ν for some $\nu \in M(m_1, m_2)$ and a nonnegative simple function f_1 in $S_{L_1(M_1)}$ such that $\|T - T_\nu\| < \varepsilon_1$, $\|f_1 - f\| < 3\varepsilon_1$ and $\frac{d|\nu|^1}{dm_1}(x) = 1$ for all $x \in \operatorname{supp}(f_1)$. Then

$$\|T_\nu f_1\| \geqslant \|Tf\| - \|Tf - T_\nu f\| - \|T_\nu(f - f_1)\| \geqslant 1 - \frac{\varepsilon_1^3}{2^6} - \varepsilon_1 - 3\varepsilon_1 \geqslant 1 - 5\varepsilon_1 = 1 - \frac{\varepsilon^6}{2^7}.$$

Now, by Lemma 3.4, there exist a nonnegative simple function $\tilde{f}$ and an operator $T_{\tilde{\nu}}$ in $\mathcal{L}(L_1(m_1), L_1(m_2))$ such that $\|T_{\tilde{\nu}}\tilde{f}\| = \|T_{\tilde{\nu}}\| = 1$, $\|T_\nu - T_{\tilde{\nu}}\| \leqslant 3\sqrt{\varepsilon}$ and $\|f_1 - \tilde{f}\| \leqslant 3\varepsilon$. Therefore, $\|T - T_{\tilde{\nu}}\| < 4\sqrt{\varepsilon}$ and $\|f - \tilde{f}\| < 4\varepsilon$, which complete the proof. □

4. The Bishop-Phelps-Bollobás property of $(L_1(\mu), L_\infty(\nu))$

Our aim now is to show that $(L_1(\mu), L_\infty(\nu))$ has the BPBp for any measure μ and any localizable measure ν.

Theorem 4.1. *Let μ be an arbitrary measure and let ν be a localizable measure. Then the pair $(L_1(\mu), L_\infty(\nu))$ has the BPBp. Moreover,*

$$\eta\big(L_1(\mu), L_\infty(\nu)\big)(\varepsilon) \geqslant \left(\frac{\varepsilon}{10}\right)^8 \qquad (0 < \varepsilon < 1).$$

By Corollaries 2.2 and 2.5, it is enough to prove the result in the case where μ is σ-finite and ν is the product measure on $[0,1]^\Lambda$. Therefore, we just need to prove the following result.

Theorem 4.2. *Assume μ is a σ-finite measure and ν is the product measure of Lebesgue measures on $[0,1]^\Lambda$. Let $0 < \varepsilon < 1/3$, let $T : L_1(\mu) \longrightarrow L_\infty(\nu)$ be a bounded linear operator of norm one and let $f_0 \in S_{L_1(\mu)}$ satisfy $\|T(f_0)\|_\infty > 1 - \varepsilon^8$. Then there exist $S \in \mathcal{L}(L_1(\mu), L_\infty(\nu))$ with $\|S\| = 1$ and $g_0 \in S_{L_1(\mu)}$ such that*

$$\|S(g_0)\|_\infty = 1, \quad \|T - S\| < 2\varepsilon \quad \text{and} \quad \|f_0 - g_0\|_1 < 10\varepsilon.$$

Recall that the particular case where Λ reduces to one point was established in [9]. Actually, our proof is based on the argument given there.

Prior to giving the proof of Theorem 4.2, we state the following representation result for operators from $L_1(\mu)$ into $L_\infty\big([0,1]^\Lambda\big)$ and one lemma.

Let (Ω,Σ,μ) be a σ-finite measure space and let $K=[0,1]^\Lambda$ be the product space equipped with the product measure ν of the Lebesgue measures. Let J be a countable subset of Λ and let π_J be the natural projection from K onto $[0,1]^J$. Fix a sequence (Π_n) of finite partitions of $[0,1]^J$ into sets of positive measure such that Π_{n+1} is a refinement of Π_n for each n, and the σ-algebra generated by $\bigcup_{n=1}^\infty \Pi_n$ is the Borel σ-algebra of $[0,1]^J$. For each $y\in K$ and $n\in\mathbb{N}$, let $B(n,\pi_J(y))$ be the set in Π_n containing $\pi_J(y)$. Then, given a Borel set F of the form $F_0\times[0,1]^{\Lambda\setminus J}$ with $F_0\subset[0,1]^J$, define

$$\delta(F)=\left\{y\in K:\lim_{n\to\infty}\frac{\nu(F\cap\pi_J^{-1}(B(n,\pi_J(y))))}{\nu(\pi_J^{-1}(B(n,\pi_J(y))))}=1\right\}.$$

It is easy to check that $\delta(F)=\delta_J(F_0)\times[0,1]^{\Lambda\setminus J}$, where

$$\delta_J(F_0)=\left\{y\in[0,1]^J:\lim_{n\to\infty}\frac{\nu(\pi_J^{-1}(F_0\cap B(n,y)))}{\nu(\pi_J^{-1}(B(n,y)))}=1\right\}.$$

Using the martingale almost everywhere convergence theorem [22], we have

$$\nu(F\Delta\delta(F))=0$$

where $F\Delta\delta(F)$ denotes the symmetric difference of the sets F and $\delta(F)$.

On the other hand, it is well-known that the space $\mathcal{L}(L_1(\mu),L_\infty(\nu))$ is isometrically isomorphic to the space $L_\infty(\mu\otimes\nu)$, where $\mu\otimes\nu$ denotes the product measure on $\Omega\times K$. More precisely, the operator $\widehat{h}$ corresponding to $h\in L_\infty(\mu\otimes\nu)$ is given by

$$\widehat{h}(f)(t)=\int_\Omega h(\omega,t)f(\omega)\,d\mu(\omega)$$

for ν-almost every $t\in K$. For a reference, see [20]. For a measurable subset M of $\Omega\times K$, let $M_x=\{y\in K\ :\ (x,y)\in M\}$ for each $x\in\Omega$ and $M^y=\{x\in\Omega\ :\ (x,y)\in M\}$ for each $y\in K$.

Lemma 4.3. *Let M be a measurable subset of $\Omega\times K$ with positive measure, $0<\varepsilon<1$, and let f_0 be a simple function. If $\|\widehat{\chi_M}(f_0)\|_\infty>1-\varepsilon$, then there exists a simple function $g_0\in S_{L_1(\mu)}$ such that*

$$\|[\widehat{\chi_M}+\widehat{\varphi}](g_0)\|_\infty=1\quad\text{and}\quad\|f_0-g_0\|_1<4\sqrt{\varepsilon}$$

for every simple function φ in $L_\infty(\mu\otimes\nu)$ with $\|\varphi\|_\infty\leqslant 1$ and vanishing on M.

Proof. Write $f_0=\sum_{j=1}^m\alpha_j\frac{\chi_{A_j}}{\mu(A_j)}\in S_{L_1(\mu)}$, where each A_j is a measurable subset of Ω with finite positive measure, $A_k\cap A_l=\emptyset$ for $k\neq l$, and α_j is a positive real number for every $j=1,\dots,m$ with $\sum_{j=1}^m\alpha_j=1$. Since $\|\widehat{\chi_M}(f_0)\|_\infty>1-\varepsilon$, there is a measurable subset B of K such that $0<\nu(B)$ and

$$\left\langle\widehat{\chi_M}(f_0),\frac{\chi_B}{\nu(B)}\right\rangle>1-\varepsilon.$$

We may assume that there is a countable subset J of Λ such that $M=M_0\times[0,1]^{\Lambda\setminus J}$ and $B=B_0\times[0,1]^{\Lambda\setminus J}$ for some measurable subsets $M_0\subset\Omega\times[0,1]^J$ and $B_0\subset[0,1]^J$. For each $j\in\{1,\dots,m\}$, we write $M_j=M\bigcap(A_j\times B)=(M_0\cap(A_j\times B_0))\times[0,1]^{\Lambda\setminus J}$ and define

$$H_j=\{(x,y)\ :\ x\in A_j,y\in\delta\big((M_j)_x\big)\}.$$

As in the proof of [35, Proposition 5], the H_j's are disjoint measurable subsets of $\Omega\times K$. We note that for each $j\in\{1,\dots,m\}$, we have $H_j\subset A_j\times\delta(B)$ and $(\mu\otimes\nu)(M_j\Delta H_j)=0$.

Now, by Fubini theorem, we have that

$$\begin{aligned}
1-\varepsilon &< \langle \widehat{\chi}_M(f_0), \frac{\chi_B}{\nu(B)}\rangle \\
&= \sum_{j=1}^m \frac{\alpha_j}{\mu(A_j)\nu(B)} \int_{\Omega\times K} \chi_{M_j}(x,y)\ d(\mu\otimes\nu) \\
&= \sum_{j=1}^m \frac{\alpha_j}{\mu(A_j)\nu(B)} \int_{\Omega\times K} \chi_{H_j}(x,y)\ d(\mu\otimes\nu) \\
&= \sum_{j=1}^m \frac{\alpha_j}{\nu(B)} \int_{\delta(B)} \frac{\mu(H_j^y)}{\mu(A_j)}\ d\nu(y) \\
&= \frac{1}{\nu(\delta(B))} \int_{\delta(B)} \sum_{j=1}^m \alpha_j \frac{\mu(H_j^y)}{\mu(A_j)}\ d\nu(y).
\end{aligned}$$

So, there exists $y_0 \in \delta(B)$ such that

$$\sum_{j=1}^m \alpha_j \frac{\mu(H_j^{y_0})}{\mu(A_j)} > 1-\varepsilon.$$

Let $J = \Big\{ j \in \{1,\dots,m\} : \frac{\mu(H_j^{y_0})}{\mu(A_j)} > 1-\sqrt{\varepsilon} \Big\}$. For each $j \in J$, we have that $\mu(A_j \setminus H_j^{y_0}) < \sqrt{\varepsilon}\mu(A_j)$ and, by Lemma 2.6, we also have $\alpha_J := \sum_{j\in J} \alpha_j > 1-\sqrt{\varepsilon}$. Define

$$g_0 = \sum_{j\in J} \beta_j \frac{\chi_{H_j^{y_0}}}{\mu(H_j^{y_0})},$$

where $\beta_j = \alpha_j/\alpha_J$. Then

$$\begin{aligned}
\|g_0 - f_0\| &< \left\| \sum_{j\in J} \beta_j \frac{\chi_{H_j^{y_0}}}{\mu(H_j^{y_0})} - \sum_{j\in J} \alpha_j \frac{\chi_{A_j}}{\mu(A_j)} \right\| + \sqrt{\varepsilon} \\
&\leqslant \left\| \sum_{j\in J} \beta_j \frac{\chi_{H_j^{y_0}}}{\mu(H_j^{y_0})} - \sum_{j\in J} \beta_j \frac{\chi_{A_j}}{\mu(A_j)} \right\| + \left\| \sum_{j\in J} \beta_j \frac{\chi_{A_j}}{\mu(A_j)} - \sum_{j\in J} \alpha_j \frac{\chi_{A_j}}{\mu(A_j)} \right\| + \sqrt{\varepsilon} \\
&\leqslant \left\| \sum_{j\in J} \beta_j \frac{\chi_{H_j^{y_0}}}{\mu(H_j^{y_0})} - \sum_{j\in J} \beta_j \frac{\chi_{H_j^{y_0}}}{\mu(A_j)} \right\| + \left\| \sum_{j\in J} \beta_j \frac{\chi_{H_j^{y_0}}}{\mu(A_j)} - \sum_{j\in J} \beta_j \frac{\chi_{A_j}}{\mu(A_j)} \right\| + 2\sqrt{\varepsilon} \\
&\leqslant 2\frac{\mu(A_j \setminus H_j^{y_0})}{\mu(A_j)} + 2\sqrt{\varepsilon} \leqslant 4\sqrt{\varepsilon}
\end{aligned}$$

We claim that $\widehat{\chi_M} + \widehat{\varphi}$ attains its norm at g_0. Let $B_n = \pi_J^{-1}(B(n, \pi_J(y_0)))$ for each n. Note that for every $x \in H_j^{y_0}$ we have $(x, y_0) \in H_j$, which implies that

$$\lim_{n\to\infty} \frac{\nu\big((M_j)_x \cap B_n\big)}{\nu(B_n)} = 1.$$

It follows from the Lebesgue dominated convergence theorem and Fubini theorem that, for each $j \in J$,

$$\begin{aligned}
1 &= \lim_{n\to\infty} \frac{1}{\mu(H_j^{y_0})} \int_{H_j^{y_0}} \frac{\nu\big((M_j)_x \cap B_n\big)}{\nu(B_n)}\ d\mu(x) \\
&= \lim_{n\to\infty} \frac{(\mu\otimes\nu)(M_j \cap (H_j^{y_0} \times B_n))}{\mu(H_j^{y_0})\nu(B_n)}.
\end{aligned}$$

On the other hand, since the simple function φ is assumed to vanish on M and $\|\varphi\|_\infty \leqslant 1$, we have

$$\begin{aligned}\left|\langle\widehat{\varphi}\big(\frac{\chi_{H_j^{y_0}}}{\mu(H_j^{y_0})}\big), \frac{\chi_{B_n}}{\nu(B_n)}\rangle\right| &= \left|\frac{1}{\mu(H_j^{y_0})\nu(B_n)}\int_{H_j^{y_0}\times B_n}\varphi\, d(\mu\otimes\nu)\right| \\ &\leqslant \frac{(\mu\otimes\nu)((H_j^{y_0}\times B_n)\setminus M_j)}{\mu(H_j^{y_0})\nu(B_n)} \\ &= 1-\frac{(\mu\otimes\nu)(M_j\cap(H_j^{y_0}\times B_n))}{\mu(H_j^{y_0})\nu(B_n)} \longrightarrow 0,\end{aligned}$$

as $n\to\infty$. Therefore,

$$\begin{aligned}1 \geqslant \big\|[\widehat{\chi_M}+\widehat{\varphi}](g_0)\big\|_\infty &\geqslant \lim_{n\to\infty}\left|\left\langle(\widehat{\chi_M}+\widehat{\varphi})\Big(\sum_{j\in J}\beta_j\frac{\chi_{H_j^{y_0}}}{\mu(H_j^{y_0})}\Big), \frac{\chi_{B_n}}{\nu(B_n)}\right\rangle\right| \\ &= \lim_{n\to\infty}\sum_{j\in J}\beta_j\frac{(\mu\otimes\nu)(M\cap(H_j^{y_0}\times B_n))}{\mu(H_j^{y_0})\nu(B_n)} \\ &\geqslant \lim_{n\to\infty}\sum_{j\in J}\beta_j\frac{(\mu\otimes\nu)(M_j\cap(H_j^{y_0}\times B_n))}{\mu(H_j^{y_0})\nu(B_n)} = 1,\end{aligned}$$

which shows that $\widehat{\chi_M}+\widehat{\varphi}$ attains its norm at g_0. □

We are now ready to give the proof of the main result in this section.

Proof of Theorem 4.2. Since the set of all simple functions is dense in $L_1(\mu)$, we may assume

$$f_0 = \sum_{j=1}^m \alpha_j\frac{\chi_{A_j}}{\mu(A_j)} \in S_{L_1(\mu)},$$

where each A_j is a measurable subset of Ω with finite positive measure, $A_k\cap A_l=\emptyset$ for $k\neq l$, and every α_j is a nonzero complex number with $\sum_{j=1}^m|\alpha_j|=1$. We may also assume that $0<\alpha_j\leqslant 1$ for every $j=1,\ldots,m$. Indeed, there exists an isometric isomorphism $\Psi: L_1(\mu)\longrightarrow L_1(\mu)$ such that $\Psi(f_0)=|f_0|$. Hence we may replace T and f_0 by $T\circ\Psi^{-1}$ and $\Psi(f_0)$, respectively.

Let h be the element in $L_\infty(\Omega\times K,\mu\otimes\nu)$ with $\|h\|_\infty=1$ corresponding to T, that is, $T=\widehat{h}$. We may find a simple function

$$h_0\in L_\infty(\Omega\times K,\mu\otimes\nu),\quad \|h_0\|_\infty=1$$

such that $\|h-h_0\|_\infty<\|T(f_0)\|_\infty-(1-\varepsilon^8)$, hence $\|\widehat{h_0}(f_0)\|_\infty>1-\varepsilon^8$. We can write $h_0=\sum_{l=1}^p c_l\chi_{D_l}$, where each D_l is a measurable subset of $\Omega\times K$ with positive measure, $D_k\cap D_l=\emptyset$ for $k\neq l$, $|c_l|\leqslant 1$ for every $l=1,\ldots,p$, and $|c_{l_0}|=1$ for some $1\leqslant l_0\leqslant p$.

Let B be a Lebesgue measurable subset of K with $0<\nu(B)<\infty$ such that

$$\left|\left\langle\widehat{h}_0(f_0), \frac{\chi_B}{\nu(B)}\right\rangle\right|>1-\varepsilon^8.$$

Choose $\theta\in\mathbb{R}$ so that

$$\begin{aligned}1-\varepsilon^8 &< \big|\langle\widehat{h}_0(f_0), \frac{\chi_B}{\nu(B)}\rangle\big| \\ &= \mathrm{e}^{i\theta}\langle\widehat{h}_0(f_0), \frac{\chi_B}{\nu(B)}\rangle \\ &= \sum_{j=1}^m\alpha_j\ \mathrm{e}^{i\theta}\left\langle\widehat{h}_0\big(\frac{\chi_{A_j}}{\mu(A_j)}\big), \frac{\chi_B}{\nu(B)}\right\rangle.\end{aligned}$$

Set

$$J = \left\{ j \in \{1, \dots, m\} \,:\, \mathrm{Re}\, \left[\, \mathrm{e}^{i\theta} \langle \widehat{h}_0(\frac{\chi_{A_j}}{\mu(A_j)}), \frac{\chi_B}{\nu(B)} \rangle \right] \, > 1 - \varepsilon^4 \right\}.$$

By Lemma 2.6, we have

$$\alpha_J = \sum_{j \in J} \alpha_j > 1 - \frac{\varepsilon^8}{1 - (1 - \varepsilon^4)} = 1 - \varepsilon^4.$$

We define

$$f_1 = \sum_{j \in J} \left(\frac{\alpha_j}{\alpha_J} \right) \frac{\chi_{A_j}}{\mu(A_j)}.$$

We have that $\|f_1\|_1 = 1$,

$$\begin{aligned} \|f_0 - f_1\|_1 &\leqslant \Big\| \sum_{j \notin J} \alpha_j \frac{\chi_{A_j}}{\mu(A_j)} \Big\|_1 + \Big(\frac{1}{\alpha_J} - 1 \Big) \Big\| \sum_{j \in J} \alpha_j \frac{\chi_{A_j}}{\mu(A_j)} \Big\|_1 \\ &= \sum_{j \notin J} \alpha_j + (1 - \alpha_J) = 2(1 - \alpha_J) < 2\varepsilon^4. \end{aligned}$$

and

$$\begin{aligned} \left| \langle \widehat{h}_0(f_1), \frac{\chi_B}{\nu(B)} \rangle \right| &\geqslant \mathrm{Re} \left[\mathrm{e}^{i\theta} \left\langle \widehat{h}_0(f_1), \frac{\chi_B}{\nu(B)} \right\rangle \right] \\ &= \frac{1}{\alpha_J} \sum_{j \in J} \alpha_j \ \mathrm{Re} \left[\mathrm{e}^{i\theta} \left\langle \widehat{h}_0(\frac{\chi_{A_j}}{\mu(A_j)}), \frac{\chi_B}{\nu(B)} \right\rangle \right] \\ &> \frac{1}{\alpha_J} \sum_{j \in J} \alpha_j (1 - \varepsilon^4) = 1 - \varepsilon^4. \end{aligned}$$

Let $L = \left\{ l \in \{1, \dots, p\} \,:\, \mathrm{Re}\,(\mathrm{e}^{i\theta}\, c_l) > 1 - \frac{\varepsilon^2}{2} \right\}$. On the other hand, for each $j \in J$, we have

$$\begin{aligned} 1 - \varepsilon^4 &< \mathrm{Re} \left[\mathrm{e}^{i\theta} \left\langle \widehat{h}_0(\frac{\chi_{A_j}}{\mu(A_j)}), \frac{\chi_B}{\nu(B)} \right\rangle \right] \\ &= \sum_{l=1}^{p} \mathrm{Re}\,(\mathrm{e}^{i\theta}\, c_l) \ \frac{(\mu \otimes \nu)(D_l \cap (A_j \times B))}{\mu(A_j)\nu(B)} \\ &\leqslant \sum_{l \in \{1,\dots,p\} \setminus L} (1 - \frac{\varepsilon^2}{2}) \frac{(\mu \otimes \nu)(D_l \cap (A_j \times B))}{\mu(A_j)\nu(B)} \\ &\qquad + \sum_{l \in L} \frac{(\mu \otimes \nu)(D_l \cap (A_j \times B))}{\mu(A_j)\nu(B)} \\ &\leqslant 1 - \frac{\varepsilon^2}{2} \sum_{l \in \{1,\dots,p\} \setminus L} \frac{(\mu \otimes \nu)(D_l \cap (A_j \times B))}{\mu(A_j)\nu(B)}. \end{aligned}$$

This implies that for each $j \in J$

$$\sum_{l \in \{1,\dots,p\} \setminus L} \frac{(\mu \otimes \nu)(D_l \cap (A_j \times B))}{\mu(A_j)\nu(B)} \leqslant 2\varepsilon^2.$$

Since

$$\sum_{l=1}^{p} \frac{(\mu \otimes \nu)(D_l \cap (A_j \times B))}{\mu(A_j)\nu(B)} > 1 - \varepsilon^4,$$

for every $j \in J$ we have that

$$\sum_{l \in L} \frac{(\mu \otimes \nu)(D_l \cap (A_j \times B))}{\mu(A_j)\nu(B)} \geqslant (1 - \varepsilon^4 - 2\varepsilon^2) \geqslant 1 - 3\varepsilon^2.$$

Set $D = \bigcup_{l\in L} D_l$. Then we have

$$\langle \widehat{\chi}_D(f_1), \frac{\chi_B}{\nu(B)}\rangle \;=\; \sum_{j\in J} \big(\frac{\alpha_j}{\alpha_J}\big) \cdot \sum_{l\in L} \frac{\mu\otimes\nu(D_l \cap (A_j \times B))}{\mu(A_j)\nu(B)} \geqslant 1 - 3\varepsilon^2.$$

By Lemma 4.3, there is $g_0 \in S_{L_1(\mu)}$ such that $\|(\widehat{\chi}_D + \widehat{\varphi})(g_0)\|_\infty = 1$ and $\|f_1 - g_0\| < 4\sqrt{3\varepsilon^2} < 8\varepsilon$ for every simple function φ in $L_\infty(\mu\otimes\nu)$ vanishing on D with $\|\varphi\|_\infty \leqslant 1$. Therefore, we have

$$\|f_0 - g_0\|_1 \leqslant \|f_0 - f_1\|_1 + \|f_1 - g_0\|_1 \leqslant 2\varepsilon^4 + 8\varepsilon < 10\varepsilon.$$

Define

$$h_1 = \mathrm{e}^{-i\theta}\ \chi_D + \sum_{l\notin L} c_l\ \chi_{D_l} \in L_\infty(\mu\otimes\nu).$$

Let S be the operator in $\mathcal{L}(L_1(\mu), L_\infty(m))$ corresponding to h_1. Then we get

$$\|S(g_0)\|_\infty = \|\widehat{h_1}(g_0)\|_\infty = 1$$

and

$$\|h_0 - h_1\|_\infty = \max_{l\in L} |c_l - \mathrm{e}^{-i\theta}\,| = \max_{l\in L} |\,\mathrm{e}^{i\theta}\, c_l - 1|.$$

As $\mathrm{Re}\,(\mathrm{e}^{i\theta}\, c_l) > 1 - \frac{\varepsilon^2}{2}$ for every $l \in L$, we have that

$$\big(\mathrm{Im}\,(\mathrm{e}^{i\theta}\, c_l)\big)^2 \;\leqslant\; 1 - \big(\mathrm{Re}\,(\mathrm{e}^{i\theta}\, c_l)\big)^2 < 1 - (1 - \frac{\varepsilon^2}{2})^2 = \varepsilon^2 - \frac{\varepsilon^4}{4}.$$

Since

$$\begin{aligned} |\,\mathrm{e}^{i\theta}\, c_l - 1| \;&=\; \sqrt{\big(1 - \mathrm{Re}\,(\mathrm{e}^{i\theta}\, c_l)\big)^2 + \big(\mathrm{Im}\,(\mathrm{e}^{i\theta}\, c_l)\big)^2} \\ &<\; \sqrt{\varepsilon^4/4 + (\varepsilon^2 - \varepsilon^4/4)} = \varepsilon, \end{aligned}$$

we conclude that

$$\|h_0 - h_1\|_\infty < \varepsilon$$

and

$$\|T - S\|_\infty \leqslant \|h - h_0\|_\infty + \|h_0 - h_1\|_\infty < \varepsilon^8 + \varepsilon < 2\varepsilon. \qquad \square$$

5. The Bishop-Phelps-Bollobás Property for some operators from $L_1(\mu)$ into $C(K)$

Throughout this section, we consider only a finite measure μ on a measurable space (Ω, Σ) and **real** Banach spaces $L_1(\mu)$ and $C(K)$. Our aim is to obtain the Bishop-Phelps-Bollobás property for some classes of operators from $L_1(\mu)$ to $C(K)$, sharpening the results about denseness of norm-attaining operators given by Iwanik in 1982 [26].

We use the following standard representation of operators into $C(K)$ [23, Theorem 1 in p. 490].

Lemma 5.1. *Given a bounded linear operator $T : X \longrightarrow C(K)$, define $F : K \longrightarrow X^*$ by $F(s) = T^*(\delta_s)$, where δ_s is the point measure at $s \in K$. Then, for $x \in X$, the relation $Tx(s) = \langle x, F(s)\rangle$ defines an isometric isomorphism of $\mathcal{L}(X, C(K))$ onto the space of weak* continuous functions from K to X^* with the supremum norm. Moreover, compact operators correspond to norm continuous functions.*

Iwanik [26] considered operators $T \in \mathcal{L}(L_1(\mu), C(K))$ satisfying one of the following conditions:

(1) The map $s \longmapsto T^*\delta_s$ is continuous in measure.
(2) There exists a co-meager set $G \subset K$ such that $\{T^*\delta_s : s \in G\}$ is norm separable in $L_\infty(\mu)$.

We recall that a subset A is said to be a co-meager subset of K if the set $K \setminus A$ is meager, that is, of first category.

Theorem 5.2. *Let $0 < \varepsilon < 1$. Suppose that $T \in \mathcal{L}(L_1(\mu), C(K))$ (real case) has norm one and satisfies condition (1). If $\|Tf\| > 1 - \frac{\varepsilon^2}{6}$ for some $f \in S_{L_1(\mu)}$, then there exist $S \in \mathcal{L}(L_1(\mu), C(K))$ with $\|S\| = 1$ and $g \in S_{L_1(\mu)}$ such that $\|Sg\| = 1$, $\|S - T\| < \varepsilon$, and $\|f - g\| < \varepsilon$. Moreover, S also satisfies condition (1).*

Proof. Without loss of generality, we assume that there exists $s_0 \in K$ such that

$$Tf(s_0) > 1 - \frac{\varepsilon^2}{6}.$$

Consider the function $G : L_\infty(\mu) \longrightarrow L_\infty(\mu)$ given by

$$G(h) = \Big(h \wedge (1 - \varepsilon/3)\Big) \vee (-1 + \varepsilon/3) \qquad \big(h \in L_\infty(\mu)\big).$$

Since the lattice operation G is continuous in the L_∞ norm and T satisfies condition (1), we can see that the mapping $s \longmapsto GT^*\delta_s$ is continuous in measure, hence weak*-continuous. Let $\bar{S}$ be the element of $\mathcal{L}(L_1(\mu), C(K))$ represented by the function $F(s) := GT^*\delta_s$. Then

$$\|\bar{S} - T\| = \sup_{s \in K} \|F(s) - T^*\delta_s\| \leqslant \frac{\varepsilon}{3}.$$

Let

$$C = \Big\{\omega \in \Omega \ : \ \operatorname{sign}\big(f(\omega)\big) T^*\delta_s(\omega) > 1 - \frac{\varepsilon}{3}\Big\}$$

and define $S = \overline{S}/\|\overline{S}\|$ and $g = f|_C/\|f|_C\|$, where $f|_C$ is the restriction of f to the subset C. It is easy to see that S satisfies condition (1) and

$$\|S - T\| \leqslant \|S - \overline{S}\| + \|\overline{S} - T\| = |\|\bar{S}\| - 1| + \|\bar{S} - T\| \leqslant 2\|\bar{S} - T\| < \varepsilon.$$

Moreover, we get

$$\begin{aligned} 1 - \frac{\varepsilon^2}{6} &< Tf(s_0) = \langle T^*\delta_{s_0}, f\rangle = \int_\Omega T^*\delta_{s_0}(\omega) f(\omega)\, d\mu \\ &= \int_C \operatorname{sign}\big(f(\omega)\big) T^*\delta_{s_0}(\omega) |f(\omega)|\, d\mu + \int_{\Omega\setminus C} \operatorname{sign}\big(f(\omega)\big) T^*\delta_{s_0}(\omega) |f(\omega)|\, d\mu \\ &\leqslant \int_C |f(x)|\, d\mu + (1 - \frac{\varepsilon}{3}) \int_{\Omega\setminus C} |f(\omega)|\, d\mu \\ &= 1 - \frac{\varepsilon}{3} \int_{\Omega\setminus C} |f(x)| d\mu, \end{aligned}$$

which implies that

$$\int_{\Omega\setminus C} |f(x)|\, d\mu < \frac{\varepsilon}{2}.$$

Therefore,

$$\begin{aligned} \|g - f\| &\leqslant \|g - f|_C\| + \|f|_C - f\| = 2(1 - \|f_C\|) \\ &= 2\int_{\Omega\setminus C} |f(x)| d\mu < \varepsilon \end{aligned}$$

On the other hand, we see that $Sg(s_0) = \langle S^*\delta_{s_0}, g\rangle = 1$ because $S^*\delta_{s_0}(\omega) = \operatorname{sign}\big(f(x)\big) = \operatorname{sign}\big(g(\omega)\big)$ for every $\omega \in C$. This completes the proof. □

We do not know, and it is clearly of interest, for which topological compact Hausdorff spaces K all operators in $\mathcal{L}(L_1(\mu), C(K))$ satisfy condition (1).

We recall that a bounded linear operator T from $L_1(\mu)$ into a Banach space X is said to be *Bochner representable* if there is a bounded strongly measurable function $g : \Omega \longrightarrow X$ such that

$$Tf = \int f(\omega) g(\omega)\, d\mu(\omega) \qquad \big(f \in L_1(\mu)\big).$$

The Dunford-Pettis-Phillips Theorem [21, Theorem 12 p. 75] says that $T \in \mathcal{L}(L_1(\mu), X)$ is weakly compact if and only if T is Bochner representable by a function g which has an essentially relatively weakly compact range. Iwanik [26] showed that every Bochner representable operator from $L_1(\mu)$ into $C(K)$ satisfies condition (1). Moreover, we get the following result which has been independently obtained in [6, Corollary 2.4].

Corollary 5.3. *Let $0 < \varepsilon < 1$. Suppose that $T \in \mathcal{L}(L_1(\mu), C(K))$ (real case) has norm-one and it is Bochner representable (resp. weakly compact). If $\|Tf\| > 1 - \frac{\varepsilon^2}{6}$ for some $f \in S_{L_1(\mu)}$, then there exist a Bochner representable (resp. weakly compact) operator $S \in \mathcal{L}(L_1(\mu), C(K))$ with $\|S\| = 1$ and $g \in S_{L_1(\mu)}$ such that $\|Sg\| = 1$, $\|S - T\| < \varepsilon$, and $\|f - g\| < \varepsilon$.*

Proof. By Theorem 5.2, it is enough to show that if T is a Bochner representable operator from $L_1(\mu)$ into $C(K)$, then $F(s) = T^*\delta_s$ is continuous in measure and that the operator S defined in the proof is Bochner representable.

Let $g : \Omega \longrightarrow C(K)$ be a bounded strongly measurable function which represents T. It is easy to check that $F(s) = g(\cdot)(s)$ for all $s \in K$. Since the range of g is separable, the range of T is separable and contained in a separable sub-algebra A of $C(K)$ with unit. By the Gelfand representation theorem, A is isometrically isomorphic to $C(\bar{K})$ for some compact metrizable space $\bar{K}$. So, we may assume that K is metrizable. To show that the mapping $F(s) = T^*\delta_s = g(\omega)(s)$ is continuous in measure, assume that a sequence (s_n) converges to s in K. Then for all $\omega \in \Omega$,

$$\lim_{n\to\infty} |g(\omega)(s_n) - g(\omega)(s)| = 0.$$

By the dominated convergence theorem, we have that

$$\lim_{n\to\infty} \sup_{f\in S_{L_\infty(\mu)}} \int f(\omega)(g(\omega)(s_n) - g(\omega)(s))\, d\mu(\omega) \leqslant \lim_{n\to\infty} \int |g(\omega)(s_n) - g(\omega)(s)|\, d\mu(\omega) = 0.$$

Hence the sequence $(g(\cdot)(s_n))_n$ converges to $g(\cdot)(s)$ in measure. That is, $(F(s_n))_n$ converges to $F(s)$.

We note that the operator $\bar{S}$ in the proof of Theorem 5.2 is determined by $GT^*\delta_s = G(g(\cdot)(s))$. Since the mapping

$$s \longmapsto G(g(\cdot)(s))(\omega) = (g(\omega)(s) \wedge (1 - \varepsilon/3)) \vee (-1 + \varepsilon/3))$$

is continuous for each $\omega \in \Omega$, the operator $\bar{S}$ is Bochner representable by this mapping. Finally, if T is weakly compact, then the proof is done by the Dunford-Pettis-Phillips theorem. □

As observed in [26], the operator $T : L_1[0,1] \longrightarrow C[0,1]$ determined by $T^*\delta_s = \chi_{[0,s]}$ is not Bochner representable, but satisfies condition (1).

For condition (2), we have the following result.

Theorem 5.4. *Let $0 < \varepsilon < 1$. Suppose that $T \in \mathcal{L}(L_1(\mu), C(K))$ (real case) has norm-one and satisfies condition (2). If $\|Tf\| > 1 - \frac{\varepsilon^2}{4}$ for some $f \in S_{L_1(\mu)}$, then there exist $S \in \mathcal{L}(L_1(\mu), C(K))$ with $\|S\| = 1$ and $g \in S_{L_1(\mu)}$ such that $\|Sg\| = 1$, $\|S - T\| < \varepsilon$, and $\|f - g\| < \varepsilon$. Moreover, S also satisfies condition (2).*

Proof. By using a suitable isometric isomorphism, we may first assume that f is nonnegative. Let G be the co-meager set in the condition (2) and $(T^*\delta_{s_k})_k$ be a sequence which is $\|\cdot\|_\infty$-dense in the closure of $\{T^*\delta_s \ : \ s \in G\} \subset L_\infty(\mu)$. Observe that the sets

$$\{\omega \in \Omega : a < T^*\delta_{s_k}(\omega) < b\}$$

where $a, b \in \mathbb{Q}$ and $k \geqslant 1$, form a countable family $\{A_i\}_i$ of measurable subsets of Ω. We define, for each i, the functions

$$u_i(s) = \text{ess. inf}\{T^*\delta_s(\omega) : \omega \in A_i\} \quad \text{and} \quad v_i(s) = \text{ess. sup}\{T^*\delta_s(\omega) : \omega \in A_i\}.$$

Let U_i and V_i be the set of all continuity points of u_i and v_i for all i, respectively. Let F be the intersection of all subsets U_i's and V_i's. We claim that the functions u_i's are upper semi-continuous and the functions v_i's are lower semi-continuous. Indeed, recall (see [36, 3.7 Definition] for a reference) that

$$v_i(s) = \inf\Big\{\lambda \in \mathbb{R} : \mu\{\omega \in A_i : T^*\delta_s(\omega) > \lambda\} = 0\Big\},$$

where $\inf \emptyset = \infty$ and $\inf \mathbb{R} = -\infty$. To show that the set $\{s : \lambda < v_i(s)\}$ is open in K for all $\lambda \in \mathbb{R}$, suppose that $v_i(s_0) > \lambda_0$ for some $s_0 \in K$ and $\lambda_0 \in \mathbb{R}$. It suffice to prove that there is an open neighborhood V of s_0 such that $V \subset \{s : v_i(s) > \lambda_0\}$. We note that $\mu\{\omega \in A_i : T^*\delta_{s_0}(\omega) > \lambda_0\} > 0$ and there exists $\lambda_1 > \lambda_0$ such that

$$\mu\{\omega \in A_i : T^*\delta_{s_0}(\omega) > \lambda_1\} > 0.$$

Let $E = \{\omega \in A_i : T^*\delta_{s_0}(\omega) > \lambda_1\}$. Then

$$\frac{1}{\mu(E)} \int_E T^*\delta_{s_0}(\omega) d\mu(\omega) > \lambda_1 > \lambda_0.$$

Since the map $s \longmapsto T^*\delta_s$ is weak* continuous on $L_\infty(\mu)$, the set

$$V := \left\{ s \in K : \frac{1}{\mu(E)} \int_E T^*\delta_s(\omega) d\mu(\omega) > \lambda_1 \right\}$$

is an open subset containing s_0. We note that $\mu\{\omega \in A_i : T^*\delta_s(\omega) > \lambda_1\} > 0$ for all $s \in V$. Otherwise, there is $s_1 \in V$ such that $\mu\{\omega \in A_i : T^*\delta_{s_1}(\omega) > \lambda_1\} = 0$. Then $T^*\delta_{s_1}(\omega) \leqslant \lambda_1$ almost everywhere $\omega \in A_i$ and

$$\frac{1}{\mu(E)} \int_E T^*\delta_{s_1}(\omega) d\mu(\omega) \leqslant \lambda_1.$$

This is a contradiction to the fact that s_1 is an element of V, which implies that $v_i(s) > \lambda_0$ for all $s \in V$ and $V \subset \{s : v_i(s) > \lambda_0\}$. This gives the lower semi-continuity of v_i. The upper semi-continuity of u_i follows from the fact that $-u_i$ is lower semi-continuous. The claim is proved.

We deduce then that the set F is co-meager (c.f. see [31, § 32 II. p. 400]). Since the set $\{s : s \in K, |Tf(s)| > 1 - \frac{\varepsilon^2}{4}\}$ is nonempty and open, there exists $s_0 \in F \cap G$ such that $|Tf(s_0)| > 1 - \frac{\varepsilon^2}{4}$. Without loss of generality, we may assume that

$$Tf(s_0) = \langle T^*\delta_{s_0}, f\rangle > 1 - \frac{\varepsilon^2}{4}.$$

Because of the denseness of the sequence $(T^*\delta_{s_k})_k$, there exists $k_0 \in \mathbb{N}$ such that

$$Tf(s_{k_0}) = \left\langle T^*\delta_{s_{k_0}}, f\right\rangle > 1 - \frac{\varepsilon^2}{4} \qquad \text{and} \qquad \|T^*\delta_{s_0} - T^*\delta_{s_{k_0}}\| < \frac{\varepsilon}{4}.$$

Fix $q \in \mathbb{Q}$ such that $1 - \frac{3}{4}\varepsilon < q < 1 - \frac{\varepsilon}{2}$ and let

$$C = \left\{\omega \in \Omega \ : \ T^*\delta_{s_{k_0}}(\omega) > q\right\}.$$

Then

$$\begin{aligned}
1 - \frac{\varepsilon^2}{4} < \left\langle T^*\delta_{s_{k_0}}, f\right\rangle &= \int_\Omega T^*\delta_{s_{k_0}}(\omega) f(\omega)\, d\mu \\
&= \int_C T^*\delta_{s_{k_0}}(\omega) f(\omega)\, d\mu + \int_{\Omega\setminus C} T^*\delta_{s_{k_0}}(\omega) f(\omega)\, d\mu \\
&\leqslant \int_C f(\omega)\, d\mu + \left(1 - \frac{\varepsilon}{2}\right) \int_{\Omega\setminus C} f(\omega)\, d\mu \\
&= 1 - \frac{\varepsilon}{2} \int_{\Omega\setminus C} f(\omega)\, d\mu.
\end{aligned}$$

Hence we have that

$$\int_{\Omega\setminus C} f(\omega)\, d\mu < \frac{\varepsilon}{2} \quad \text{and} \quad \int_C f(\omega)\, d\mu > 1 - \frac{\varepsilon}{2}.$$

Let $B_n = \{\omega : q < T^*\delta_{s_{k_0}}(\omega) < n\}$ for each n. Then $C = \bigcup_{n=1}^{\infty} B_n$ and there exists n_0 such that

$$\int_{B_{n_0}} f(\omega)\, d\mu > 1 - \frac{\varepsilon}{2}.$$

Hence $B_{n_0} = A_{i_0}$ for some i_0 and $\mu(A_{i_0}) > 0$. This implies that $u_{i_0}(s_{k_0}) \geqslant q$ and $u_{i_0}(s_0) \geqslant q - \frac{\varepsilon}{4} > 1 - \varepsilon$. Setting $A = A_{i_0}$, it is also clear that

$$\left\| \frac{f|_A}{\|f|_A\|} - f \right\| < \varepsilon.$$

Since u_{i_0} is continuous at s_0, there exist an open neighborhood U of s_0 and a continuous function $h : K \longrightarrow [0,1]$ such that $u_{i_0}(s) > 1 - \varepsilon$ for all $s \in U$, $h(s_0) = 1$ and $h(U^c) = 0$. We define a weak*-continuous map $M : K \longrightarrow L_\infty(\mu)$ by

$$M(s)(\omega) = T^*\delta_s(\omega) + \chi_A(\omega)h(s)(1 - T^*\delta_s(\omega)) \qquad \big(\omega \in \Omega,\ s \in K\big).$$

We note that $M(s_0) = 1$ for all $\omega \in A$. It is also easy to get that

$$\|M(s)(\omega) - T^*\delta_s(\omega)\| = \|\chi_A(\omega)h(s)(1 - T^*\delta_s(\omega))\| < \varepsilon \quad \text{and} \quad \sup_{s\in K} \|M(s)\| = 1.$$

Let S be the operator represented by the function M. Then S satisfies condition (2), $S\left(\frac{f|_A}{\|f|_A\|}\right)(s_0) = 1$ and $\|S - T\| < \varepsilon$. □

As shown in [26], the Dunford-Pettis-Phillips Theorem implies that every weakly compact operator T in from $L_1(\mu)$ to an arbitrary Banach space Y has separable range, hence the range of its weakly compact adjoint T^* is also separable and so T satisfies condition (2). On the other hand, there are Bochner representable operators which do not satisfy the condition (2) (see [26]). Indeed, let μ be a strictly positive probability measure on $\mathbb{N}$ and consider the operator $T \in \mathcal{L}(L_1(\mu), C(\{0,1\}^{\mathbb{N}})$ defined by $Tf(s) = \int f(n)\pi_n(s)\, d\mu(n)$, where π_n be the n-th natural projection on $\{0,1\}^{\mathbb{N}}$. Then T is Bochner representable, while $\{T^*\delta_s : s \in G\}$ is non-separable in $L_\infty(\mu)$ for every uncountable subset G of $\{0,1\}^{\mathbb{N}}$.

Finally, let us comment that it is also observed in [26] that if K has a countable dense subset of isolated points, then condition (2) is automatically satisfied for all $T \in \mathcal{L}(L_1(\mu), C(K))$. Actually, in this case, $C(K)$ has the so-called property (β) and then the pair $(X, C(K))$ has the BPBp for all Banach spaces X [3, Theorem 2.2].

It would be of interest to characterize those topological Hausdorff compact spaces K such that $(X, C(K))$ has the BPBp for every Banach space X.

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