BISHOP-PHELPS-BOLLOBÁS MODULI OF A BANACH SPACE

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Dedicated to the memory of Robert R. Phelps (1926–2013)

Abstract. We introduce two Bishop-Phelps-Bollobás moduli of a Banach space which measure, for a given Banach space, what is the best possible Bishop-Phelps-Bollobás theorem in this space. We show that there is a common upper bound for these moduli for all Banach spaces and we present an example showing that this bound is sharp. We prove the continuity of these moduli and an inequality with respect to duality. We calculate the two moduli for Hilbert spaces and also present many examples for which the moduli have the maximum possible value (among them, there are $C(K)$ spaces and $L_1(\mu)$ spaces). Finally, we show that if a Banach space has the maximum possible value of any of the moduli, then it contains almost isometric copies of the real space $\ell_\infty^{(2)}$ and present an example showing that this condition is not sufficient.

1. Introduction

The classical Bishop-Phelps theorem of 1961 [4] states that the set of norm attaining functionals on a Banach space is norm dense in the dual space. A few years later, B. Bollobás [5] gave a sharper version of this theorem allowing to approximate at the same time a functional and a vector in which it almost attains the norm (see the result below). The main aim of this paper is to study the best possible approximation of this kind that one may have in each Banach space, measuring it by using two moduli which we define.

We first present the original result by Bollobás which nowadays is known as the *Bishop-Phelps-Bollobás theorem*. We need to fix some notation. Given a (real or complex) Banach space X, we write B_X and S_X to denote the closed unit ball and the unit sphere of the space, and X^* denotes the (topological) dual of X. We will also use the notation

$$\Pi(X) := \big\{(x,x^*) \in X \times X^* \,:\, \|x\| = \|x^*\| = x^*(x) = 1\big\}.$$

Theorem 1.1 (Bishop-Phelps-Bollobás theorem [5])**.**
Let X be a Banach space. Suppose $x \in S_X$ and $x^* \in S_{X^*}$ satisfy $|1-x^*(x)| \leqslant \varepsilon^2/2$ for some $0<\varepsilon<1/2$. Then there exists $(y,y^*) \in \Pi(X)$ such that $\|x-y\| < \varepsilon+\varepsilon^2$ and $\|x^*-y^*\| \leqslant \varepsilon$.

The idea is that given $(x,x^*) \in S_X \times S_{X^*}$ such that $x^*(x) \sim 1$, there exist $y \in S_X$ close to x and $y^* \in S_{X^*}$ close to x^* for which $y^*(y)=1$. This result has many applications, especially for the theory of numerical ranges, see [5, 6].

Our objective is to introduce two moduli which measure, for a given Banach space, what is the best possible Bollobás theorem in this space, that is, how close can be y to x and y^* to x^* in the result above depending on how close is $x^*(x)$ to 1. In the first modulus, we allow the vector and the functional to

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have norm less than or equal to one, whereas in the second modulus we only consider norm-one vectors and functionals.

Definitions 1.2 (Bishop-Phelps-Bollobás moduli)**.**
Let X be a Banach space. The *Bishop-Phelps-Bollobás modulus* of X is the function $\Phi_X : (0,2) \longrightarrow \mathbb{R}^+$ such that given $\delta \in (0,2)$, $\Phi_X(\delta)$ is the infimum of those $\varepsilon > 0$ satisfying that for every $(x, x^*) \in B_X \times B_{X^*}$ with $\operatorname{Re} x^*(x) > 1 - \delta$, there is $(y, y^*) \in \Pi(X)$ with $\|x - y\| < \varepsilon$ and $\|x^* - y^*\| < \varepsilon$.

The *spherical Bishop-Phelps-Bollobás modulus* of X is the function $\Phi_X^S : (0,2) \longrightarrow \mathbb{R}^+$ such that given $\delta \in (0,2)$, $\Phi_X^S(\delta)$ is the infimum of those $\varepsilon > 0$ satisfying that for every $(x, x^*) \in S_X \times S_{X^*}$ with $\operatorname{Re} x^*(x) > 1 - \delta$, there is $(y, y^*) \in \Pi(X)$ with $\|x - y\| < \varepsilon$ and $\|x^* - y^*\| < \varepsilon$.

Evidently, $\Phi_X^S(\delta) \leqslant \Phi_X(\delta)$, so any estimation from above for $\Phi_X(\delta)$ is also valid for $\Phi_X^S(\delta)$ and, viceversa, any estimation from below for $\Phi_X^S(\delta)$ is also valid for $\Phi_X(\delta)$.

Recall that the dual of a complex Banach space X is isometric (taking real parts) to the dual of the real subjacent space $X_{\mathbb{R}}$. Also, $\Pi(X)$ does not change if we consider X as a real Banach space (indeed, if $(x, x^*) \in \Pi(X)$ then $x^* \in S_{X^*}$ and $x \in S_X$ satisfies $x^*(x) = 1$ so, obviously, $\operatorname{Re} x^*(x) = 1$ and $(x, \operatorname{Re} x^*) \in \Pi(X_{\mathbb{R}})$). Therefore, only the real structure of the space is playing a role in the above definitions. Nevertheless, we prefer to develop the theory for real and complex spaces which, actually, does not suppose much more effort. This is mainly because for classical sequence or function spaces, the real space underlying the complex version of the space is not equal, in general, to the real version of the space. Unless otherwise is stated, the (arbitrary or concrete) spaces we are dealing with will be real or complex and the results work in both cases.

The following notations will help to the understanding and further use of Definitions 1.2. Let X be a Banach space and fix $0 < \delta < 2$. Write

$$A_X(\delta) := \big\{(x, x^*) \in B_X \times B_{X^*} \ : \ \operatorname{Re} x^*(x) > 1 - \delta\big\},$$
$$A_X^S(\delta) := \big\{(x, x^*) \in S_X \times S_{X^*} \ : \ \operatorname{Re} x^*(x) > 1 - \delta\big\}.$$

It is clear that

$$\Phi_X(\delta) = \sup_{(x,x^*)\in A_X(\delta)} \ \inf_{(y,y^*)\in\Pi(X)} \ \max\{\|x - y\|, \|x^* - y^*\|\},$$
$$\Phi_X^S(\delta) = \sup_{(x,x^*)\in A_X^S(\delta)} \ \inf_{(y,y^*)\in\Pi(X)} \ \max\{\|x - y\|, \|x^* - y^*\|\}.$$

We denote $d_H(A, B)$ the Hausdorff distance between $A, B \subset X \times X^*$ associated to the ℓ_∞-distance $\operatorname{dist}_\infty$ in $X \times X^*$, that is,

$$\operatorname{dist}_\infty\big((x, x^*), (y, y^*)\big) = \max\{\|x - y\|, \|x^* - y^*\|\}$$

for $(x, x^*), (y, y^*) \in X \times X^*$, and

$$d_H(A, B) = \max\left\{\sup_{a\in A}\ \inf_{b\in B} \operatorname{dist}_\infty(a, b)\,,\ \sup_{b\in B}\ \inf_{a\in A} \operatorname{dist}_\infty(a, b)\right\}$$

for $A, B \subset X \times X^*$. We clearly have that

$$\Phi_X(\delta) = d_H\big(A_X(\delta), \Pi(X)\big) \qquad \text{and} \qquad \Phi_X^S(\delta) = d_H\big(A_X^S(\delta), \Pi(X)\big)$$

for every $0 < \delta < 2$ (observe that $\Pi(X) \subset A_X(\delta)$ and $\Pi(X) \subset A_X^S(\delta)$ for every δ).

The following result is immediate.

Remark 1.3. *Let X be a Banach space. Given $\delta_1, \delta_2 \in (0,2)$ with $\delta_1 < \delta_2$, one has*

$$A_X(\delta_1) \subset A_X(\delta_2) \quad \textit{and} \quad A_X^S(\delta_1) \subset A_X^S(\delta_2).$$

Therefore, the functions $\Phi_X(\cdot)$ and $\Phi_X^S(\cdot)$ are increasing.

Routine computations and the fact that the Hausdorff distance does not change if we take closure in one of the sets, provide the following observations.

Remark 1.4. *Let X be a Banach space. Then, for every $\delta \in (0,2)$, one has*

$$\begin{aligned}\Phi_X(\delta) := \inf\bigl\{\varepsilon > 0 \,:\, &\forall (x,x^*) \in B_X \times B_{X^*} \text{ with } \operatorname{Re} x^*(x) > 1-\delta,\\ &\exists (y,y^*) \in \Pi(X) \text{ with } \operatorname{dist}_\infty((x,x^*),(y,y^*)) < \varepsilon\bigr\}\\ = \inf\bigl\{\varepsilon > 0 \,:\, &\forall (x,x^*) \in B_X \times B_{X^*} \text{ with } \operatorname{Re} x^*(x) \geqslant 1-\delta,\\ &\exists (y,y^*) \in \Pi(X) \text{ with } \operatorname{dist}_\infty((x,x^*),(y,y^*)) < \varepsilon\bigr\}\\ = \inf\bigl\{\varepsilon > 0 \,:\, &\forall (x,x^*) \in B_X \times B_{X^*} \text{ with } \operatorname{Re} x^*(x) > 1-\delta,\\ &\exists (y,y^*) \in \Pi(X) \text{ with } \operatorname{dist}_\infty((x,x^*),(y,y^*)) \leqslant \varepsilon\bigr\}\\ = \inf\bigl\{\varepsilon > 0 \,:\, &\forall (x,x^*) \in B_X \times B_{X^*} \text{ with } \operatorname{Re} x^*(x) \geqslant 1-\delta,\\ &\exists (y,y^*) \in \Pi(X) \text{ with } \operatorname{dist}_\infty((x,x^*),(y,y^*)) \leqslant \varepsilon\bigr\},\end{aligned}$$

and

$$\begin{aligned}\Phi_X^S(\delta) := \inf\bigl\{\varepsilon > 0 \,:\, &\forall (x,x^*) \in S_X \times S_{X^*} \text{ with } \operatorname{Re} x^*(x) > 1-\delta,\\ &\exists (y,y^*) \in \Pi(X) \text{ with } \operatorname{dist}_\infty((x,x^*),(y,y^*)) < \varepsilon\bigr\}\\ = \inf\bigl\{\varepsilon > 0 \,:\, &\forall (x,x^*) \in S_X \times S_{X^*} \text{ with } \operatorname{Re} x^*(x) \geqslant 1-\delta,\\ &\exists (y,y^*) \in \Pi(X) \text{ with } \operatorname{dist}_\infty((x,x^*),(y,y^*)) < \varepsilon\bigr\}\\ = \inf\bigl\{\varepsilon > 0 \,:\, &\forall (x,x^*) \in S_X \times S_{X^*} \text{ with } \operatorname{Re} x^*(x) > 1-\delta,\\ &\exists (y,y^*) \in \Pi(X) \text{ with } \operatorname{dist}_\infty((x,x^*),(y,y^*)) \leqslant \varepsilon\bigr\}\\ = \inf\bigl\{\varepsilon > 0 \,:\, &\forall (x,x^*) \in S_X \times S_{X^*} \text{ with } \operatorname{Re} x^*(x) \geqslant 1-\delta,\\ &\exists (y,y^*) \in \Pi(X) \text{ with } \operatorname{dist}_\infty((x,x^*),(y,y^*)) \leqslant \varepsilon\bigr\}.\end{aligned}$$

Observe that *the smaller are the functions $\Phi_X(\cdot)$ and $\Phi_X^S(\cdot)$, the better is the approximation on the space*. It can be deduced from the Bishop-Phelps-Bollobás theorem that there is a common upper bound for $\Phi_X(\cdot)$ and $\Phi_X^S(\cdot)$ for all Banach spaces X. In the next section we present the best possible upper bound. We will show that

$$\Phi_X^S(\delta) \leqslant \Phi_X(\delta) \leqslant \sqrt{2\delta} \qquad \bigl(0 < \delta < 2,\ X \text{ Banach space}\bigr). \tag{1}$$

This follows from a result by R. Phelps [13]. A version for $\Phi_X^S(\delta)$ for small δ's can be also deduced from the Brøndsted-Rockafellar variational principle [14, Theorem 3.17], as claimed in [7]. The sharpness of (1) can be verified by considering the real space $X = \ell_\infty^{(2)}$.

In section 3, we prove that for every Banach space X, the moduli $\Phi_X(\delta)$ and $\Phi_X^S(\delta)$ are continuous in δ. We prove that $\Phi_X(\delta) \leqslant \Phi_{X^*}(\delta)$ and $\Phi_X^S(\delta) \leqslant \Phi_{X^*}^S(\delta)$. Finally, we show that $\Phi_X(\delta) = \sqrt{2\delta}$ if and only if $\Phi_X^S(\delta) = \sqrt{2\delta}$.

Examples of spaces for which the two moduli are computed are presented in section 4. Among other results, the moduli of $\mathbb{R}$ and of every real or complex Hilbert space of (real)-dimension greater than one are calculated, and there are presented a number of spaces for which the value of both moduli are $\sqrt{2\delta}$ (i.e. the maximal possible value) for small δ's: namely c_0, ℓ_1 and, more in general, $L_1(\mu)$, $C_0(L)$, unital C^*-algebras with non-trivial centralizer…

The main result of section 5 states that if a Banach space X satisfies $\Phi_X(\delta_0) = \sqrt{2\delta_0}$ (equivalently, $\Phi_X^S(\delta_0) = \sqrt{2\delta_0}$) for some $\delta_0 \in (0,1/2)$, then X contains almost isometric copies of the real space $\ell_\infty^{(2)}$. We provide, for every $\delta \in (0,1/2)$, an example of a three dimensional real space Z containing an isometric copy of $\ell_\infty^{(2)}$ for which $\Phi_Z(\delta) < \sqrt{2\delta}$. This is the content of section 6.

2. The upper bound of the moduli

Our first result is the promised best upper bound of the Bishop-Phelps-Bollobás moduli.

Theorem 2.1. *For every Banach space X and every $\delta \in (0,2)$, $\Phi_X(\delta) \leqslant \sqrt{2\delta}$ and so, $\Phi_X^S(\delta) \leqslant \sqrt{2\delta}$*

We deduce the above result from [13, Corollary 2.2], which was stated for general bounded convex sets on real Banach spaces. Particularizing the result to the case of the unit ball of a Banach space, using a routine argument to change non-strict inequalities to strict inequalities, and taking into account that the dual of a complex Banach space is isometric (taking real parts) to the dual of the real subjacent space, we get the following result.

Proposition 2.2 (Particular case of [13, Corollary 2.2])**.** *Let X be Banach space. Suppose that $z^* \in S_{X^*}$, $z \in B_X$ and $\eta > 0$ are given such that $\operatorname{Re} z^*(z) > 1 - \eta$. Then, for any $k \in (0,1)$ there exist $\tilde{y}^* \in X^*$ and $\tilde{y} \in S_X$ such that*

$$\|\tilde{y}^*\| = \tilde{y}^*(\tilde{y}), \qquad \|z - \tilde{y}\| < \frac{\eta}{k}, \qquad \|z^* - \tilde{y}^*\| < k.$$

Proof of Theorem 2.1. We have to show that given $(x, x^*) \in B_X \times B_{X^*}$ with $\operatorname{Re} x^*(x) > 1 - \delta$, there exists $(y, y^*) \in \Pi(X)$ such that $\|x - y\| < \sqrt{2\delta}$ and $\|x^* - y^*\| < \sqrt{2\delta}$. We first prove the case of $\delta \in (0,1)$. In this case,

$$0 < 1 - \delta < \|x^*\| \leqslant 1,$$

so, if we write $\eta = \dfrac{\|x^*\| - 1 + \delta}{\|x^*\|} > 0$, $z^* = x^*/\|x^*\|$ and $z = x$, one has

$$\operatorname{Re} z^*(z) > 1 - \eta.$$

Next, we consider $k = \eta/\sqrt{2\delta}$ and claim that $0 < k < 1$. Indeed, as the function

$$\varphi(t) = \frac{t - 1 + \delta}{\sqrt{2\delta}\, t} \qquad (t \in \mathbb{R}^+) \tag{2}$$

is strictly increasing, $k = \varphi(\|x^*\|)$ and $1 - \delta < \|x^*\| \leqslant 1$, we have that

$$0 = \varphi(1 - \delta) < k \leqslant \varphi(1) = \frac{\sqrt{\delta}}{\sqrt{2}} < 1,$$

as desired. Therefore, we apply Proposition 2.2 with $z^* \in S_{X^*}$, $z \in B_X$, $\eta > 0$ and $0 < k < 1$ to obtain $\tilde{y}^* \in X^*$ and $\tilde{y} \in S_X$ satisfying

$$\|\tilde{y}^*\| = \tilde{y}^*(\tilde{y}), \qquad \|z - \tilde{y}\| < \frac{\eta}{k} = \sqrt{2\delta}, \qquad \left\| \frac{x^*}{\|x^*\|} - \tilde{y}^* \right\| < k = \frac{\|x^*\| - 1 + \delta}{\|x^*\|\sqrt{2\delta}}.$$

As $k < 1$, we get $\tilde{y}^* \neq 0$ and we write $y^* = \frac{\tilde{y}^*}{\|\tilde{y}^*\|}$, $y = \tilde{y}$, to get that $(y, y^*) \in \Pi(X)$. We already have that $\|x - y\| < \sqrt{2\delta}$. On the other hand, we have

$$\begin{aligned}
\|x^* - y^*\| = \left\| x^* - \frac{\tilde{y}^*}{\|\tilde{y}^*\|} \right\| &\leqslant \big\| x^* - \|x^*\|\tilde{y}^* \big\| + \left\| \|x^*\|\tilde{y}^* - \frac{\tilde{y}^*}{\|\tilde{y}^*\|} \right\| \\
&\leqslant \|x^*\| \left\| \frac{x^*}{\|x^*\|} - \tilde{y}^* \right\| + \big| \|x^*\|\|\tilde{y}^*\| - 1 \big| \\
&\leqslant \|x^*\| \left\| \frac{x^*}{\|x^*\|} - \tilde{y}^* \right\| + \big| \|x^*\|\|\tilde{y}^*\| - \|x^*\| \big| + \big| 1 - \|x^*\| \big| \\
&\leqslant \|x^*\| \left[\left\| \frac{x^*}{\|x^*\|} - \tilde{y}^* \right\| + \big| \|\tilde{y}^*\| - 1 \big| \right] + 1 - \|x^*\| \\
&\leqslant 2\|x^*\| \left\| \frac{x^*}{\|x^*\|} - \tilde{y}^* \right\| + 1 - \|x^*\| \\
&< \frac{2}{\sqrt{2\delta}} \big(\|x^*\| - 1 + \delta \big) + 1 - \|x^*\|.
\end{aligned}$$

Now, as the function

$$\gamma(t) = \frac{2}{\sqrt{2\delta}} \big(t - 1 + \delta \big) + 1 - t \qquad \big(t \in [0,1] \big)$$

is strictly increasing (for this we only need $0<\delta<2$), we get $\gamma(\|x^*\|)\leqslant\gamma(1)=\frac{2\delta}{\sqrt{2\delta}}=\sqrt{2\delta}$. It follows that $\|x^*-y^*\|<\sqrt{2\delta}$, as desired.

Let us now prove the case when $\delta\in[1,2)$. Here, it can be routinely verified that

$$\frac{\delta-1}{\sqrt{2\delta}-1}<\sqrt{2\delta}-1$$

so, writing

$$\psi(\delta)=\frac{1}{2}\left(\frac{\delta-1}{\sqrt{2\delta}-1}+\sqrt{2\delta}-1\right)$$

we get

$$\frac{\delta-1}{\sqrt{2\delta}-1}<\psi(\delta)<\sqrt{2\delta}-1\qquad\big(\delta\in[1,2)\big). \tag{3}$$

Now, we distinguish two situations. First suppose that $\|x^*\|\leqslant\psi(\delta)$. Then, we take any $y\in S_X$ such that $\|x-y\|\leqslant 1$ and take $y^*\in S_{X^*}$ such that $y^*(y)=1$. Then, $(y,y^*)\in\Pi(X)$, $\|x-y\|\leqslant 1<\sqrt{2\delta}$ and

$$\|x^*-y^*\|\leqslant 1+\|x^*\|\leqslant 1+\psi(\delta)<\sqrt{2\delta}$$

by (3). Otherwise, suppose $\|x^*\|>\psi(\delta)$. We then write $\eta=\dfrac{\|x^*\|-1+\delta}{\|x^*\|}>0$ and $k=\eta/\sqrt{2\delta}$ as in the previous case, and we show that $k<1$. This is trivial for the case $\delta=1$ and for $\delta>1$, we use that the function φ defined in (2) is now strictly decreasing to get that

$$k=\varphi(\|x^*\|)<\varphi\big(\psi(\delta)\big)<\varphi\left(\frac{\delta-1}{\sqrt{2\delta}-1}\right)=1.$$

Then, the rest of the proof follows the same lines of the case when $\delta\in(0,1)$ since this hypothesis is no longer used. □

Notice that the above proof is much simpler if we restrict to $x^*\in S_{X^*}$ (in particular, to the spherical modulus $\Phi^S_X(\delta)$), but the result for non-unital functionals is stronger. Actually, the following stronger version can be deduced by modifying the selection of k in the proof of Theorem 2.1.

Remark 2.3. *For every $0<\theta<1$ and every $0<\delta<2$, there is $\rho=\rho(\delta,\theta)>0$ such that for every Banach space X, if $x^*\in B_{X^*}$ with $\|x^*\|\leqslant\theta$, $x\in B_X$ satisfy that $\operatorname{Re} x^*(x)>1-\delta$, then there is a pair $(y,y^*)\in\Pi(X)$ satisfying*

$$\|x-y\|<\sqrt{2\delta}-\rho\qquad\text{and}\qquad\|x^*-y^*\|<\sqrt{2\delta}-\rho.$$

Let us observe that, given $0<\theta<1$, the hypothesis above is not empty only when $1-\theta<\delta$. On the other hand, in the proof it is sufficient to consider only the case of $\delta<1+\theta$. Otherwise, the evident inequality $\operatorname{Re} x^*(x)>-\theta=1-(1+\theta)$ implies that there is a pair $(y,y^*)\in\Pi(X)$ satisfying $\|x-y\|<\sqrt{2(1+\theta)}$ and $\|x^*-y^*\|<\sqrt{2(1+\theta)}$. Hence the statement of our remark holds true with $\rho:=\sqrt{2\delta}-\sqrt{2(1+\theta)}$.

Next, we rewrite Theorem 2.1 in two equivalent ways.

Corollary 2.4. *Let X be a Banach space.*

(a) *Let $0<\varepsilon<2$ and suppose that $x\in B_X$ and $x^*\in B_{X^*}$ satisfy*

$$\operatorname{Re} x^*(x)>1-\varepsilon^2/2.$$

Then, there exists $(y,y^*)\in\Pi(X)$ such that

$$\|x-y\|<\varepsilon\qquad\textit{and}\qquad\|x^*-y^*\|<\varepsilon.$$

(b) *Let $0<\delta<2$ and suppose that $x\in B_X$ and $x^*\in B_{X^*}$ satisfy*

$$\operatorname{Re} x^*(x)>1-\delta.$$

Then, there exists $(y,y^*)\in\Pi(X)$ such that

$$\|x-y\|<\sqrt{2\delta}\qquad\textit{and}\qquad\|x^*-y^*\|<\sqrt{2\delta}.$$

As the last result of this section, we present an example of a Banach space for which the estimate in Theorem 2.1 is sharp.

Example 2.5. *Let X be the real space $\ell_\infty^{(2)}$. Then, $\Phi_X^S(\delta) = \Phi_X(\delta) = \sqrt{2\delta}$ for all $\delta \in (0, 2)$.*

Proof. Fix $0 < \delta < 2$. We consider

$$z = (1 - \sqrt{2\delta}, 1) \in S_X \quad \text{and} \quad z^* = \left(\frac{\sqrt{2\delta}}{2}, 1 - \frac{\sqrt{2\delta}}{2}\right) \in S_{X^*},$$

and observe that $z^*(z) = 1 - \delta$. Now, suppose we may find $(y, y^*) \in \Pi(X)$ such that $\|z - y\| < \sqrt{2\delta}$ and $\|z^* - y^*\| < \sqrt{2\delta}$. By the shape of B_X, we only have two possibilities: either y is an extreme point of B_X or y^* is an extreme point of B_{X^*} (this is actually true for all two-dimensional real spaces). Suppose first that y is an extreme point of B_X, which has the form $y = (a, b)$ with $a, b \in \{-1, 1\}$. As

$$\|z - y\| = \max\{|1 - \sqrt{2\delta} - a|, |1 - b|\} < \sqrt{2\delta},$$

we are forced to have $b = 1$ and $a = -1$. Now, we have $y^* = (-t, 1 - t)$ for some $0 \leqslant t \leqslant 1$ and

$$\|z^* - y^*\| = \frac{\sqrt{2\delta}}{2} + t + \left|t - \frac{\sqrt{2\delta}}{2}\right| = \max\left\{\sqrt{2\delta}, 2t\right\} \geqslant \sqrt{2\delta},$$

a contradiction. On the other hand, if y^* is an extreme point of B_{X^*}, then either $y^* = (a, 0)$ or $y^* = (0, b)$ for suitable $a, b \in \{-1, 1\}$. In the first case, as

$$\|z^* - y^*\| = \left|\frac{\sqrt{2\delta}}{2} - a\right| + 1 - \frac{\sqrt{2\delta}}{2} < \sqrt{2\delta},$$

we are forced to have $a = 1$ and so, $y = (1, s)$ for suitable $s \in [-1, 1]$. But then $\|z - y\| \geqslant \sqrt{2\delta}$, which is impossible. In case $y^* = (0, b)$ with $b = \pm 1$, we have

$$\|z^* - y^*\| = \frac{\sqrt{2\delta}}{2} + \left|1 - \frac{\sqrt{2\delta}}{2} - b\right| < \sqrt{2\delta},$$

so $b = -1$ and therefore, $y = (s, -1)$ for suitable $s \in [-1, 1]$, giving $\|z - y\| \geqslant 2$, a contradiction. □

3. Basic properties of the moduli

Our first result is the continuity of the Bishop-Phelps-Bollobás moduli.

Proposition 3.1. *Let X be a Banach space. Then, the functions*

$$\delta \longmapsto \Phi_X(\delta) \qquad \text{and} \qquad \delta \longmapsto \Phi_X^S(\delta)$$

are continuous in $(0, 2)$.

We need the following three lemmas which could be of independent interest.

Lemma 3.2. *For every pair $(x_0, x_0^*) \in B_X \times B_{X^*}$ there is a pair $(y, y^*) \in \Pi(X)$ with*

$$\operatorname{Re}\left[y^*(x_0) + x_0^*(y)\right] \geqslant 0.$$

Moreover, if actually $\operatorname{Re} x_0^*(x_0) > 0$ then $(y, y^*) \in \Pi(X)$ can be selected to satisfy

$$\operatorname{Re}\left[y^*(x_0) + x_0^*(y)\right] \geqslant 2\sqrt{\operatorname{Re}\left(x_0^*(x_0)\right)}.$$

Proof. 1. Take $y_0 \in S_X \cap \ker x_0^*$ and let y_0^* be a supporting functional at y_0. Then

$$\mathrm{Re}\,\big[y_0^*(x_0) + x_0^*(y_0)\big] = \mathrm{Re}\; y_0^*(x_0)$$

If the right hand side is positive we can take $y = y_0$, $y^* = y_0^*$, in the opposite case take $y = -y_0$, $y^* = -y_0^*$.

2. Take $y = \frac{x_0}{\|x_0\|}$ and let y^* be a supporting functional at y. Then, since for a fixed $a > 0$ the minimum of $f(t) := t + \frac{a}{t}$ for $t > 0$ equals $2\sqrt{a}$, we get

$$\mathrm{Re}\,\big[y^*(x_0) + x_0^*(y)\big] = \|x_0\| + \frac{1}{\|x_0\|}\mathrm{Re}\; x_0^*(x_0) \geqslant 2\sqrt{\mathrm{Re}\; x_0^*(x_0)},$$

as desired. □

The above lemma allows us to prove the following result which we will use to show the continuity of the Bishop-Phelps-Bollobás modulus.

Lemma 3.3. *Let X be a Banach space. Suppose $(x_0, x_0^*) \in A_X(\delta_0)$ with $0 < \delta < \delta_0 < 2$. Then:*

Case 1: *If $\delta, \delta_0 \in]0, 1]$ then*

$$\mathrm{dist}_\infty\big((x_0, x_0^*), A_X(\delta)\big) \leqslant 2\frac{\sqrt{1-\delta} - \sqrt{1-\delta_0}}{1 - \sqrt{1-\delta_0}}.$$

Case 2: *If $\delta, \delta_0 \in [1, 2)$ then*

$$\mathrm{dist}_\infty\big((x_0, x_0^*), A_X(\delta)\big) \leqslant 2\frac{2-\delta_0}{\delta_0} \cdot \frac{\delta_0 - \delta}{\delta_0 - 1 + \sqrt{1 - 2\delta + \delta\delta_0}}.$$

Proof. Denote $t = \mathrm{Re}\; x_0^*(x_0)$. Let $(y, y^*) \in \Pi(X)$ be from the previous lemma (in case 1 we use part 2 of the lemma, in case 2 we use part 1). For every $\lambda \in [0, 1]$ we define $x_\lambda = (1-\lambda)x_0 + \lambda y$ and $x_\lambda^* = (1-\lambda)x_0^* + \lambda y^*$. Both x_λ and x_λ^* belong to corresponding balls, and $\mathrm{dist}_\infty\left((x_0, x_0^*), (x_\lambda, x_\lambda^*)\right) \leqslant 2\lambda$. We have

$$\mathrm{Re}\; x_\lambda^*(x_\lambda) = (1-\lambda)^2 t + \lambda(1-\lambda)\,\mathrm{Re}\,\big[y^*(x_0) + x_0^*(y)\big] + \lambda^2, \tag{4}$$

so in case 1

$$\mathrm{Re}\; x_\lambda^*(x_\lambda) \geqslant (1-\lambda)^2 t + 2\lambda(1-\lambda)\sqrt{t} + \lambda^2 = \Big((1-\lambda)\sqrt{t} + \lambda\Big)^2.$$

Now we are looking for a possibly small value of λ, for which $(x_\lambda, x_\lambda^*) \in A_X(\delta)$. If $\delta \geqslant 1 - t$, the value $\lambda = 0$ is already ok and $\mathrm{dist}_\infty\left((x_0, x_0^*), A_X(\delta)\right) = 0$. If $0 < \delta < 1 - t$ then the positive solution in λ of the equation $\big((1-\lambda)\sqrt{t} + \lambda\big)^2 = 1 - \delta$ is

$$\lambda_t = \frac{\sqrt{1-\delta} - \sqrt{t}}{1 - \sqrt{t}}.$$

Evidently, $\lambda_t \in [0, 1]$, so $(x_{\lambda_t}, x_{\lambda_t}^*) \in A_X(\delta)$. Since λ_t decreases in t,

$$\mathrm{dist}_\infty\left((x_0, x_0^*), A_X(\delta)\right) \leqslant 2\lambda_t \leqslant 2\lambda_{1-\delta_0} = 2\frac{\sqrt{1-\delta} - \sqrt{1-\delta_0}}{1 - \sqrt{1-\delta_0}}.$$

This completes the proof of case 1.

In the case 2 we may assume $t \leqslant 1 - \delta$ (otherwise the corresponding distance is 0 and the job is done), so $t \leqslant 0$. By part 1 of the previous lemma and (4)

$$\mathrm{Re}\; x_\lambda^*(x_\lambda) \geqslant (1-\lambda)^2 t + \lambda^2,$$

so we are solving in λ the equation

$$(1-\lambda)^2 t + \lambda^2 - 1 + \delta = 0, \qquad \text{i.e.} \quad (1+t)\lambda^2 - 2t\lambda + (t - 1 + \delta) = 0.$$

The discriminant of this equation is $D = -t\delta - \delta + 1$. Note that $D \geqslant -(1-\delta)\delta - \delta + 1 = (1-\delta)^2 \geqslant 0$ and $t - 1 + \delta \leqslant 0$, so there is a positive solution of our equation given by

$$\lambda_t = \frac{1}{1+t}(t + \sqrt{D}) = \frac{1}{1+t}(t + \sqrt{1 - t\delta - \delta}).$$

This λ_t decreases in t, so

$$\lambda_t \leqslant \lambda_{1-\delta_0} = \frac{1}{\delta_0}(1-\delta_0+\sqrt{1-2\delta+\delta\delta_0}) = \frac{2+\delta_0}{\delta_0}\cdot\frac{\delta_0-\delta}{\delta_0-1+\sqrt{1-2\delta+\delta\delta_0}}$$

which finishes the proof. □

For the continuity of the spherical modulus, we need the following result.

Lemma 3.4. *Let X be a Banach space. Suppose $(x_0, x_0^*) \in A_X^S(\delta_0)$ with $0<\delta<\delta_0<2$.*

Case 1: *If $\delta < 1$, then*

$$\operatorname{dist}_\infty\big((x_0,x_0^*), A_X^S(\delta)\big) \leqslant \frac{4(\delta_0-\delta)}{\delta_0}.$$

Case 2: *If $\delta\in[1,2)$ and $2-\sqrt{2-\delta_0}<\delta<\delta_0$, then*

$$\operatorname{dist}_\infty\big((x_0,x_0^*), A_X^S(\delta)\big) \leqslant \frac{2(\delta_0-\delta)}{2-\delta}.$$

Proof. Let us start with case 1. Fix $\xi\in(0,\delta)$. As $\|x_0^*\|=1$, we may find $y_\xi\in S_X$ satisfying $x_0^*(y_\xi)>1-\xi$. For every $\lambda\in[0,1]$ we define

$$x(\lambda,\xi)=\lambda x_0+(1-\lambda)y_\xi.$$

Consider $\lambda_\xi=\frac{\delta-\xi}{\delta_0-\xi}\in[0,1]$ and write $x_\xi=x(\lambda_\xi,\xi)$. A straightforward verification shows that

$$\operatorname{Re} x_0^*(x_\xi)>1-\delta$$

and so, as $1-\delta\geqslant 0$, we have that $x_\xi\neq 0$ and also that

$$\operatorname{Re} x_0^*\left(\frac{x_\xi}{\|x_\xi\|}\right)>1-\delta.$$

Therefore, $\left(\frac{x_\xi}{\|x_\xi\|}, x_0^*\right)\in A_X^S(\delta)$. We have

$$\left\|x_0-\frac{x_\xi}{\|x_\xi\|}\right\| \leqslant \|x_0-x_\xi\|+\left\|x_\xi-\frac{x_\xi}{\|x_\xi\|}\right\| \leqslant 2\left(\frac{\delta_0-\delta}{\delta_0-\xi}\right)+|\|x_\xi\|-1| \leqslant$$
$$\leqslant 2\left(\frac{\delta_0-\delta}{\delta_0-\xi}\right)+|\|x_\xi\|-\|x_0\||\leqslant 2\left(\frac{\delta_0-\delta}{\delta_0-\xi}\right)+\|x_\xi-x_0\|\leqslant 4\left(\frac{\delta_0-\delta}{\delta_0-\xi}\right).$$

We get the result by just letting $\xi\longrightarrow 0$.

Let us prove case 2. If $\operatorname{Re} x_0^*(x_0)>1-\delta$, then the proof is done. Suppose that

$$1-\delta\geqslant \operatorname{Re} x_0^*(x_0)>1-\delta_0.$$

Fix $\xi\in\left(0,\min\{2-\delta_0,\frac{4\delta-2-\delta_0-\delta^2}{\delta-1}\}\right)$ (observe that $\frac{4\delta-2-\delta_0-\delta^2}{\delta-1}>0$ by the conditions on δ). As $\|x_0^*\|=1$, we may find $y_\xi\in S_X$ satisfying $x_0^*(y_\xi)>1-\xi$. Now, we consider

$$\lambda_\xi=\frac{\delta_0-\delta}{2-\delta-\xi}\qquad\text{and}\qquad x_\xi=x_0+\lambda_\xi y_\xi.$$

Notice that $\lambda_\xi\in(0,1)$ (since $\delta<\delta_0$ and $\xi<2-\delta_0$) and

$$\|x_\xi\|\geqslant\|x_0\|-\lambda\|y_\xi\|=1-\lambda_\xi>0.$$

Also, observe that

$$\operatorname{Re} x_0^*(x_\xi)\leqslant 1-\delta+\lambda_\xi=\frac{(1-\delta)(2-\delta-\xi)+\delta_0-\delta}{2-\delta-\xi}$$

so, $\operatorname{Re} x_0^*(x_\xi)\leqslant 0$ since $\xi\leqslant\frac{4\delta-2-\delta_0-\delta^2}{\delta-1}$. Now,

$$\operatorname{Re} x_0^*\left(\frac{x_\xi}{\|x_\xi\|}\right)\geqslant \operatorname{Re} x_0^*\left(\frac{x_\xi}{1-\lambda_\xi}\right)>\frac{1-\delta_0+\lambda_\xi(1-\xi)}{1-\lambda_\xi}=1-\delta.$$

Therefore, $\left(\frac{x_\xi}{\|x_\xi\|}, x_0^*\right) \in A_X^S(\delta)$. We have

$$\left\|x_0 - \frac{x_\xi}{\|x_\xi\|}\right\| \leqslant \|x_0 - x_\xi\| + \left\|x_\xi - \frac{x_\xi}{\|x_\xi\|}\right\| \leqslant \frac{\delta_0 - \delta}{2-\delta-\xi} + \big|\|x_\xi\| - 1\big| \leqslant$$
$$\leqslant \frac{\delta_0 - \delta}{2-\delta-\xi} + \big|\|x_\xi\| - \|x_0\|\big| \leqslant \frac{\delta_0 - \delta}{2-\delta-\xi} + \|x_\xi - x_0\| \leqslant 2\left(\frac{\delta_0 - \delta}{2-\delta-\xi}\right).$$

Consequently, letting $\xi \longrightarrow 0$, we get

$$\operatorname{dist}_\infty\big((x_0, x_0^*), A_X^S(\delta)\big) \leqslant \frac{2(\delta_0 - \delta)}{2-\delta}$$

as we desired. □

Proof of Proposition 3.1. Let us give the proof for $\Phi_X(\delta)$. Observe that for $\delta_1, \delta_2 \in (0,2)$ with $\delta_1 < \delta_2$, one has

$$0 < \Phi_X(\delta_2) - \Phi_X(\delta_1) = d_H\left(A_X(\delta_2), \Pi(X)\right) - d_H\left(A_X(\delta_1), \Pi(X)\right) \leqslant d_H\left(A_X(\delta_2), A_X(\delta_1)\right).$$

Now, the continuity follows routinely from Lemma 3.3.

An analogous argument allows to prove the continuity of $\Phi_X^S(\delta)$ from Lemma 3.4. □

The following lemma will be used to show that the approximation in the space is not worse than the approximation in the dual. It is actually an easy application of the Principle of Local Reflexivity.

Lemma 3.5. *For $\varepsilon > 0$, let $(x, x^*) \in B_X \times B_{X^*}$ and let $(\tilde{y}^*, \tilde{y}^{**}) \in \Pi(X^*)$ such that*

$$\|x^* - \tilde{y}^*\| < \varepsilon \quad \textit{and} \quad \|x - \tilde{y}^{**}\| < \varepsilon.$$

Then there is a pair $(y, y^*) \in \Pi(X)$ such that

$$\|x - y\| < \varepsilon \quad \textit{and} \quad \|x^* - y^*\| < \varepsilon.$$

Proof. First chose $\varepsilon' < \varepsilon$ such that still

$$\|x^* - \tilde{y}^*\| < \varepsilon' \quad \text{and} \quad \|x - \tilde{y}^{**}\| < \varepsilon'.$$

Now, we consider $\xi > 0$ such that

$$(1+\xi)\varepsilon' + \xi + \sqrt{\frac{2\xi}{1+\xi}} < \varepsilon,$$

and use the Principle of Local Reflexivity (see [1, Theorem 11.2.4], for instance) to get an operator $T : \operatorname{Lin}\{x, \tilde{y}^{**}\} \longrightarrow X$ satisfying

$$\|T\|, \|T^{-1}\| \leqslant 1+\xi, \quad T(x) = x, \quad \tilde{y}^*(T(\tilde{y}^{**})) = y^{**}(\tilde{y}^*) = 1.$$

Next, we consider $\tilde{x} = \dfrac{T(\tilde{y}^{**})}{\|T(\tilde{y}^{**})\|} \in S_X$ and $\tilde{x}^* = \tilde{y}^* \in S_{X^*}$, observe that

$$\operatorname{Re} \tilde{x}^*(\tilde{x}) > \frac{1}{1+\xi} = 1 - \frac{\xi}{1+\xi},$$

and we use Corollary 2.4 to get $(y, y^*) \in \Pi(X)$ satisfying that

$$\|\tilde{x} - y\| < \sqrt{\frac{2\xi}{1+\xi}} \quad \text{and} \quad \|\tilde{x}^* - y^*\| < \sqrt{\frac{2\xi}{1+\xi}}.$$

Let us show that $(y, y^*) \in \Pi(X)$ fulfill our requirements.

$$\|x - y\| \leqslant \|T(x) - T(\tilde{y}^{**})\| + \|T(\tilde{y}^{**}) - \tilde{x}\| + \|\tilde{x} - y\|$$
$$< (1+\xi)\varepsilon' + \xi + \sqrt{\frac{2\xi}{1+\xi}} < \varepsilon$$

and, analogously,

$$\|x^* - y^*\| \leqslant \|x^* - \tilde{y}^*\| + \|\tilde{y}^* - y^*\| < \varepsilon' + \sqrt{\frac{2\xi}{1+\xi}} < \varepsilon,$$

getting the desired result. □

Proposition 3.6. *Let X be a Banach space. Then*

$$\Phi_X(\delta) \leqslant \Phi_{X^*}(\delta) \quad \textit{and} \quad \Phi^S_X(\delta) \leqslant \Phi^S_{X^*}(\delta)$$

for every $\delta \in (0, 2)$.

Proof. The proof is the same for both moduli, so we are only giving the case of $\Phi_X(\delta)$. Fix $\delta \in (0, 2)$. We consider any $\varepsilon > 0$ such that $\Phi_{X^*}(\delta) < \varepsilon$ and for a given $(x, x^*) \in A_X(\delta)$ consider $(x^*, x) \in A_{X^*}(\delta)$ (we identify X as a subspace of X^{**}) and so we may find $(\tilde{y}^*, \tilde{y}^{**}) \in \Pi(Y^*)$ such that

$$\|x^* - \tilde{y}^*\| < \varepsilon \quad \text{and} \quad \|x - \tilde{y}^{**}\| < \varepsilon.$$

From Lemma 3.5, we find $(y, y^*) \in \Pi(X)$ such that

$$\|x - y\| < \varepsilon \quad \text{and} \|x^* - y^*\| < \varepsilon.$$

This means that $\Phi_X(\delta) \leqslant \varepsilon$ and, therefore, $\Phi_X(\delta) \leqslant \Phi_{X^*}(\delta)$, as desired. □

We do not know whether the inequalities in Proposition 3.6 can be strict. Of course, this can not be the case when the space is reflexive.

Corollary 3.7. *For every reflexive Banach space X, one has $\Phi_X(\delta) = \Phi_{X^*}(\delta)$ and $\Phi^S_X(\delta) = \Phi^S_{X^*}(\delta)$ for every $0 < \delta < 2$.*

Our last result in this section states that when the Bishop-Phelps-Bollobás modulus is the worst possible, then the spherical Bishop-Phelps-Bollobás modulus is also the worst possible.

Proposition 3.8. *Let X be a Banach space. For every $\delta \in (0, 2)$, the condition $\Phi_X(\delta) = \sqrt{2\delta}$ is equivalent to the condition $\Phi^S_X(\delta) = \sqrt{2\delta}$.*

Proof. Since $\Phi^S_X(\delta) \leqslant \Phi_X(\delta) \leqslant \sqrt{2\delta}$, the implication $\left[\Phi^S_X(\delta) = \sqrt{2\delta}\right] \Rightarrow \left[\Phi_X(\delta) = \sqrt{2\delta}\right]$ is evident. Let us prove the inverse implication. Let $\Phi_X(\delta) = \sqrt{2\delta}$. Then there is a sequence of pairs $(x_n, x^*_n) \in B_X \times B_{X^*}$ such that $\operatorname{Re} x^*_n(x_n) > 1 - \delta$ but for every $(y, y^*) \in \Pi(X)$ we have

$$\|x_n - y\| \geqslant \sqrt{2\delta} - \frac{1}{n} \quad \text{or } \|x^*_n - y^*\| \geqslant \sqrt{2\delta} - \frac{1}{n}.$$

An application of Remark 2.3 gives us that $\|x^*_n\| \longrightarrow 1$ as $n \to \infty$. As the duality argument given in Lemma 3.5 implies the dual version of Remark 2.3, we also have $\|x_n\| \longrightarrow 1$ as $n \to \infty$. Denote $\tilde{x_n} = \frac{x_n}{\|x_n\|}$, $\tilde{x}^*_n = \frac{x^*_n}{\|x^*_n\|}$. In the case when $\delta \in (0, 1]$, we have $\operatorname{Re} \tilde{x}^*_n(\tilde{x}_n) > 1 - \delta$ but for every $(y, y^*) \in \Pi(X)$

$$\|\tilde{x}_n - y\| \geqslant \sqrt{2\delta} - \frac{1}{n} - \|x_n - \tilde{x}_n\| \quad \text{or } \|\tilde{x}^*_n - y^*\| \geqslant \sqrt{2\delta} - \frac{1}{n} - \|\tilde{x}^*_n - x^*_n\|.$$

Since the right-hand sides of the above inequalities go to $\sqrt{2\delta}$, we get the condition $\Phi^S_X(\delta) = \sqrt{2\delta}$.

In the case of $\delta \in (1, 2)$, we no longer know that $\operatorname{Re} \tilde{x}^*_n(\tilde{x}_n) > 1 - \delta$, but what we do know is that $\liminf \operatorname{Re} \tilde{x}^*_n(\tilde{x}_n) \geqslant 1 - \delta$, and that gives us the desired condition $\Phi^S_X(\delta) = \sqrt{2\delta}$ thanks to the continuity of the spherical modulus (Proposition 3.1). □

4. Examples

We start with the simplest example of $X = \mathbb{R}$.

Example 4.1. $\Phi_{\mathbb{R}}(\delta) = \begin{cases} \delta & \text{if } 0 < \delta \leqslant 1 \\ \sqrt{\delta - 1} + 1 & \text{if } 1 < \delta < 2 \end{cases}, \qquad \Phi^S_{\mathbb{R}}(\delta) = 0$ for every $\delta \in (0, 2)$.

Proof. We first fix $\delta \in (0, 1]$. First observe that taking $x = 1 - \delta$, $x^* = 1$, it is evident that $\Phi_{\mathbb{R}}(\delta) \geqslant \delta$. For the other inequality, we fix $x, x^* \in [-1, 1]$ with $x^*x > 1 - \delta$. Then, x and x^* have the same sign and we have that $|x| > 1 - \delta$ and $|x^*| > 1 - \delta$. Indeed, if $|x| < 1 - \delta$, as $|x^*| \leqslant 1$, one has $x^*x = |x^*x| < 1 - \delta$, a contradiction; the other inequality follows in the same manner. Finally, one deduces that $|x - \operatorname{sign}(x)| < \delta$ and $|x^* - \operatorname{sign}(x^*)| < \delta$, as desired.

Second, fix $\delta \in (1, 2)$. On the one hand, taking $x = \sqrt{\delta - 1}$, $x^* = -\sqrt{\delta - 1}$, one has $x^*x = 1 - \delta$. As $|x + 1| = \sqrt{\delta - 1} + 1$ and $|x^* - 1| = \sqrt{\delta - 1} + 1$, it follows that $\Phi_{\mathbb{R}}(\delta) \geqslant \sqrt{\delta - 1} + 1$. For the other inequality, we fix $x, x^* \in [-1, 1]$ with $x^*x > 1 - \delta$. If x and x^* have the same sign, which we may and do suppose positive, then $|x - 1| \leqslant 1 < \delta$ and $|x^* - 1| \leqslant 1 < \delta$ and the same is true if one of them is null. Therefore, to prove the last case we may and do suppose that $x > 0$ and $x^* < 0$. Now, if we suppose, for the sake of contradiction, that

$$|x - (-1)| \geqslant \sqrt{\delta - 1} + 1 \quad \text{and} \quad |x^* - 1| \geqslant \sqrt{\delta - 1} + 1,$$

we get $x \geqslant \sqrt{\delta - 1}$ and $-x^* \geqslant \sqrt{\delta - 1}$, so $-x^*x \geqslant \delta - 1$ or, equivalently, $x^*x \leqslant 1 - \delta$, a contradiction. Therefore, either $|x - (-1)| < \sqrt{\delta - 1} + 1$ and $|x^* - (-1)| < 1 < \sqrt{\delta - 1} + 1$ or $|x^* - 1| < \sqrt{\delta - 1} + 1$ and $|x - 1| < 1 < \sqrt{\delta - 1} + 1$.

The result for $\Phi^S_{\mathbb{R}}$ is an obvious consequence of the fact that $S_{\mathbb{R}} = \{-1, 1\}$. □

Let us observe that the above proof gives actually a lower bound for $\Phi_X(\delta)$ for every Banach space X when $\delta \in (0, 1]$.

Remark 4.2. *Let X be a Banach space. Then $\Phi_X(\delta) \geqslant \delta$ for every $\delta \in (0, 1]$.* Indeed, consider $x_0 \in S_X$ and $x_0^* \in S_{X^*}$ with $x_0^*(x_0) = 1$ and write $x = (1 - \delta)x_0$ and $x^* = x_0^*$. Then $\operatorname{Re} x^*(x) = 1 - \delta$ and $\operatorname{dist}(x, S_X) = \delta$.

We do not know a result giving a lower bound for $\Phi_X(\delta)$ when $\delta > 1$, outside of the trivial one $\Phi_X(\delta) \geqslant 1$. Also, we do not know if the lower bound for the behavior of $\Phi_X(\delta)$ in a neighborhood of 0 given in the remark above can be improved for Banach spaces of dimension greater than or equal to two.

We next calculate the moduli of a Hilbert space of (real) dimension greater than one.

Example 4.3. Let H be a Hilbert space of dimension over $\mathbb{R}$ greater than or equal to two. Then:

(a) $\Phi^S_H(\delta) = \sqrt{2 - \sqrt{4 - 2\delta}}$ for every $\delta \in (0, 2)$.
(b) For $\delta \in (0, 1]$, $\Phi_H(\delta) = \max\left\{\delta, \sqrt{2 - \sqrt{4 - 2\delta}}\right\}$. For $\delta \in (1, 2)$, $\Phi_H(\delta) = \sqrt{\delta}$.

Proof. As we commented in the introduction, both Φ_H and Φ^S_H only depend on the real structure of the space, so we may and do suppose that H is a real Hilbert space of dimension greater than or equal to 2. Let us also recall that H^* identifies with H and that the action of a vector $y \in H$ on a vector $x \in H$ is nothing but their inner product denoted by $\langle x, y\rangle$. In particular,

$$\Pi(H) = \big\{(z, z) \in S_H \times S_H\big\}.$$

Therefore, for every $\delta \in (0, 2)$, $\Phi_H(\delta)$ (resp. $\Phi^S_H(\delta)$) is the infimum of those $\varepsilon > 0$ such that whenever $x, y \in B_H$ (resp. $x, y \in S_H$) satisfies $\langle x, y\rangle \geqslant 1 - \delta$, there is $z \in S_H$ such that $\|x - z\| \leqslant \varepsilon$ and $\|y - z\| \leqslant \varepsilon$.

We will use the following (easy) claim in both the proofs of (a) and (b).

Claim: Given $x, y \in S_H$ with $x+y \neq 0$, write $z = \frac{x+y}{\|x+y\|}$ to denote the normalized midpoint. Then

$$\|x-z\| = \|y-z\| = \sqrt{2-\sqrt{2+2\langle x,y\rangle}}.$$

Indeed, we have $\|x-z\|^2 = 2 - 2\langle x, z\rangle$ and

$$2\langle x,z\rangle = \frac{2\langle x, x+y\rangle}{\|x+y\|} = \frac{2+2\langle x,y\rangle}{\sqrt{2+2\langle x,y\rangle}},$$

giving $\|x-z\| = \sqrt{2-\sqrt{2+2\langle x,y\rangle}}$, being the other equality true by symmetry.

(a). We first prove that $\Phi^S_H(\delta) \leqslant \sqrt{2-\sqrt{4-2\delta}}$. Take $x, y \in S_H$ with $\langle x,y\rangle \geqslant 1-\delta$ (so $x+y\neq 0$), consider $z = \frac{x+y}{\|x+y\|} \in S_H$ and use the claim to get that

$$\|x-z\| = \|y-z\| = \sqrt{2-\sqrt{2+2\langle x,y\rangle}} \leqslant \sqrt{2-\sqrt{4-2\delta}}.$$

To get the other inequality, we fix an ortonormal basis $\{e_1, e_2, \ldots\}$ of H, consider

$$x = \sqrt{1-\delta/2}\, e_1 + \sqrt{\delta/2} e_2 \in S_H \quad \text{and} \quad y = \sqrt{1-\delta/2}\, e_1 - \sqrt{\delta/2} e_2 \in S_H$$

and observe that $\langle x,y\rangle = 1-\delta$. Now, given $z \in S_H$, we write $z_1 = \langle z, e_1\rangle$, $z_2 = \langle z, e_2\rangle$, and observe that

$$\begin{aligned}
\max\{\|z-x\|^2, \|z-y\|^2\} &= \max_{\pm}\left\{|z_1 - \sqrt{1-\delta/2}|^2 + |z_2 \pm \sqrt{\delta/2}|^2 + 1 - z_1^2 - z_2^2\right\} \\
&= z_1^2 + 1 - \delta/2 - 2z_1\sqrt{1-\delta/2} + \max_{\pm} |z_2 \pm \sqrt{\delta/2}|^2 + 1 - z_1^2 - z_2^2 \\
&= 2 - 2z_1\sqrt{1-\delta/2} + 2|z_2|\sqrt{\delta/2} \geqslant 2 - 2\sqrt{1-\delta/2}.
\end{aligned}$$

It follows that $\Phi^S_H(\delta) \geqslant \sqrt{2-\sqrt{4-2\delta}}$, as desired.

(b). We first fix $\delta \in (0,1)$ and write $\varepsilon_0 = \max\left\{\delta, \sqrt{2-\sqrt{4-2\delta}}\right\}$. The inequality $\Phi_H(\delta) \geqslant \varepsilon_0$ follows from Remark 4.2, the fact that $\Phi_H(\delta) \geqslant \Phi^S_H(\delta)$ and the result in item (a). To get the other inequality, we first observe that

$$\Phi_H(\delta) \leqslant \Phi_{\mathrm{Lin}\,\{x,y\}}(\delta) \qquad \forall x, y \in B_H \quad \text{with} \quad \langle x,y\rangle = 1-\delta. \tag{5}$$

This follows from the obvious fact that $\Phi_\cdot(\delta)$ increases when we restrict to subspaces. This implies that it is enough to show that for $P = (p_1, 0), Q = (q_1, q_2) \in B_{\ell_2^{(2)}}$ such that $p_1, q_2 > 0$, $\|P\| > \|Q\|$, and $\langle P, Q\rangle \geqslant 1-\delta$ where $\ell_2^{(2)}$ is the 2-dimensional Hilbert space, there exists $z \in S_{\ell_2^{(2)}}$ so that $\|P-z\| \leqslant \varepsilon_0$ and $\|Q-z\| \leqslant \varepsilon_0$. Now, it is straightforward to check that we have $\|P\| \in \left[\sqrt{1-\delta}, 1\right]$, and $q_1 = \frac{1-\delta}{\|P\|} \in \left[1-\delta, \sqrt{1-\delta}\right]$. Figure 1 helps to the better understanding of the rest of the proof.

Consider $M = \left(\sqrt{\frac{1-\delta+\|P\|}{2\|P\|}}, \sqrt{\frac{\|P\|-(1-\delta)}{2\|P\|}}\right)$, which is the normalized midpoint between $A = (1,0)$ and $B = \left(\frac{1-\delta}{\|P\|}, \sqrt{1-(\frac{1-\delta}{\|P\|})^2}\right)$ and write Δ to denote the arc of the unit sphere of H between A and M. We claim that $Q \in \bigcup_{z\in\Delta} B(z,\varepsilon_0)$ and $P \in \bigcap_{z\in\Delta} B(z,\varepsilon_0)$. Observe that this gives that there is $z \in \Delta \subset S_H$ whose distance to P and Q is less than or equal to ε_0, finishing the proof. Let us prove the claim. First, we show that $Q = (q_1, q_2) \in \bigcup_{z\in\Delta} B(z,\varepsilon_0)$. If $q_2 \leqslant \sqrt{\frac{\|P\|-(1-\delta)}{2\|P\|}}$, the ball of radius ε_0 centered in the point of Δ with second coordinate equal to q_2 contains the point Q since $\varepsilon_0 \geqslant \mathrm{dist}_\infty((q_1,0),A) \geqslant \mathrm{dist}(Q,\Delta)$. For greater values of q_2, write first $C = \left(q_1, \sqrt{\frac{\|P\|-(1-\delta)}{2\|P\|}}\right)$, which belongs to $B(M,\varepsilon_0)$ by the previous argument. Also, as M is the normalized mid point between A and B, we have by the claim at the beginning of this proof that

$$\|M-B\| = \sqrt{2-\sqrt{2+2\langle A,B\rangle}} = \sqrt{2-\sqrt{2+2\frac{1-\delta}{\|P\|}}} \leqslant \sqrt{2-\sqrt{4-2\delta}} \leqslant \varepsilon_0$$

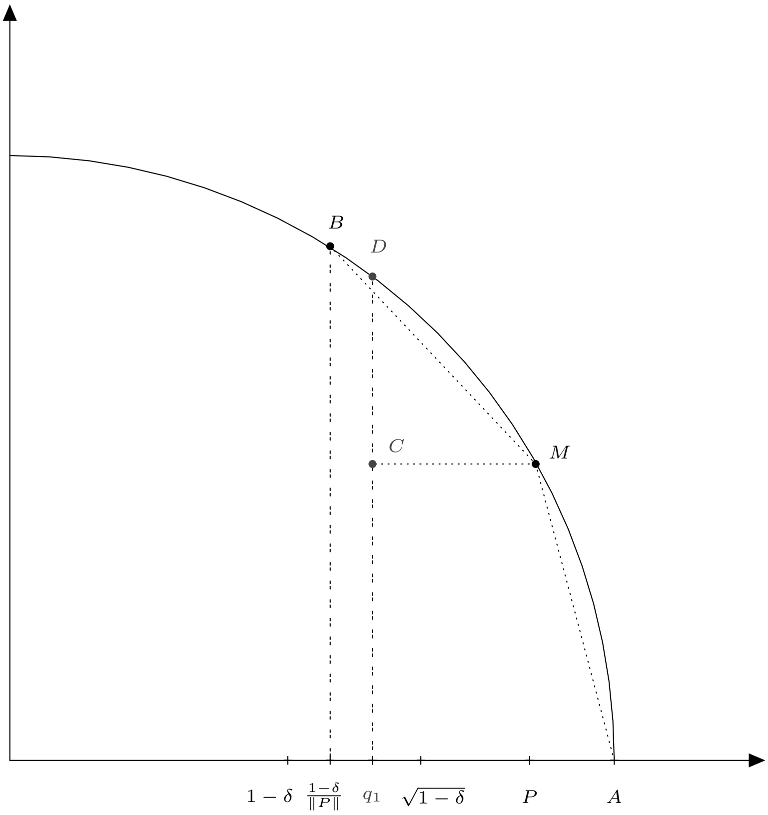

FIGURE 1. Calculating $\Phi_H(\delta)$ for $\delta \in (0,1)$

so, also, $\|M - D\| \leqslant \varepsilon_0$. Therefore, both the points C and D belong to $B(M,\varepsilon_0)$, so also the whole segment $[C,D]$ is contained there, and this proves the first part of the claim. To show the second part of the claim, that $P \in \bigcap_{z\in\Delta} B(z,\varepsilon_0)$, we consider the function

$$f(p) := 1 + p^2 - \sqrt{2p(p+1-\delta)} \qquad \big(p \in [\sqrt{1-\delta}, 1]\big)$$

and observe that it is a convex function, so

$$f(p) \leqslant \max\{f(1), f(\sqrt{1-\delta}\} \leqslant \varepsilon_0^2.$$

It follows that

$$\|P - M\| = \sqrt{1 + \|P\|^2 - \sqrt{2\|P\|(\|P\| + 1 - \delta)}} \leqslant \varepsilon_0,$$

hence $M \in B(P,\varepsilon_0)$. As also $A \in B(P,\varepsilon_0)$, it follows that the whole circular arc Δ is contained in $B(P,\varepsilon_0)$ or, equivalently, that $P \in \bigcap_{z\in\Delta} B(z,\varepsilon_0)$.

Fix $\delta \in (1,2)$. Analogously to what we did before in equation (5), to show that $\Phi_H(\delta) \leqslant \sqrt{\delta}$, it is enough to consider the two-dimensional case and that, given $p = (\|p\|, 0) \in B_H$, $q = (q_1, q_2) \in B_H$ with $q_2 \geqslant 0$, to find $z \in S_H$ such that $\|z - P\|, \|z - Q\| \leqslant \sqrt{\delta}$. Routine computations show that

$$z = \left(\frac{\|p\| + q_1}{2}, \sqrt{1 - (\frac{\|p\| + q_1}{2})^2}\right) \in S_H$$

does the job. For the other inequality, we fix an orthonormal basis $\{e_1, e_2, \dots\}$ of H, consider

$$P = \sqrt{\delta - 1}\, e_1 \in B_H, \qquad Q = -\sqrt{\delta - 1}\, e_1 \in B_H$$

and observe that $\langle P, Q\rangle = 1-\delta$. For any $z \in S_H$, we write $z_1 = \langle z, e_1\rangle$ and we compute

$$\begin{aligned}\max\{\|z-P\|^2, \|z-Q\|^2\} &= \max\left\{|z_1-\sqrt{\delta-1}|^2+1-|z_1|^2\,,\ |z_1+\sqrt{\delta-1}|^2+1-|z_1|^2\right\}\\ &= \max_{\pm}|z_1\pm\sqrt{\delta-1}|^2+1-|z_1|^2 = (|z_1|+\sqrt{\delta-1})^2+1-|z_1|^2\\ &= \delta+2\sqrt{\delta-1}|z_1| \geqslant \delta.\end{aligned}$$

It follows that $\Phi_H(\delta) \geqslant \sqrt{\delta}$, as desired. □

Our next aim is to present a number of examples for which the values of the Bishop-Phelps-Bollobás moduli are the maximum possible, namely $\Phi^S_X(\delta) = \Phi_X(\delta) = \sqrt{2\delta}$ for small δ's. As we always have $\Phi^S_X(\delta) \leqslant \Phi_X(\delta) \leqslant \sqrt{2\delta}$, it is enough if we prove the formally stronger result that $\Phi^S_X(\delta) = \sqrt{2\delta}$ for small δ's (actually, the two facts are equivalent, see Proposition 3.8), and this is what we will show. It happens that all of the examples have in common that they contains an isometric copy of the real space $\ell_\infty^{(2)}$ or $\ell_1^{(2)}$. In the next section we will show that the latter is a necessary condition that it is not actually sufficient.

The first result is about Banach spaces admitting an L-decomposition. As a consequence we will calculate the moduli of $L_1(\mu)$ spaces.

Proposition 4.4. *Let X be a Banach space. Suppose that there are two (non-trivial) subspaces Y and Z such that $X = Y\oplus_1 Z$. Then $\Phi_X(\delta) = \Phi^S_X(\delta) = \sqrt{2\delta}$ for every $\delta \in (0, 1/2]$.*

Proof. Fix $\delta \in (0, 1/2]$ and consider $(y_0, y_0^*) \in \Pi(Y)$ and $(z_0, z_0^*) \in \Pi(Z)$ and write

$$x_0 = \left(\frac{\sqrt{2\delta}}{2}\,y_0\,,\ \left(1-\frac{\sqrt{2\delta}}{2}\right)z_0\right) \in S_X \qquad x_0^* = \left((1-\sqrt{2\delta})y_0^*\,,\ z_0^*\right) \in S_{X^*}.$$

It is clear that $\operatorname{Re} x_0^*(x_0) = 1-\delta$. Now, suppose that we may choose $(x, x^*) \in \Pi(X)$ such that

$$\|x_0 - x\| < \sqrt{2\delta} \quad \text{and} \quad \|x_0^* - x^*\| < \sqrt{2\delta}.$$

Write $x = (y, z) \in Y \oplus_1 Z$, $x^* = (y^*, z^*) \in Y^* \oplus_\infty Z^*$ and observe that

$$1 = \operatorname{Re} x^*(x) = \operatorname{Re} y^*(y) + \operatorname{Re} z^*(z) \leqslant \|y^*\|\|y\| + \|z^*\|\|z\| \leqslant \|y\| + \|z\| = 1,$$

therefore, we have

$$\operatorname{Re} y^*(y) = \|y^*\|\|y\|. \tag{6}$$

Now, we have

$$\Big|(1-\sqrt{2\delta}) - \|y^*\|\Big| \leqslant \Big\|(1-\sqrt{2\delta})y_0^* - y^*\Big\| < \sqrt{2\delta}$$

from which follows that $\|y^*\| < 1$ and so, $y = 0$ by (6), giving $\|z\| = \|x\| = 1$. But then,

$$\|x_0 - x\| = \left\|\frac{\sqrt{2\delta}}{2}\,y_0\right\| + \left\|\left(1-\frac{\sqrt{2\delta}}{2}\right)z_0 - z\right\| \geqslant \frac{\sqrt{2\delta}}{2} + \left|\left(1-\frac{\sqrt{2\delta}}{2}\right) - \|z\|\right| = \sqrt{2\delta},$$

a contradiction. We have proved that $\Phi_X(\delta) \geqslant \sqrt{2\delta}$, being the other inequality always true. □

The result above produces the following example.

Example 4.5. *Let (Ω, Σ, μ) be a measure space such that $L_1(\mu)$ has dimension greater than one and let E be any non-zero Banach space. Then, $\Phi_{L_1(\mu,E)}(\delta) = \Phi^S_{L_1(\mu,E)}(\delta) = \sqrt{2\delta}$ for every $\delta \in (0, 1/2]$.*
Indeed, we may find two measurable sets $A, B \subset \Omega$ with empty intersection such that $\Omega = A \cup B$. Then $Y = L_1(\mu|_A, E)$ and $Z = L_1(\mu|_B, E)$ are non-null, $L_1(\mu, E) = Y \oplus_1 Z$ and so the results follows from Proposition 4.4.

Particular cases of the above example are ℓ_1 and $L_1[0,1]$.

It is immediate that, using a dual argument than the one given in Proposition 4.4, it is possible to deduce the same result for a Banach space which decomposes as an ℓ_∞-sum. Actually, in this case we will get a better result using ideals instead of subspaces.

Proposition 4.6. *Let X be a Banach space. Suppose that $X^* = Y \oplus_1 Z$ where Y and Z are (non-trivial) subspaces of X^* such that $\overline{Y}^{w^*} \neq X^*$ and $\overline{Z}^{w^*} \neq X^*$ (w^* is the weak*-topology $\sigma(X^*, X)$). Then $\Phi_X(\delta) = \Phi_X^S(\delta) = \sqrt{2\delta}$ for every $\delta \in (0, 1/2]$.*

Proof. We claim that there are $y_0, z_0 \in S_X$ and $y_0^* \in S_Y$ and $z_0^* \in S_Z$ such that

$$\text{Re } y_0^*(y_0) = 1, \quad \text{Re } z_0^*(z_0) = 1, \quad y^*(z_0) = 0 \ \forall y^* \in Y, \quad z^*(y_0) = 0 \ \forall z^* \in Z.$$

Indeed, we define y_0 and y_0^*, being z_0 and z_0^* analogous. By assumption there is $y_0 \in S_X$ such that $z^*(y_0) = 0$ for every $z^* \in Z$ and we may choose $x^* \in S_{X^*}$ such that $\text{Re } x^*(y_0) = 1$ and we only have to prove that $x^* \in Y$ and then write $y_0^* = x^*$. But we have $x^* = y^* + z^*$ with $y^* \in Y$, $z^* \in Z$ and

$$1 = \text{Re } x^*(y_0) = \text{Re } y^*(y_0) \leqslant \|y^*\| \leqslant \|y^*\| + \|z^*\| = 1,$$

so $z^* = 0$ and $x^* \in Y$.

We now define

$$x_0^* = \left(\frac{\sqrt{2\delta}}{2}\, y_0^*\,,\ \left(1 - \frac{\sqrt{2\delta}}{2}\right) z_0^* \right) \in S_{X^*} \qquad x_0 = \left(1 - \sqrt{2\delta}\right) y_0 + z_0 \in X$$

and first observe that $\|x_0\| \leqslant 1$. Indeed, for every $x^* = y^* + z^* \in S_{X^*}$ one has

$$|x^*(x_0)| = \left| \left(1 - \sqrt{2\delta}\right) y^*(y_0) + z^*(z_0) \right| \leqslant \left(1 - \sqrt{2\delta}\right) \|y^*\| + \|z^*\| \leqslant \|y^*\| + \|z^*\| = 1.$$

It is clear that $\text{Re } x_0^*(x_0) = 1 - \delta$. Now, suppose that we may choose $(x, x^*) \in \Pi(X)$ such that

$$\|x_0 - x\| < \sqrt{2\delta} \quad \text{and} \quad \|x_0^* - x^*\| < \sqrt{2\delta}.$$

We consider the semi-norm $\|\cdot\|_Y$ defined on X by $\|x\|_Y := \sup\{|y^*(x)| \,:\, y^* \in S_Y\}$ which is smaller than or equal to the original norm, write $x^* = y^* + z^*$ with $y^* \in Y$ and $z^* \in Z$, and observe that

$$1 = \text{Re } x^*(x) = \text{Re } y^*(x) + \text{Re } z^*(x) \leqslant \|y^*\| \|x\|_Y + \|z^*\| \|x\| \leqslant \|y^*\| + \|z^*\| = 1.$$

Therefore, we have, in particular, that

$$\text{Re } y^*(x) = \|y^*\| \|x\|_Y. \tag{7}$$

Now, we have

$$\left| \left(1 - \sqrt{2\delta}\right) - \|x\|_Y \right| = \left| \left(1 - \sqrt{2\delta}\right) \|y_0\|_Y - \|x\|_Y \right| \leqslant \left\| \left(1 - \sqrt{2\delta}\right) y_0 \, - \, x \right\|_Y < \sqrt{2\delta}$$

from which follows that $\|x\|_Y < 1$ and so, $y^* = 0$ by (7) and $\|z^*\| = \|x^*\| = 1$. But then,

$$\|x_0^* - x^*\| = \left\| \frac{\sqrt{2\delta}}{2}\, y_0^* \right\| + \left\| \left(1 - \frac{\sqrt{2\delta}}{2}\right) z_0^* \, - \, z^* \right\| \geqslant \frac{\sqrt{2\delta}}{2} + \left| \left(1 - \frac{\sqrt{2\delta}}{2}\right) \, - \, \|z^*\| \right| = \sqrt{2\delta},$$

a contradiction. Again, we have proved that $\Phi_X(\delta) \geqslant \sqrt{2\delta}$, the other inequality always being true. □

Of course, the first consequence of the above result is to Banach spaces which decompose as ℓ_∞-sum of two subspaces. Indeed, if $X = Y \oplus_\infty Z$ for two (non-trivial) subspaces Y and Z, then $X^* = Y^\perp \oplus_1 Z^\perp$ and $Y^\perp$ and $Z^\perp$ are w^*-closed, so far away of being dense. Therefore, Proposition 4.6 applies. We have proved the following result.

Corollary 4.7. *Let X be a Banach space. Suppose that there are two (non-trivial) subspaces Y and Z such that $X = Y \oplus_\infty Z$. Then $\Phi_X(\delta) = \Phi_X^S(\delta) = \sqrt{2\delta}$ for every $\delta \in (0, 1/2]$.*

As a consequence, we obtain the following examples, analogous to the ones presented in Example 4.5.

Examples 4.8. (a) Let (Ω, Σ, μ) a measure space such that $L_\infty(\Omega)$ has dimension greater than one and let E be any non-zero Banach space. Then,

$$\Phi_{L_\infty(\mu,E)} = \Phi^S_{L_\infty(\mu,E)}(\delta) = \sqrt{2\delta} \qquad \big(\delta \in (0, 1/2]\big).$$

(b) Let Γ be a set with more than one point and let E be any non-zero Banach space. Then,

$$\Phi_{c_0(\Gamma,E)} = \Phi^S_{c_0(\Gamma,E)}(\delta) = \sqrt{2\delta} \quad \text{and} \quad \Phi_{c(\Gamma,E)} = \Phi^S_{c(\Gamma,E)}(\delta) = \sqrt{2\delta} \qquad \big(\delta \in (0, 1/2]\big).$$

Our next aim is to deduce from Proposition 4.6 that also arbitrary $C(K)$ spaces have the maximum moduli and for this we have to deal with the concept of M-ideal. Given a subspace J of a Banach space X, J is called *M-ideal* if $J^\perp$ is a L-summand on X^* (use [10] for background). In this case, $X^* = J^\perp \oplus_1 J^\sharp$ where $J^\sharp = \{x^* \in X^* \,:\, \|x^*\| = \|x^*|_J\|\} \equiv J^*$. Now, if X contain a non-trivial M-ideal J, one has $X^* = J^\perp \oplus_1 J^\sharp$ and to apply Proposition 4.6 we need that $J^\sharp$ to be not $\sigma(X^*, X)$-dense. Actually, $J^\sharp$ is not dense in X^* if and only if there is $x_0 \in X \setminus \{0\}$ such that $\|x_0 + y\| = \max\{\|x_0\|, \|y\|\}$ for every $y \in J$ (this is easy to verify and a proof can be found in [3]). Let us enunciate what we have shown.

Corollary 4.9. *Let X be a Banach space. Suppose that there is a non-trivial M-ideal J of X and a point $x_0 \in X \setminus \{0\}$ such that $\|x_0 + y\| = \max\{\|x_0\|, \|y\|\}$ for every $y \in J$. Then, $\Phi_X(\delta) = \Phi^S_X(\delta) = \sqrt{2\delta}$ for every $\delta \in (0, 1/2]$.*

With the above corollary we are able to prove that the moduli of any non-trivial $C_0(L)$ space are maximum.

Example 4.10. *Let L be a locally compact Hausdorff topological space with at least two points and let E be any non-zero Banach space. Then $\Phi_{C_0(L,E)}(\delta) = \Phi^S_{C_0(L,E)}(\delta) = \sqrt{2\delta}$ for every $\delta \in (0, 1/2]$.*
Indeed, we may find a non-empty non-dense open subset U of L and consider the subspace

$$J = \big\{f \in C_0(L, E) \,:\, f|_U = 0\big\},$$

which is an M-ideal of $C_0(L, E)$ by [10, Corollary VI.3.4] (use the simpler [10, Example I.1.4.a] for the scalar-valued case) and it is non-zero since $L \setminus U$ has non-empty interior. As U is open and non-empty, we may find a non-null function $x_0 \in C_0(L, E)$ whose support is contained in U. It follows that $\|x_0 + y\| = \max\{\|x_0\|, \|y\|\}$ for every $y \in J$ by disjointness of the supports.

A sufficient condition to be in the hypotheses of Corollary 4.9 is that a Banach space X contains two non-trivial M-ideals J_1 and J_2 such that $J_1 \cap J_2 = \{0\}$. In this case, J_1 and J_2 are complementary M-summands in $J_1 + J_2$ [10, Proposition I.1.17]. Let us comment that this is actually what happens in $C(K)$ when K has more than one point.

Corollary 4.11. *Let X be a Banach space. Suppose there are two non-trivial M-ideals J_1 and J_2 such that $J_1 \cap J_2 = \{0\}$. Then $\Phi_X(\delta) = \Phi^S_X(\delta) = \sqrt{2\delta}$ for every $\delta \in (0, 1/2]$.*

A sufficient condition for a Banach space to have two non-intersecting M-ideals is that its centralizer is non-trivial (i.e. has dimension at least two). We are not going into details, but roughly speaking, the *centralizer* $Z(X)$ of a Banach space X is a closed subalgebra of $L(X)$ isometrically isomorphic to $C(K_X)$ where K_X is a Hausdorff topological space, and it is possible to see X as a $C(K_X)$-submodule of $\prod_{k\in K_X} X_k$ for suitable X_k's. We refer to [2, §3.B] and [10, §I.3] for details. It happens that every M-ideal of $C(K_X)$ produces an M-ideal of X in a suitable way (see [2, §4.A]) and if $Z(X)$ contains more than one point, then two non-intersecting M-ideals appear in X, so our corollary above applies.

Corollary 4.12. *Let X a Banach space. If $Z(X)$ has dimension greater than one, then $\Phi_X(\delta) = \Phi^S_X(\delta) = \sqrt{2\delta}$ for every $\delta \in (0, 2]$.*

To give some new examples coming from this corollary, we recall that the centralizer of a unital (complex) C^*-algebra identifies with its center (see [10, Theorem V.4.7] or [2, Example 3 in page 63]).

Example 4.13. *Let A be a unital C^*-algebra with non-trivial center. Then, $\Phi_A(\delta) = \Phi^S_A(\delta) = \sqrt{2\delta}$ for every $\delta \in (0, 1/2]$.*

It would be interesting to see whether the algebra $L(H)$ for a finite- or infinite-dimensional Hilbert space H has the maximum Bishop-Phelps-Bollobás moduli. None of the results of this section applies to it since its center is trivial and, despite it containing $K(H)$ as an M-ideal, there is no element $x_0 \in L(H)$ satisfying the requirements of Corollary 4.9 (see [3, page 538]). Let us also comment that the bidual of $L(H)$ is a C^*-algebra with non-trivial centralizer, so $\Phi_{L(H)^{**}}(\delta) = \Phi^S_{L(H)^{**}}(\delta) = \sqrt{2\delta}$ for every $\delta \in (0, 1/2]$. If there is $\delta \in (0, 1/2]$ such that $\Phi_{L(H)}(\delta) < \sqrt{2\delta}$, then this would be an example when the inequality in Proposition 3.6 is strict.

We finish this section with two pictures: one with the Bishop-Phelps-Bollobás moduli of $\mathbb{R}$, $\mathbb{C}$ and $\ell_\infty^{(2)}$, and another one with the corresponding values of the spherical Bishop-Phelps-Bollobás moduli.

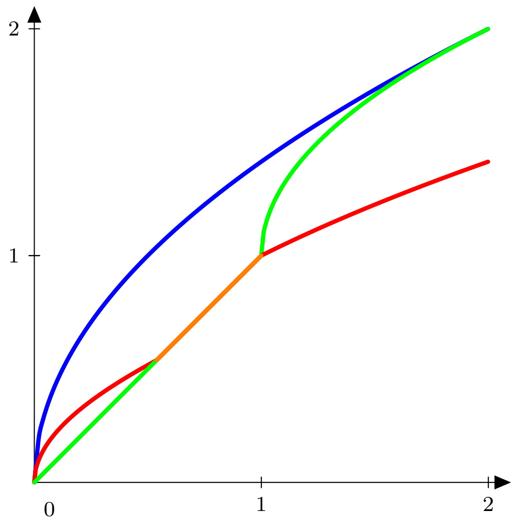

FIGURE 2. The value of $\Phi_X(\delta)$ for $\mathbb{R}$, $\mathbb{C}$, $\boldsymbol{\ell_\infty^{(2)}}$

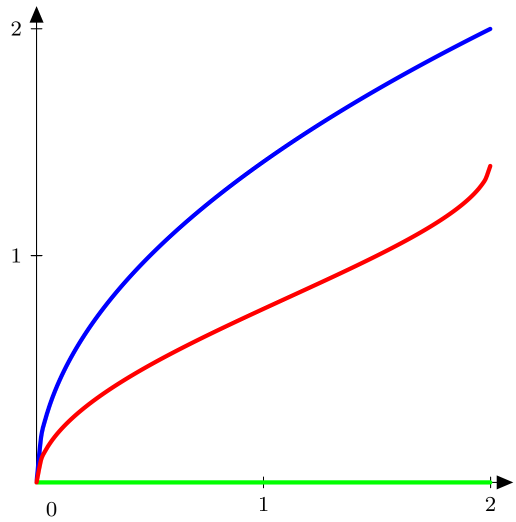

FIGURE 3. The value of $\Phi^S_X(\delta)$ for $\mathbb{R}$, $\mathbb{C}$, $\boldsymbol{\ell_\infty^{(2)}}$

5. Banach spaces with the greatest possible modulus

Our goal in this section is to show that Banach spaces with the greatest possible moduli contain almost isometric copies of the real ℓ_∞^2. Let us first recall the following definition.

Definition 5.1. Let X, E be Banach spaces. X is said to contain almost isometric copies of E if, for every $\varepsilon > 0$ there is a subspace $E_\varepsilon \subset X$ and there is a bijective linear operator $T : E \longrightarrow E_\varepsilon$ with $\|T\| < 1 + \varepsilon$ and $\|T^{-1}\| < 1 + \varepsilon$.

The next result is well-known and has a straightforward proof.

Lemma 5.2. *A real Banach space E contains an isometric copy of $\ell_\infty^{(2)}$ if and only if there are elements $u, v \in S_E$ such that $\|u - v\| = \|u + v\| = 2$. E contains almost isometric copies of $\ell_\infty^{(2)}$ if and only if there are elements $u_n, v_n \in S_E$, $n \in \mathbb{N}$ such that $\|u_n - v_n\| \longrightarrow 2$ and $\|u_n + v_n\| \longrightarrow 2$ as $n \to \infty$.*

The class of spaces X that do not contain almost isometric copies of $\ell_\infty^{(2)}$ was deeply studied by James [11] (see also the exposition in Van Dulst's book [9]), who gave to such spaces the name "uniformly non-square". He proved in particular, that every uniformly non-square space must be reflexive, that this property is stable under passing to subspaces, quotient spaces and duals. In fact, a general result is true [12]: for every 2-dimensional space E if a real Banach space X does not contain almost isometric copies of E then X is reflexive.

The aim of this section is to prove that if a real Banach space X satisfies that its Bishop-Phelps-Bollobás modulus is $\sqrt{2\delta}$ in at least one point $\delta \in (0, 1/2)$, then X (and, equivalently, the dual space) contains almost isometric copies of $\ell_\infty^{(2)}$. Actually, as shown in Remark 3.8, $\Phi_X(\delta) = \sqrt{2\delta}$ if and only if $\Phi_X^S(\delta) = \sqrt{2\delta}$. Therefore, we may use the formally stronger hypothesis of $\Phi_X^S(\delta) = \sqrt{2\delta}$.

We will use some lemmas and ideas of Bishop and Phelps [4], but for the reader's convenience we will refer to the corresponding lemmas in Diestel's already classical book [8].

From now on, X will denote a *real* Banach space. For $t > 1$ and $x^* \in S_{X^*}$, we denote

$$K(t, x^*) := \{x \in X \,:\, \|x\| \leqslant t\, x^*(x)\}.$$

Observe that $K(t, x^*)$ is a convex cone with non-empty interior.

Lemma 5.3 (Particular case of [8, Chapter 1, Lemma 1])**.** *For every $z \in B_X$, every $x^* \in S_{X^*}$ and every $t > 1$, there is $x_0 \in S_X$ such that $x_0 - z \in K(t, x^*)$ and $[K(t, x^*) + x_0] \cap B_X = \{x_0\}$.*

Lemma 5.4 ([8, Chapter 1, Lemma 2] with a little modification that follows from the proof there)**.** *Let $x^*, y^* \in S_{X^*}$ and suppose that $x^*\big(\ker y^* \cap S_X\big) \subset (-\infty, \varepsilon/2]$. Then*

$$\operatorname{dist}\big(x^*, \operatorname{Lin} y^*\big) \leqslant \varepsilon/2 \qquad \textit{and} \qquad \min\{\|x^* - y^*\|, \|x^* + y^*\|\} \leqslant \varepsilon.$$

Lemma 5.5. *Let $z \in B_X$, $x^* \in S_{X^*}$, $t > 1$, and let $x_0 \in S_X$ be from Lemma 5.3. Denote $y^* \in S_{X^*}$ a functional that separates $x_0 + K(t, x^*)$ from B_X, so $y^*(x_0) = 1$ and $y^*\big(K(t, x^*)\big) \subset [0, \infty)$. Then $x^*\big(\ker y^* \cap S_X\big) \subset (-\infty, 1/t]$ and so, $\operatorname{dist}\big(x^*, \operatorname{Lin} y^*\big) \leqslant 1/t$ and $\min\{\|x^* - y^*\|, \|x^* + y^*\|\} \leqslant 2/t$.*

Proof. This also can be extracted from [8, Chapter 1], but it is better to give a proof. For every $w \in \ker y^* \cap S_X$ we have that w does not belong to the interior of $K(t, x^*)$, so $1 = \|w\| \geqslant t\, x^*(w)$, i.e. $x^*(\ker y^* \cap S_X) \subset (-\infty, 1/t]$. An application of Lemma 5.4 completes the proof. □

Now we are passing to our results. At first, for the sake of simplicity, we consider the easier finite-dimensional case.

Lemma 5.6. *Let X be a finite-dimensional real space. Fix $\varepsilon \in (0, 1)$. Suppose that $(x, x^*) \in S_X \times S_{X^*}$ satisfies that $x^*(x) = 1 - \frac{\varepsilon^2}{2}$ and that*

$$\max\{\|y - x\|, \|y^* - x^*\|\} \geqslant \varepsilon$$

for every pair $(y, y^*) \in \Pi(X)$. Then for $t = \frac{2}{\varepsilon}$, there exists $y_0 \in [x + K(t, x^*)] \cap S_X$ such that $x^*(y_0) = 1$.

Proof. Consider a sequence $t_n > t$, $n \in \mathbb{N}$, with $\lim_n t_n = t$. Using Lemma 5.3, we get $y_n \in S_X$ such that

$$y_n - x \in K(t_n, x^*) \quad \text{and} \quad (K(t_n, x^*) + y_n) \cap B_X = \{y_n\}. \tag{8}$$

Let $y_n^* \in X^*$ be a functional that separates $K(t_n, x^*) + y_n$ from B_X, i.e. $y_n^*(y_n) = 1$ and $y_n^*(K(t_n, x^*)) \subset [0, \infty)$. Then, according to Lemma 5.5,

$$\min\{\|x^* - y_n^*\|, \|x^* + y_n^*\|\} \leqslant 2/t_n < \varepsilon. \tag{9}$$

But

$$\|x^* + y_n^*\| \geqslant (x^* + y_n^*)(y_n) = 1 + x^*(y_n) = 1 + x^*(x) + x^*(y_n - x) = 2 - \frac{\varepsilon^2}{2} + x^*(y_n - x).$$

Since $(y_n - x) \in K(t_n, x^*)$, we have $x^*(y_n - x) \geqslant \|(y_n - x)\|/t_n \geqslant 0$ so

$$\|x^* + y_n^*\| \geqslant 2 - \frac{\varepsilon^2}{2} > \varepsilon$$

(we have used here that $0 < \varepsilon < 1$). Comparing with (9), we get $\|x^* - y_n^*\| < \varepsilon$, so the condition of our lemma says that $\|x - y_n\| \geqslant \varepsilon$. Without loss of generality (passing to a subsequence if necessary) we can assume that y_n tend to some y_0. Then

$$\begin{aligned}\varepsilon &\leqslant \lim_n \|y_n - x\| \leqslant \lim_n t_n x^*(y_n - x) = t(x^*(y_0) - x^*(x)) \\ &\leqslant \frac{2}{\varepsilon}(x^*(y_0) - 1 + \frac{\varepsilon^2}{2}) \leqslant \frac{2}{\varepsilon}(1 - 1 + \frac{\varepsilon^2}{2}) = \varepsilon.\end{aligned}$$

This means that all the inequalities in the above chain are in fact equalities. In particular, $x^*(y_0) = 1$ and

$$\|y_0 - x\| = \lim_n \|y_n - x\| = t\big(x^*(y_0) - x^*(x)\big),$$

i.e. $y_0 \in [x + K(t, x^*)] \cap S_X$. □

Lemma 5.7. *Under the conditions of Lemma 5.6, there are $y^* \in S_{X^*}$ and $\alpha \geqslant 1 - \frac{\varepsilon}{2}$ with*

$$\|x^* - \alpha y^*\| \leqslant \frac{\varepsilon}{2} \quad \text{and} \quad \|x^* - y^*\| \geqslant \varepsilon, \tag{10}$$

and there is $v \in S_X$ such that

$$x^*(v) = y^*(v) = 1. \tag{11}$$

Proof. Let y_0 be from Lemma 5.6. Fix a strictly increasing sequence of $t_n > 1$ with $\lim_n t_n = t$ and let us consider two cases.

Case 1: Suppose there exists $m_0 \in \mathbb{N}$ with $\operatorname{int}\big[K(t_{m_0}, x^*) + x\big] \cap B_X \neq \emptyset$. Then, using the fact that for every closed convex set with non-empty interior, the closure of the interior is the whole set, we get

$$y_0 \in \big[x + K(t, x^*)\big] \cap B_X = \overline{\operatorname{int}\big[x + K(t, x^*)\big] \cap B_X} = \overline{\bigcup_{n \geqslant m_0} \operatorname{int}\big[x + K(t_n, x^*)\big] \cap B_X}.$$

So, we can pick

$$z_n \in \big[x + K(t_n, x^*)\big] \cap B_X \tag{12}$$

such that $z_n \longrightarrow y_0$. In particular, $x^*(z_n) \longrightarrow 1$. Let us apply Lemma 5.3: there are $v_n \in S_X$ such that

$$v_n - z_n \in K(t_n, x^*) \quad \text{and} \quad \big[K(t_n, x^*) + v_n\big] \cap B_X = \{v_n\}. \tag{13}$$

Then $x^*(v_n - z_n) \geqslant 0$, i.e. $1 \geqslant x^*(v_n) \geqslant x^*(z_n) \longrightarrow 1$, so $x^*(v_n) \longrightarrow 1$. Condition (12) implies that $z_n - x \in K(t_n, x^*)$ which, together with (13), mean that $v_n - x \in K(t_n, x^*)$. Consequently,

$$\|v_n - x\| \leqslant t_n x^*(v_n - x) \leqslant t_n \frac{\varepsilon^2}{2} < \varepsilon.$$

If we denote $y_n^* \in S_{X^*}$ to the functional that separates $v_n + K(t_n, x^*)$ from B_X, then $(v_n, y_n^*) \in \Pi(X)$. Since we are working under the conditions of Lemma 5.6, it follows that

$$\|y_n^* - x^*\| \geqslant \varepsilon.$$

Also, by Lemma 5.5, $\operatorname{dist}(x^*, \operatorname{Lin} y_n^*) \leqslant 1/t_n$, so there are $\alpha_n \in \mathbb{R}$ such that

$$\|x^* - \alpha_n y_n^*\| \leqslant 1/t_n.$$

Again, without loss of generality, we may assume that the sequences (α_n), (v_n) and (y_n^*) have limits. Let us denote $\alpha := \lim_n \alpha_n$, $y^* := \lim_n y_n^*$, and $v := \lim_n v_n$. Then $\|v\| = 1$, $\|y^*\| = 1$, $x^*(v) = \lim_n x^*(v_n) = 1$, and $y^*(v) = \lim_n y_n^*(v_n) = 1$. This proves (11). Also,

$$\|x^* - \alpha y^*\| = \lim_n \|x^* - \alpha_n y_n^*\| \leqslant \frac{1}{t} = \frac{\varepsilon}{2}.$$

Consequently,

$$\frac{\varepsilon}{2} \geqslant \|x^* - \alpha y^*\| \geqslant (x^* - \alpha y^*)(v) = 1 - \alpha, \tag{14}$$

so, $\alpha \geqslant 1 - \frac{\varepsilon}{2}$.

Case 2: Assume that for every $n \in \mathbb{N}$ we have $\operatorname{int}\big[K(t_n, x^*) + x\big] \cap B_X = \emptyset$. Let us separate $x + \operatorname{int}(K(t_n, x^*))$ from B_X by a norm-one functional y_n^*, that is,

$$y_n^*\left(x + \operatorname{int}\big[K(t_n, x^*)\big]\right) > 1,$$

so, in particular, $y_n^*(x) \geqslant 1$.

Again, passing to a subsequence, we can assume that there exists $y^* = \lim_n y_n^*$ which satisfies $\|y^*\| = 1$, $1 \geqslant y^*(x) \geqslant \lim_n y_n^*(x) \geqslant 1$. So, $y^*(x) = 1$, i.e. $(x, y^*) \in \Pi(X)$. By the conditions of our lemma, this implies that

$$\|y^* - x^*\| = \max\{\|x - x\|, \|y^* - x^*\|\} \geqslant \varepsilon.$$

Since

$$y_0 \in x + K(t, x^*) = \overline{\bigcup_{n\in\mathbb{N}} \operatorname{int}\big[x + K(t_n, x^*)\big]},$$

we can select $z_n \in \operatorname{int}\big[x + K(t_n, x^*)\big]$ in such a way that $z_n \longrightarrow y_0$. Then

$$y^*(y_0) = \lim_n y_n^*(z_n) \geqslant 1,$$

hence, $y^*(y_0) = 1$. This means that condition (11) works for $v := y_0$. The remaining conditions can be deduced from Lemma 5.5 the same way as in the case 1. ∎

We state and prove the main result of the section in the finite-dimensional case.

Theorem 5.8. *Let X be a finite-dimensional real Banach space. Suppose that there is a $\delta \in (0, 1/2)$ such that $\Phi_X(\delta) = \sqrt{2\delta}$ (or, equivalently, $\Phi_X^S(\delta) = \sqrt{2\delta}$). Then X^* contains an isometric copy of $\ell_\infty^{(2)}$ (hence, X also contains an isometric copy of $\ell_\infty^{(2)}$).*

Proof. Denote $\varepsilon := \sqrt{2\delta} \in (0, 1)$. There is a sequence of pairs $(x_n, x_n^*) \in S_X \times S_{X^*}$ such that $x_n^*(x_n) > 1 - \delta = 1 - \frac{\varepsilon^2}{2}$ and

$$\max\{\|y - x_n\|, \|y^* - x_n^*\|\} \geqslant \varepsilon - \frac{1}{n}$$

for every pair $(y, y^*) \in \Pi(X)$. Since the space is finite-dimensional, we can find a subsequence of (x_n, x_n^*) that converges to a pair $(x, x^*) \in S_X \times S_{X^*}$. This pair satisfies that $x^*(x) \geqslant 1 - \delta$ and for every $(y, y^*) \in \Pi(X)$,

$$\begin{aligned}\max\{\|y - x\|, \|y^* - x^*\|\} &\geqslant \max\{\|y - x_n\|, \|y^* - x_n^*\|\} - \max\{\|x - x_n\|, \|x^* - x_n^*\|\} \\ &\geqslant \varepsilon - \frac{1}{n} - \max\{\|x - x_n\|, \|x^* - x_n^*\|\} \longrightarrow \varepsilon.\end{aligned}$$

Since by Theorem 2.1, $x^*(x)$ cannot be strictly smaller than $1 - \delta$, we have $x^*(x) = 1 - \delta$. Therefore, we may apply Lemma 5.7 to find $y^* \in S_{X^*}$ and $\alpha \geqslant 1 - \frac{\varepsilon}{2}$ for which conditions (10) and (11) are fulfilled. Now we *claim* that in fact there is only one number $\gamma \in \mathbb{R}$ for which

$$\|x^* - \gamma y^*\| \leqslant \frac{\varepsilon}{2} \tag{15}$$

and this γ equals $1 - \frac{\varepsilon}{2}$. So $\alpha = 1 - \frac{\varepsilon}{2}$ and, we also *claim* that

$$\|x^* - \alpha y^*\| = \frac{\varepsilon}{2} \quad \text{and} \quad \|x^* - y^*\| = \varepsilon. \tag{16}$$

Indeed, when we were proving equation (14), we proved that every $\gamma \in \mathbb{R}$ that fulfill (15) satisfies $\gamma \geqslant 1 - \frac{\varepsilon}{2}$. On the other hand, the function $\gamma \longmapsto \|x^* - \gamma y^*\|$ is convex, so the set G of those $\gamma \in \mathbb{R}$ satisfying (15) is also convex; but $1 \notin G$, so $\gamma < 1$. Finally, according to (10),

$$\frac{\varepsilon}{2} \geqslant 1 - \gamma = \|y^* - \gamma y^*\| \geqslant \|x^* - y^*\| - \|x^* - \gamma y^*\| \geqslant \frac{\varepsilon}{2}.$$

This means that all the inequalities above are equalities, so $\gamma \leqslant 1 - \frac{\varepsilon}{2}$, and also (16) is true. The claim is proved.

Now, let us define

$$u^* := \frac{x^* - \alpha y^*}{\|x^* - \alpha y^*\|} = \frac{2}{\varepsilon}(x^* - (1 - \frac{\varepsilon}{2})y^*),$$

and let us show that functionals u^* and y^* span a subspace of X^* isometric to $\ell_\infty^{(2)}$. According to Lemma 5.2, it is sufficient to show that $\|u^* - y^*\| = \|u^* + y^*\| = 2$. At first,

$$\|u^* - y^*\| = \left\|\frac{2}{\varepsilon}(x^* - (1 - \frac{\varepsilon}{2})y^*) - y^*\right\| = \frac{2}{\varepsilon}\|x^* - y^*\| = 2.$$

At second,

$$2 \geqslant \|u^* + y^*\| = \|\frac{2}{\varepsilon}(x^* - (1 - \frac{\varepsilon}{2})y^*) + y^*\| = \frac{2}{\varepsilon}\|x^* - y^* + \varepsilon y^*\| \geqslant \frac{2}{\varepsilon}(x^* - y^* + \varepsilon y^*)(v) = 2. \qquad \square$$

Let us comment that for complex Banach spaces, we cannot expect that Theorem 5.8 provides a *complex* copy of $\ell_\infty^{(2)}$ in the dual of the space. Namely, the two-dimensional complex space $X = \ell_1^{(2)}$ satisfies $\Phi_X(\delta) = \sqrt{2\delta}$ for $\delta \in (0, 1/2)$ but it does not contain the complex space $\ell_\infty^{(2)}$ (of course, it contains the real space $\ell_\infty^{(2)}$ as a subspace since $\ell_1^{(2)}$ and $\ell_\infty^{(2)}$ are isometric in the real case). We do not know whether it is true a result saying that if a complex space X satisfies $\Phi_X(\delta) = \sqrt{2\delta}$ for some $\delta \in (0, 1/2)$, then X contains a copy of the complex space $\ell_1^{(2)}$ or a copy of the complex space $\ell_\infty^{(2)}$.

Let us extend the result of Theorem 5.8 to the infinite-dimensional case. Roughly speaking, we proceed as in the proof of such theorem, but instead of selecting convergent subsequences, we select subsequences such that their numerical characteristics (like norms of elements, pairwise distances, or values of some important functionals) have limits.

Theorem 5.9. *Let X be an infinite-dimensional Banach space. Suppose that there is $\delta \in (0, 1/2)$ such that $\Phi_X(\delta) = \sqrt{2\delta}$ (or, equivalently, $\Phi_X^S(\delta) = \sqrt{2\delta}$). Then X^* (and hence also X) contains almost isometric copies of $\ell_\infty^{(2)}$.*

Proof. Denote $\varepsilon := \sqrt{2\delta}$. There is a sequence of pairs $(x_n, x_n^*) \in S_X \times S_{X^*}$ such that $x_n^*(x_n) > 1 - \delta = 1 - \frac{\varepsilon^2}{2}$ and

$$\max\{\|y - x_n\|, \|y^* - x_n^*\|\} \geqslant \varepsilon - \frac{1}{n} \tag{17}$$

for every pair $(y, y^*) \in \Pi(X)$. Since we have $x_n^*(x_n) \leqslant 1 - (\varepsilon - \frac{1}{n})^2/2$ by Theorem 2.1, we deduce that $\lim_n x_n^*(x_n) = 1 - \delta$. Denote $t = \frac{2}{\varepsilon}$. As in the proof of Lemma 5.6, we find a sequence (y_n) of elements in S_X such that

$$\lim_n \|y_n - x_n\| \leqslant t \lim_n x_n^*(y_n - x_n) \quad \text{and} \quad \lim_n x_n^*(y_n) = 1. \tag{18}$$

Pick a sequence (t_n) with $t_n > t$, $n \in \mathbb{N}$ and $\lim_n t_n = t$. Using Lemma 5.3, for every $n \in \mathbb{N}$ we get $y_n \in S_X$ such that

$$y_n - x_n \in K(t_n, x_n^*) \quad \text{and} \quad (K(t_n, x_n^*) + y_n) \cap B_X = \{y_n\}. \tag{19}$$

For given $n \in \mathbb{N}$, let $u_n^* \in S_{X^*}$ be a functional that separates $K(t_n, x_n^*) + y_n$ from B_X, that is, satisfying $u_n^*(y_n) = 1$ and $u_n^*(K(t_n, x_n^*)) \subset [0, \infty)$. Then, according to Lemma 5.5, we have

$$\min\{\|x_n^* - u_n^*\|, \|x_n^* + u_n^*\|\} \leqslant 2/t_n < \varepsilon.$$

As we have

$$\|x_n^* + u_n^*\| \geqslant (x_n^* + u_n^*)(y_n) = 1 + x_n^*(y_n) = 1 + x_n^*(x_n) + x_n^*(y_n - x_n) \geqslant 2 - \frac{\varepsilon^2}{2} > \varepsilon,$$

we get $\|x_n^* - u_n^*\| < \varepsilon$, so (17) says that $\|x_n - y_n\| \geqslant \varepsilon - \frac{1}{n}$. Without loss of generality, passing to a subsequence if necessary, we can assume that the following limits exist: $\lim_n \|x_n - y_n\|$, $\lim_n x_n^*(y_n - x_n)$

and $\lim_n x_n^*(y_n)$. Then

$$\varepsilon \leqslant \lim_n \|y_n - x_n\| \leqslant \lim_n t_n x_n^*(y_n - x_n) = t \lim_n x_n^*(y_n - x_n)$$
$$\leqslant \frac{2}{\varepsilon}(\lim_n x_n^*(y_n) - 1 + \frac{\varepsilon^2}{2}) \leqslant \frac{2}{\varepsilon}(1 - 1 + \frac{\varepsilon^2}{2}) = \varepsilon.$$

This means that all the inequalities in the above chain are in fact equalities. In particular, $\lim_n x_n^*(y_n) = 1$, and

$$\varepsilon = \lim_n \|y_n - x_n\| = t \lim_n x_n^*(y_n - x_n), \tag{20}$$

so the analogue of Lemma 5.6 is proved.

Now, we proceed with analogue of Lemma 5.7: we need to show that there are $y_n^* \in S_{X^*}$ and $\alpha_n \geqslant 0$, $\alpha_n \longrightarrow 1 - \frac{\varepsilon}{2}$ with

$$\|x_n^* - \alpha_n y_n^*\| \leqslant \frac{\varepsilon}{2} \quad \text{and} \quad \|x_n^* - y_n^*\| \geqslant \varepsilon, \tag{21}$$

and there is a sequence of $v_n \in S_X$ such that

$$\lim_n x_n^*(v_n) = \lim_n y_n^*(v_n) = 1. \tag{22}$$

Case 1: Assume that there exist $r > 0$ and $n \in \mathbb{N}$ such that, for all $m > n$,

$$\big([K(t - r, x_m^*) + x_m] \cap B_X\big) \setminus (x_m + rB_X) \neq \emptyset.$$

This means that for all $m > n$ there is z_m such that

$$\|z_m - x_m\| > r, \quad \|z_m\| \leqslant 1 \quad \text{and} \quad \|z_m - x_m\| \leqslant (t - r)x_m^*(z_m - x_m).$$

For $\lambda \in (0, 1)$ denote $y_{m,\lambda} := \lambda z_m + (1 - \lambda)y_m$. Clearly, $y_{m,\lambda} \in B_X$. Denote also

$$\lambda_m = \inf\{\lambda : y_{m,\lambda} \in x_m + K(t, x_m^*)\},$$

and let us show that

$$\lim_m \lambda_m = 0. \tag{23}$$

Observe first that λ_m is smaller than every value of λ for which

$$\|y_{m,\lambda} - x_m\| \leqslant t x^*(y_{m,\lambda} - x_m).$$

On the one hand, if $\|y_m - x_m\| - t x_m^*(y_m - x_m) \leqslant 0$, then $\lambda = 0$ belongs to the set in question, and the job is done. On the other hand, if $\|y_m - x_m\| - t x_m^*(y_m - x_m) \geqslant 0$, then there is λ for which

$$\lambda\|z_m - x_m\| + (1 - \lambda)\|y_m - x_m\| = t\lambda x_m^*(z_m - x_m) + t(1 - \lambda)x_m^*(y_m - x_m)$$

is positive and belongs to the set in question. This means that

$$\lambda_m \leqslant \frac{\|y_m - x_m\| - t x_m^*(y_m - x_m)}{\|y_m - x_m\| - t x_m^*(y_m - x_m) + t x_m^*(z_m - x_m) - \|z_m - x_m\|},$$

but the limit of the right-hand side equals 0 thanks to (20). So condition (23) is proved. This means that $y_{m,\lambda_m} \in x_m + K(t, x_m^*)$ and $\|y_{m,\lambda_m} - y_m\| \leqslant 2\lambda_m \longrightarrow 0$. Let us pick a little bit bigger $\tilde{\lambda}_m > \lambda_m$ in such a way that we still have $\|y_{m,\tilde{\lambda}_m} - y_m\| \longrightarrow 0$, but for some $\tilde{t}_n < t$ with $\tilde{t}_n \longrightarrow t$, we have

$$\|y_{m,\tilde{\lambda}_m} - x_m\| \leqslant \tilde{t}_n x_m^*(y_{m,\tilde{\lambda}_m} - x_m). \tag{24}$$

Then, in particular, $\lim_n x_n^*(y_{n,\tilde{\lambda}_n}) = \lim_n x_n^*(y_n) = 1$. Let us apply Lemma 5.3. There are $v_n \in S_X$ such that

$$v_n - y_{n,\tilde{\lambda}_n} \in K(\tilde{t}_n, x_n^*) \quad \text{and} \quad [K(\tilde{t}_n, x_n^*) + v_n] \cap B_X = \{v_n\}. \tag{25}$$

Then $x_n^*(v_n - y_{n,\tilde\lambda_n}) \geqslant 0$, i.e. $1 \geqslant x_n^*(v_n) \geqslant x_n^*(y_{n,\tilde\lambda_n}) \longrightarrow 1$, so $x_n^*(v_n) \longrightarrow 1$. This proves the first part of (22). Condition (24) implies that $y_{n,\tilde\lambda_n} - x_n \in K(\tilde t_n, x_n^*)$ which, together with (25), mean that $v_n - x_n \in K(\tilde t_n, x_n^*)$. Consequently,

$$\|v_n - x_n\| \leqslant \tilde t_n x_n^*(v_n - x_n) \leqslant \tilde t_n \frac{\varepsilon^2}{2} < \varepsilon.$$

If we denote by $y_n^* \in S_{X^*}$ the functional that separates $v_n + K(\tilde t_n, x^*)$ from B_X, then $(v_n, y_n^*) \in \Pi(X)$ (this proves the second part of (22) even in a stronger form) so, thanks to (17),

$$\|y_n^* - x_n^*\| \geqslant \varepsilon - \frac{1}{n}.$$

Also, by Lemma 5.5, $\operatorname{dist}(x_n^*, \operatorname{Lin} y_n^*) \leqslant 1/\tilde t_n$, so there are $\alpha_n \in \mathbb{R}$ such that

$$\|x^* - \alpha_n y_n^*\| \leqslant 1/\tilde t_n.$$

Again, without loss of generality, we may assume that the sequences (α_n) and $\|x_n^* - \alpha_n y_n^*\|$ converge. Then,

$$\lim_n \|x_n^* - \alpha_n y_n^*\| \leqslant \frac{1}{t} = \frac{\varepsilon}{2}.$$

Consequently,

$$\frac{\varepsilon}{2} \geqslant \lim_n \|x_n^* - \alpha_n y_n^*\| \geqslant \lim_n (x_n^* - \alpha_n y_n^*)(v) = 1 - \lim_n \alpha_n\,,$$

so $\lim_n \alpha_n \geqslant 1 - \frac{\varepsilon}{2}$. Starting at this point, (21) can be deduced in the same way as it was done for (16).

Case 2: Assume that there is a sequence of $r_n > 0$, $r_n \longrightarrow 0$ and that there is a subsequence of (x_m, x_m^*) (that we will again denote (x_m, x_m^*)) such that

$$\big(\big[K(t - r_m, x_m^*) + x_m\big] \cap B_X\big) \setminus (x_m + r_m B_X) = \emptyset \qquad (\text{for all } m \in \mathbb{N}).$$

Then also

$$\big[K(t - r_m, x_m^*) + x_m\big] \cap (1 - r_m) B_X = \emptyset \qquad (\text{for all } m \in \mathbb{N}).$$

Let us separate

$$\frac{1}{1 - r_m}\big[K(t - r_m, x_m^*) + x_m\big]$$

from B_X by a norm-one functional y_n^*, that is,

$$y_n^*\big(K(t - r_m, x_m^*) + x_m\big) > 1 - r_m \tag{26}$$

so, in particular, $y_m^*(x_m) \geqslant 1 - r_m$ and $\lim_m y_m^*(x_m) = 1$. By the Bishop-Phelps-Bollobás theorem, there is a sequence $(\tilde x_n, \tilde y_n^*) \in \Pi(X)$, such that

$$\max\{\|\tilde x_n - x_n\|, \|\tilde y_n^* - y_n^*\|\} \longrightarrow 0 \ \text{ as } \ n \to \infty.$$

Again, passing to a subsequence, we can assume that all the numerical characteristics that appear here have the corresponding limits. According to (17), for n big enough, we have

$$\|\tilde y_n^* - x_n^*\| = \max\{\|\tilde x_n - x_n\|, \|\tilde y_n^* - x_n^*\|\} \geqslant \varepsilon - \frac{1}{n},$$

so $\lim_n \|y_n^* - x_n^*\| \geqslant \varepsilon$. We can select $z_n \in x_n + K(t - r_n, x_n^*)$ in such a way that $\|z_n - y_n\| \longrightarrow 0$. Then

$$1 \geqslant \lim_n y_n^*(y_n) = \lim_n y_n^*(z_n) \geqslant \lim_n (1 - r_n) = 1.$$

This means that condition (22) works for $v_n := y_n$.

Now consider an arbitrary $w \in \ker y_n^* \cap S_X$. Taking a convex combination with an element h of the unit sphere where $y_n^*(h)$ almost equals -1, we can construct an element $\tilde w \in B_X$ such that $\|\tilde w - w\| \leqslant 2r_n$ and $y_n^*(\tilde w) = -r_n$. Then, by (26), $\tilde w \notin \operatorname{int}\big(K(t - r_n, x_n^*)\big)$, so $\|\tilde w\| \geqslant (t - r_n) x_n^*(\tilde w)$. Consequently,

$$x_n^*(w) \leqslant x_n^*(\tilde w) + 2r_n \leqslant \frac{1}{t - r_n} + 2r_n.$$

Observe that we have shown that the values of the functional x_n^* on $\ker y_n^* \cap S_X$ do not exceed $\frac{1}{t-r_n} + 2r_n$. Therefore, by Lemma 5.4,

$$\operatorname{dist}(x_n^*, \operatorname{Lin} y_n^*) \leqslant \frac{1}{t-r_n} + 2r_n \longrightarrow \frac{1}{t}$$

and so there are $\alpha_n \in \mathbb{R}$ such that

$$\lim_n \|x_n^* - \alpha_n y_n^*\| \leqslant \frac{1}{t}.$$

The remaining conditions in (21) and (22) can be deduced from the same way as in the case 1.

Finally, (21) and (22) imply that $\lim_n \|x_n^* - y_n^*\| = \lim_n \|x_n^* - y_n^*\| = 2$: the proof does not differ much from the corresponding part of the Theorem 5.8 demonstration. □

Corollary 5.10. *Let X be a uniformly non-square Banach space. Then, $\Phi_X^S(\delta) \leqslant \Phi_X(\delta) < \sqrt{2\delta}$ for every $\delta \in (0, 1/2)$. Consequently, every superreflexive Banach space can be equivalently renormed in such a way that, in the new norm, $\Phi_X^S(\delta) \leqslant \Phi_X(\delta) < \sqrt{2\delta}$ for all $\delta \in (0, 1/2)$.*

It would be interesting to obtain a quantitative version of the above corollary.

6. A three-dimensional space E containing $\ell_\infty^{(2)}$ with $\Phi_E(\delta) < \sqrt{2\delta}$

Like in the previous section, for every $\delta \in (0, 1/2)$ we denote $\varepsilon = \sqrt{2\delta}$, so $0 < \varepsilon < 1$. We denote $B_\varepsilon^3 \subset \mathbb{R}^3$ the absolute convex hull of the following 11 points $A_k, k = 1, \ldots, 11$ (or, what is the same, the convex hull of 22 points $\pm A_k, k = 1, \ldots, 11$):

$$A_1 = (0, 0, \frac{3}{4}),$$

$$A_2 = (1-\varepsilon, 1, \frac{\varepsilon}{2}),\ A_3 = (1-\varepsilon, -1, \frac{\varepsilon}{2}),\ A_4 = (\varepsilon - 1, 1, \frac{\varepsilon}{2}),\ A_5 = (\varepsilon - 1, -1, \frac{\varepsilon}{2}),$$

$$A_6 = (1, 1-\varepsilon, \frac{\varepsilon}{2}),\ A_7 = (-1, 1-\varepsilon, \frac{\varepsilon}{2}),\ A_8 = (1, \varepsilon - 1, \frac{\varepsilon}{2}),\ A_9 = (-1, \varepsilon - 1, \frac{\varepsilon}{2}),$$

$$A_{10} = (1, 1, 0),\ A_{11} = (1, -1, 0).$$

Denote D_ε ("D" from "Diamond") the normed space $\big(\mathbb{R}^3, \|\cdot\|\big)$, for which B_ε^3 is its unit ball. Then D_ε^* can be viewed as $\mathbb{R}^3$ with the polar of B_ε^3 as the unit ball, and the action of $x^* \in D_\varepsilon^*$ on $x \in D_\varepsilon$ is just the standard inner product in $\mathbb{R}^3$. Let us list, without proof, some properties of D_ε whose verification is straightforward:

- The subspace of D_ε formed by vectors of the form $(x_1, x_2, 0)$ is canonically isometric to $\ell_\infty^{(2)}$.
- There are no other isometric copies of $\ell_\infty^{(2)}$ in D_ε.
- The subspace of D_ε^* formed by vectors of the form $(x_1, x_2, 0)$ is canonically isometric to $\ell_1^{(2)}$ (and so, is isometric to $\ell_\infty^{(2)}$).
- There are no other isometric copies of $\ell_\infty^{(2)}$ in D_ε^*.
- The following operators act as isometries both on D_ε and D_ε^*: $(x_1, x_2, x_3) \longmapsto (x_2, x_1, x_3)$, $(x_1, x_2, x_3) \longmapsto (x_1, -x_2, x_3)$. In other words, changing the sign of one coordinate or rearranging the first two coordinates do not change the norm of an element.

The following theorem shows that the existence of an $\ell_\infty^{(2)}$-subspace does not imply that $\Phi_X(\delta) = \sqrt{2\delta}$, even in dimension 3.

Theorem 6.1. *Let $\delta \in (0, 1/2)$, $\varepsilon = \sqrt{2\delta}$, and $X = D_\varepsilon$. Then $\Phi_X(\delta) < \sqrt{2\delta}$.*

Proof. Assume contrary that $\Phi_X(\delta) = \sqrt{2\delta}$. Like in the proof of Theorem 5.8, this implies the existence of a pair $(x, x^*) \in S_X \times S_{X^*}$ with the following properties: $x^*(x) = 1 - \delta$ and

$$\max\{\|z - x\|, \|z^* - x^*\|\} \geqslant \varepsilon \text{ for every pair } (z, z^*) \in \Pi(X). \tag{27}$$

Also, repeating the proof of Theorem 5.8 for this $x^* \in S_{X^*}$, we can find $u^*, y^* \in S_{X^*}$ such that the pair (u^*, y^*) is 1-equivalent to the canonical basis of $\ell_1^{(2)}$ and

$$u^* = \frac{2}{\varepsilon}(x^* - (1 - \frac{\varepsilon}{2})y^*).$$

This means that $x^* = \frac{\varepsilon}{2}u^* + (1 - \frac{\varepsilon}{2})y^*$. What can be this (u^*, y^*) if we take into account that there is only one isometric copy of $\ell_1^{(2)}$ in X^*? It can be either $u^* = (1, 0, 0)$, $y^* = (0, 1, 0)$, or a pair of vectors that can be obtained from this one by application of isometries, i.e. just 8 possibilities. Consequently, x^* either equals to the vector $(\varepsilon/2, 1 - \varepsilon/2, 0)$, or to a vector that can be obtained from this one by application of isometries, again just 8 possibilities.

By duality argument, there are $u, y \in S_X$ such that the pair (u, y) is 1-equivalent to the canonical basis of $\ell_1^{(2)}$ and

$$x = \frac{\varepsilon}{2}u + (1 - \frac{\varepsilon}{2})y.$$

Since the only (up to isometries) pair $u, y \in S_X$ of this kind is $u = (1, 1, 0)$, $y = (1, -1, 0)$, we get $x = (1, 1 - \varepsilon, 0)$, or can be obtained from this one by application of isometries. So there are $8 \times 8 = 64$ possibilities for the pair (x, x^*). Taking into account that $x^*(x) = 1 - \delta$ we reduce this number to 8 possibilities: $x = (1 - \varepsilon, 1, 0)$, $x^* = (\varepsilon/2, 1 - \varepsilon/2, 0)$ and images of this pair under remaining 7 reflections and rotations of the underlying $\mathbb{R}^2$. If we show that this choice of (x, x^*) do not satisfy condition (27) then, by symmetry, the remaining choices would not satisfy (27) neither, and this would give us the desired contradiction.

Indeed, the pair $(z, z^*) \in \Pi(X)$ that do not satisfy (27) for $x = (1 - \varepsilon, 1, 0)$, $x^* = (\varepsilon/2, 1 - \varepsilon/2, 0)$ is the following one: $z = (1 - \varepsilon, 1, \varepsilon/2)$, $z^* = (\varepsilon/2, 1 - \varepsilon/2, \varepsilon)$. Let us check the required properties. At first, $z = A_2 \in S_X$. Then, $z^*(z) = 1$. The last property means, that $\|z^*\| \geqslant 1$, so in order to check that $\|z^*\| = 1$ it remains to show that $|z^*(A_k)| \leqslant 1$ for all k. This is true for $\varepsilon < 1$. Finally, $\|z - x\| = \|(0, 0, \varepsilon/2)\| = \frac{\varepsilon}{2}\|\frac{4}{3}A_1\| = \frac{2}{3}\varepsilon < \varepsilon$, and $\|z^* - x^*\| = \|(0, 0, \varepsilon)\| = \langle(0, 0, \varepsilon), A_1\rangle = \frac{3}{4}\varepsilon < \varepsilon$. □

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