THE BISHOP-PHELPS-BOLLOBÁS PROPERTY FOR OPERATORS BETWEEN SPACES OF CONTINUOUS FUNCTIONS

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Abstract. We show that the space of bounded linear operators between spaces of continuous functions on compact Hausdorff topological spaces has the Bishop-Phelps-Bollobás property. A similar result is also proved for the class of compact operators from the space of continuous functions vanishing at infinity on a locally compact and Hausdorff topological space into a uniformly convex space, and for the class of compact operators from a Banach space into a predual of an L_1-space.

1. Introduction

E. Bishop and R. Phelps proved in 1961 [7] that every (continuous linear) functional x^* on a Banach space X can be approximated by a norm attaining functional y^*. This result is called the Bishop-Phelps Theorem. Shortly thereafter, B. Bollobás [8] showed that this approximation can be done in such a way that, moreover, the point at which x^* almost attains its norm is close in norm to a point at which y^* attains its norm. This is a quantitative version of the Bishop-Phelps Theorem, known as the Bishop-Phelps-Bollobás Theorem.

For a real or complex Banach space X, we denote by S_X, B_X and X^* the unit sphere, the closed unit ball and the dual space of X, respectively.

Theorem 1.1 (Bishop-Phelps-Bollobás Theorem, [8, Theorem 1])**.** *Let X be a Banach space and $0<\varepsilon<1/2$. Given $x \in B_X$ and $x^* \in S_{X^*}$ with $|1-x^*(x)| < \dfrac{\varepsilon^2}{2}$, there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y)=1$, $\|y-x\| < \varepsilon+\varepsilon^2$ and $\|y^*-x^*\|<\varepsilon$.*

We refer the reader to the recent paper [10] for a more accurate version of the above theorem.

In 2008, M. D. Acosta, R. M. Aron, D. García and M. Maestre introduced the so-called Bishop-Phelps-Bollobás property for operators [1, Definition 1.1]. For our purposes, it will be useful to recall an appropriate version of this property for classes of operators defined in [2, Definition 1.3]. Given two Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the space of all (bounded

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and linear) operators from X into Y. The subspace of $\mathcal{L}(X,Y)$ of *finite-rank operators* $\mathcal{F}(X,Y)$; $\mathcal{K}(X,Y)$ will denote the subspace of all *compact operators*.

Definition 1.2. Let X and Y be Banach spaces and $\mathcal{M}$ a linear subspace of $\mathcal{L}(X,Y)$. We say that $\mathcal{M}$ satisfies the *Bishop-Phelps-Bollobás property* if given $\varepsilon>0$, there is $\eta(\varepsilon)>0$ such that whenever $T\in S_{\mathcal{M}}$ and $x_0\in S_X$ satisfy that $\|Tx_0\|>1-\eta(\varepsilon)$, then there exist a point $u_0\in S_X$ and an operator $S\in S_{\mathcal{M}}$ satisfying the following conditions:

$$\|Su_0\|=1,\quad \|u_0-x_0\|<\varepsilon,\quad \text{and}\quad \|S-T\|<\varepsilon.$$

In case that $\mathcal{M}=\mathcal{L}(X,Y)$ satisfies the previous property it is said that the pair (X,Y) has the *Bishop-Phelps-Bollobás property for operators* (shortly *BPBp for operators*).

Observe that the BPBp of a pair (X,Y) means that one is able to approximate any pair of an operator and a point at which the operator almost attains its norm by a new pair of a norm-attaining operator and a point at which this new operator attains its norm. In particular, if a pair (X,Y) has the BPBp, the set of norm-attaining operators is dense in $\mathcal{L}(X,Y)$. The reverse result is far from being true: there are Banach spaces Y such that the pair (ℓ_1^2,Y) does not have the BPBp (see [1]).

In [1] the authors provided the first version of the Bishop-Phelps-Bollobás Theorem for operators. Amongst them, a sufficient condition on a Banach space Y to get that for every Banach space X, the pair (X,Y) has the BPBp for operators, which is satisfied, for instance, by $Y=c_0$ or $Y=\ell_\infty$. A characterization of the Banach spaces Y such that the pair (ℓ_1,Y) has the BPBp for operators is also given. There are also positive results for operators from $L_1(\mu)$ into $L_\infty(\nu)$ [5, 12], for operators from $L_1(\mu)$ into $L_1(\nu)$ [12], for certain ideals of operators from $L_1(\mu)$ into another Banach space [11, 2], for operators from an Asplund space into $C_0(L)$ or into a uniform algebra [4, 9], and for operators from a uniformly convex space into an arbitrary Banach space [3, 16]. For some more recent results, see also [6]. Let us also point out that the set of norm attaining operators from $L_1[0,1]$ into $C[0,1]$ is not dense in $\mathcal{L}(L_1[0,1],C[0,1])$ [19].

Our aim in this paper is to provide classes of Banach spaces satisfying a version of the Bishop-Phelps-Bollobás Theorem for operators. The first result, which is the content of section 2, states that given arbitrary compact Hausdorff topological spaces K and S, the pair $(C(K),C(S))$ satisfies the BPBp for operators in the real case. This result extends the one by J. Johnson and J. Wolfe [14] that the set of norm attaining operators from $C(K)$ into $C(S)$ is dense in $\mathcal{L}(C(K),C(S))$. In section 3, we prove that the space $\mathcal{K}(C_0(L),Y)$ satisfies the Bishop-Phelps-Bollobás property whenever L is a locally compact Hausdorff topological space and Y is uniformly convex in both the real and the complex case. Let us remark that it was also proved in [14] that the set of norm-attaining weakly compact operators from $C(K)$ into Y is dense in the space of all weakly compact operators. But, as commented above, there are Banach spaces Y such that the pair (ℓ_∞^2,Y) does not satisfy the BPBp for operators (in the real case, $\ell_\infty^2\equiv\ell_1^2$), so some assumption on Y is needed to get the Bishop-Phelps-Bollobás property. Finally, we devote section 4 to show that the space $\mathcal{K}(X,Y)$ has the Bishop-Phelps-Bollobás property when X is an arbitrary Banach space and Y is a predual of an L_1-space in both the real and the complex case. This extends the result of [14] that the set of norm-attaining finite-rank operators from an arbitrary Banach space into a predual of an L_1-space is dense in the space of compact operators. In particular, for $Y=C_0(L)$ for some locally compact Hausdorff topological space L, the result is a consequence of the already cited paper [4].

2. Operators between spaces of continuous functions

Throughout this section, K and S are compact Hausdorff topological spaces. Here $C(K)$ is the space of real valued continuous functions on K. $M(K)$ denotes the space of regular Borel finite measures on K, which identifies with the dual of $C(K)$ by the Riesz representation theorem. For $s \in S$, we write δ_s to denote the point measure concentrated at s.

Lemma 2.1 ([13, Theorem 1, p. 490])**.** *Let X be a Banach space and let S be a compact Hausdorff topological space. Given an operator $A : X \longrightarrow C(S)$, define $\mu : S \longrightarrow X^*$ by $\mu(s) = A^*(\delta_s)$ for every $s \in S$. Then the relationship*

$$[Ax](s) = \mu(s)(x), \quad \forall x \in X,\, s \in S$$

defines an isometric isomorphism between $\mathcal{L}(X, C(S))$ and the space of w^*-continuous functions from S to X^*, endowed with the supremum norm, i.e. $\|\mu\| = \sup\{\|\mu(s)\| : s \in S\}$. Compact operators correspond to norm continuous functions.

Lemma 2.2 ([14, Lemma 2.2])**.** *Let $\mu : S \longrightarrow M(K)$ be w^*-continuous. Let $\varepsilon > 0$, $s_0 \in S$ and an open subset V of K be given. Then there exists an open neighborhood U of s_0 such that if $s \in U$, then $|\mu(s)|(V) \geqslant |\mu(s_0)|(V) - \varepsilon$.*

The next result is a version of [14, Lemma 2.3] in which the main difference is that we start with an operator and a function in the unit sphere of $C(K)$ where the operator almost attains its norm and construct a new operator and a new function, both close to the previous elements and satisfying additional restrictions. Condition iii) is the new ingredient that will be useful.

Lemma 2.3. *Let $\mu : S \longrightarrow M(K)$ be a w^*-continuous function satisfying $\|\mu\| = 1$ and $0 < \delta < 1$. Suppose that $s_0 \in S$ and $f_0 \in S_{C(K)}$ satisfy $\int_K f_0 \, d\mu(s_0) > 1 - \frac{\delta^2}{12}$. Then there exist a w^*-continuous mapping $\mu' : S \longrightarrow M(K)$, an open set U in S, an open set V of K and $h_0 \in C(K)$ satisfying the following conditions:*

- i) *$|\mu'(s)|(V) = 0$ for every $s \in U$.*
- ii) *$\int_K h_0 \, d\mu'(s) \geqslant \|\mu'\| - \delta$ for every $s \in U$.*
- iii) *$\|h_0 - f_0\| < \delta$.*
- iv) *$\|h_0\| = 1$ and $|h_0(t)| = 1 \quad \forall t \in K \setminus V$.*
- v) *$\|\mu' - \mu\| < \delta$.*

Proof. Let us write $\mu_0 := \mu(s_0)$. By the Hahn decomposition theorem, there is a partition of K into two measurable sets K^+ and K^- such that K^+ is a positive set for μ_0 and K^- is negative for μ_0. For every $0 < x < 1$, consider two open subsets of K given by

$$O_x^+ := \{t \in K \,:\, f_0(t) > x\}, \quad O_x^- := \{t \in K \,:\, f_0(t) < -x\},$$

and consider the set

$$D_x := \left(K^+ \cap O_x^+\right) \cup \left(K^- \cap O_x^-\right).$$

Write $\alpha = \frac{\delta^2}{12}$. By the assumption, we have

$$\begin{aligned} 1 - \alpha < \int_K f_0 \, d\mu_0 &\leqslant |\mu_0|(D_x) + x|\mu_0|(K \setminus D_x) \\ &\leqslant |\mu_0|(D_x) + x(1 - |\mu_0|(D_x)) = (1 - x)|\mu_0|(D_x) + x. \end{aligned}$$

Hence,

$$|\mu_0|(D_x) > 1 - \frac{\alpha}{1 - x}\,. \tag{1}$$

Next, consider the open subset W_x of K given by

$$W_x := O_x^+ \cup O_x^- = \{t \in K : |f_0(t)| > x\},$$

and observe that, since $D_x \subset W_x$, we have

$$|\mu_0|(W_x) \geqslant |\mu_0|(D_x) \geqslant 1 - \frac{\alpha}{1-x} \, . \tag{2}$$

Write $c := 1 - \frac{\delta}{4}$ and choose real numbers a and b with $1 - \delta < a < b < c < 1$. As the open subset W_a contains $\overline{O_b^+ \cup O_b^-}$, there is $u \in C(K)$ such that $0 \leqslant u \leqslant 1$, $u \equiv 1$ on $O_b^+ \cup O_b^-$ and $\operatorname{supp} u \subset W_a$. Since the support of u is contained in W_a (where f_0 is separated from 0), the function h_0 defined on K by

$$h_0(t) = \frac{f_0(t)}{|f_0(t)|} u(t) + (1 - u(t)) f_0(t) \quad \text{ if } \quad f_0(t) \neq 0, \quad h_0(t) = 0 \text{ otherwise,}$$

is continuous and, actually, $h_0 \in B_{C(K)}$. We claim that $\|h_0 - f_0\| < \delta$, which guarantees condition iii). Indeed, if $t \in K \setminus W_a$, then $u(s) = 0$ and so $h_0(t) = f_0(t)$; if otherwise $t \in W_a$, we have that

$$|h_0(t) - f_0(t)| = u(t) \left| \frac{f_0(t)}{|f_0(t)|} - f_0(t) \right| \leqslant 1 - |f_0(t)| < 1 - a < \delta,$$

proving the claim. On the other hand, we know that $u \equiv 1$ in $O_b^+ \cup O_b^-$, so $|h_0| = 1$ on $\overline{O_b^+ \cup O_b^-}$. Therefore, if we write $V := K \setminus \left[\overline{O_b^+ \cup O_b^-} \right]$, which is an open subset of K, the second part of condition iv) is satisfied.

Next, as the open subsets V and W_c satisfy $\overline{V} \cap \overline{W_c} = \varnothing$, there is a function $f \in C(K)$ such that $0 \leqslant f \leqslant 1$, $f \equiv 1$ on V and $\operatorname{supp} f \subset K \setminus W_c$. Since $D_c \subset W_c$, we have that

$$\begin{aligned} \int_{W_c} h_0 \, d\mu_0 &= \int_{D_c} h_0 \, d\mu_0 + \int_{W_c \setminus D_c} h_0 \, d\mu_0 = |\mu_0|(D_c) + \int_{W_c \setminus D_c} h_0 \, d\mu_0 \\ &\geqslant |\mu_0|(D_c) - |\mu_0|(W_c \setminus D_c) \geqslant |\mu_0|(D_c) - |\mu_0|(K \setminus D_c) \geqslant 2|\mu_0|(D_c) - |\mu_0|(K). \end{aligned}$$

Therefore, by using (1), we obtain that

$$\int_{W_c} h_0 \, d\mu_0 > 1 - 2\frac{\alpha}{1-c} = 1 - \frac{2}{3}\delta. \tag{3}$$

As a consequence,

$$\begin{aligned} \int_K h_0(1-f) \, d\mu_0 &= \int_{W_c} h_0(1-f) \, d\mu_0 + \int_{K \setminus W_c} h_0(1-f) \, d\mu_0 \\ &\geqslant \int_{W_c} h_0(1-f) \, d\mu_0 - |\mu_0|(K \setminus W_c) \\ &> 1 - 2\frac{\alpha}{1-c} - \big(1 - |\mu_0|(W_c)\big) \qquad \text{(by (3))} \\ &> 1 - 3\frac{\alpha}{1-c} = 1 - \delta \qquad \text{(by (2))} \end{aligned}$$

Now, in view of the w^*-continuity of μ, the previous inequality, condition (2) and Lemma 2.2, we get that there exists an open neighborhood U_0 of s_0 such that

$$\int_K h_0(1-f)d\mu(s) > 1-\delta \qquad \text{and} \qquad |\mu(s)|(W_c) > 1-\delta, \quad \forall s \in U_0. \tag{4}$$

We can also choose an open subset U of S such that $s_0 \in U$ and $\overline{U} \subset U_0$, and a function $g \in C(S)$ such that $0 \leqslant g \leqslant 1$, $g(U) = \{1\}$ and $\operatorname{supp} g \subset U_0$. Define $\mu' : S \longrightarrow M(K)$ by

$$\mu'(s) = \big(1 - g(s)f\big)\mu(s) \qquad (s \in S).$$

That $\mu'(s)$ is the unique Borel measure on K satisfying

$$\int_K \varphi \; d\mu'(s) = \int_K \big(1 - g(s)f\big)\varphi \; d\mu(s) \qquad \forall \varphi \in C(K).$$

It is clear that μ' is w^*-continuous. If $s \in U$, $g(s) = 1$ and $f(V) = \{1\}$, so condition i) is satisfied. Since $0 \leqslant f, g \leqslant 1$, then $\|\mu'\| \leqslant \|\mu\| = 1$ and hence, in view of (4), for every $s \in U$ we have that

$$\int_K h_0 \; d\mu'(s) = \int_K \big(1 - g(s)f\big)h_0 \; d\mu(s) > 1 - \delta \geqslant \|\mu'\| - \delta,$$

so condition ii) is also satisfied.

We only have to check condition v), that is, $\|\mu' - \mu\| < \delta$. Indeed, if $s \in S \setminus U_0$, then $g(s) = 0$ and so $\mu'(s) = \mu(s)$; if, otherwise, $s \in U_0$, by (4) and the fact that $f(W_c) = \{0\}$, we obtain that

$$\Big|\int_K \varphi \; d(\mu(s) - \mu'(s))\Big| = \Big|\int_K \varphi g(s) f \; d\mu(s)\Big| \leqslant |\mu(s)|(K \setminus W_c) \leqslant \|\mu\| - |\mu(s)|(W_c) < \delta$$

for every $\varphi \in B_{C(K)}$, proving the claim.

Finally, since $\|\mu\| = 1$, condition v) implies that $\mu' \neq 0$ and, in view of i), we deduce that $K \neq V$, so $K \setminus V$ is not empty and $\|h_0\| = 1$ since $|h_0(t)| = 1$ for every $t \in K \setminus V$. □

The last ingredient that we will use is the next iteration result due to Johnson and Wolfe.

Lemma 2.4 ([14, Lemma 2.4])**.** *Let $\mu : S \longrightarrow M(K)$ be w^*-continuous and $\delta > 0$. Suppose there is an open set $U \subset S$, an open set $V \subset K$, $s_0 \in U$ and $h_0 \in C(K)$ with $\|h_0\| = 1$ such that*

- *a) if $s \in U$, then $|\mu(s)|(V) = 0$,*
- *b) $\int_K h_0 \, d\mu(s_0) \geqslant \|\mu\| - \delta$,*
- *c) $|h_0(t)| = 1$ for $t \in K \setminus V$.*

Then, for any $\frac{2}{3} < r < 1$ there exist a w^*-continuous function $\mu' : S \longrightarrow M(K)$ and a point $s_1 \in U$ such that

- *a′) if $s \in U$, then $|\mu'(s)|(V) = 0$,*
- *b′) $\int_K h_0 \, d\mu'(s_1) \geqslant \|\mu'\| - r\delta$,*
- *c′) $\|\mu' - \mu\| \leqslant r\delta$.*

The next result improves [14, Theorem 1]. In the complex case, it is not known whether the same result is true or not.

Theorem 2.5. *Let K and S be compact Hausdorff topological spaces. Then the pair $(C(K), C(S))$ has the Bishop-Phelps-Bollobás property for operators in the real case. Moreover, the function η satisfying Definition 1.2 does not depend on the spaces K and S (in fact one can take $\eta(\varepsilon) = \frac{\varepsilon^2}{12 \cdot 6^2}$).*

Proof. Let us fix $\frac{2}{3} < r < 1$. Given $0 < \varepsilon < 2$ let us choose $0 < \delta < \varepsilon\frac{1-r}{2}$. Assume that $T_0 \in S_{\mathcal{L}(C(K),C(S))}$ and $f_0 \in S_{C(K)}$ satisfy that $\|T_0(f_0)\| > 1 - \frac{\delta^2}{12}$. Then, there is an element $s_1 \in S$ such that $\left|\left[T_0(f_0)\right](s_1)\right| > 1 - \frac{\delta^2}{12}$. By using $-f_0$ instead of f_0, if necessary, we may assume that $T_0(f_0)(s_1) > 1 - \frac{\delta^2}{12}$. Therefore, we can apply Lemma 2.3 to the w^*-continuous function $\mu_0 : S \longrightarrow M(K)$ associated with the operator T_0 (i.e. $\mu_0(s) = T_0^*(\delta_s)$ for every $s \in S$) to get that there exist a function $h_0 \in S_{C(K)}$, an open set U in S, an open set V of K and a w^*-continuous function $\mu_1 : S \longrightarrow M(K)$ satisfying the following conditions:

i) $|\mu_1(s)|(V) = 0$ for every $s \in U$.
ii) $\int_K h_0\, d\mu_1(s) \geqslant \|\mu_1\| - \delta$ for every $s \in U$.
iii) $\|h_0 - f_0\| < \delta$.
iv) $\|h_0\| = 1$ and $|h_0(t)| = 1 \ \ \forall t \in K \setminus V$.
v) $\|\mu_1 - \mu_0\| < \delta$.

Now, by using Lemma 2.4, we inductively construct a sequence $\{\mu_n\}$ of w^*-continuous functions from S into $M(K)$ and a sequence $\{s_n\}$ in U satisfying

$$\|\mu_{n+1} - \mu_n\| \leqslant r^n\delta,\ \|\mu_n\| \leqslant \int_K h_0\, d\mu_n(s_n) + r^n\delta \text{ and } |\mu_n(s)|(V) = 0$$

for every $s \in U$ and $n \in \mathbb{N}$.

If for every $n \in \mathbb{N}$, we write $T_n \in \mathcal{L}(C(K), C(S))$ to denote the bounded linear operator associated with the function μ_n, we may rewrite i) and ii) as

$$\|T_{n+1} - T_n\| \leqslant r^n\delta \qquad \text{and} \qquad \|T_n\| \leqslant \|T_n(h_0)\| + r^n\delta. \tag{5}$$

Since $0 < r < 1$, the previous condition implies that $\{T_n\}$ is a Cauchy sequence, so it converges to an operator $T \in \mathcal{L}(C(K), C(S))$ satisfying

$$\|T - T_0\| \leqslant \sum_{k=0}^{\infty} \|T_{k+1} - T_k\| \leqslant \sum_{k=0}^{\infty} r^k\delta = \delta\frac{1}{1-r} < \frac{\varepsilon}{2}.$$

By taking limit in the right-hand side of (5), we also have that

$$\|T\| \leqslant \|T(h_0)\|$$

and, since $h_0 \in S_{C(K)}$, T attains its norm at h_0.

Finally, we have that

$$|1 - \|T\|| \leqslant \big|\|T_0\| - \|T\|\big| \leqslant \|T_0 - T\| < \frac{\varepsilon}{2} < 1,$$

so $T \neq 0$, $\frac{T}{\|T\|}$ also attains its norm at h_0 and

$$\left\|\frac{T}{\|T\|} - T_0\right\| \leqslant \left\|\frac{T}{\|T\|} - T\right\| + \|T - T_0\| = \big|1 - \|T\|\big| + \|T - T_0\| < \varepsilon.$$

As we already knew that $\|h_0 - f_0\| < \delta < \varepsilon$, this shows that the pair $(C(K), C(S))$ satisfies the Bishop-Phelps-Bollobás Theorem for operators with $\eta = \frac{\delta^2}{12}$. ☐

3. Compact operators from a space of continuous functions into a uniformly convex space

Our purpose now is to prove the Bishop-Phelps-Bollobás property for compact operators. The following result due to Kim will play an essential role:

Lemma 3.1 ([15, Theorem 2.5])**.** *Let Y be a uniformly convex space. For every $0<\varepsilon<1$, there is $0<\gamma(\varepsilon)<1$ with the following property:*
given $n\in\mathbb{N}$, $T\in S_{\mathcal{L}(\ell_\infty^n,Y)}$ and $x_0\in S_{\ell_\infty^n}$ such that $\|Tx_0\|>1-\gamma(\varepsilon)$, there exist $S\in S_{\mathcal{L}(\ell_\infty^n,Y)}$ and $x_1\in S_{\ell_\infty^n}$ satisfying

$$\|Sx_1\|=1,\quad \|S-T\|<\varepsilon \quad \textit{and} \quad \|x_1-x_0\|<\varepsilon\ .$$

It is easy to show that the function γ in the previous result satisfies $\lim_{t\to 0+}\gamma(t)=0$.

Let L be a locally compact Hausdorff topological space. As usual, $C_0(L)$ will be the space either of real or complex continuous functions on L with limit zero at infinity. We recall that $C_0(L)^*$ can be identified with the space $M(L)$ of regular Borel measures on L by the Riesz representation theorem.

For every $f\in C_0(L)$ and every (non-empty) set $S\subset L$, we define $\mathrm{Osc}(f,S)$ by

$$\mathrm{Osc}(f,S)=\sup_{x,y\in S}|f(x)-f(y)|$$

The next result generalizes Lemma 3.1 and Proposition 3.2 of [14] to $C_0(L)$. Its proof is actually based on the proof of these results.

Proposition 3.2. *Let L be a locally compact Hausdorff topological space and let Y be a Banach space. For every $\varepsilon>0$, $T\in\mathcal{K}(C_0(L),Y)$ and $f_0\in C_0(L)$, there exist a positive regular Borel measure μ, a non-negative integer m, pairwise disjoint compact subsets K_j of L and $\varphi_j\in C_0(L)$ for $1\leqslant j\leqslant m$, satisfying the following conditions:*

1. $\mathrm{Osc}(f_0,K_j)<\varepsilon$.
2. $0\leqslant\varphi_j\leqslant 1$ *and* $\varphi_j\equiv 1$ *on* K_j.
3. $\operatorname{supp}\varphi_i\cap\operatorname{supp}\varphi_j=\emptyset$ *for* $i\neq j$.
4. *The operator $P: C_0(L)\longrightarrow C_0(L)$ given by*

$$P(f):=\sum_{j=1}^m\frac{1}{\mu(K_j)}\Big(\int_{K_j}f\ d\mu\Big)\varphi_j,\quad \forall f\in C_0(L),$$

is a norm-one projection from $C_0(L)$ onto the linear span of $\{\varphi_1,\ldots,\varphi_m\}$ that also satisfies $\|T-TP\|<\varepsilon$.

Proof. Since T is a compact operator, the adjoint operator T^* is a compact operator from Y^* into $C_0(L)^*$, so we may take a finite $\frac{\varepsilon}{4}$-net $\{\mu_1,\ldots,\mu_t\}$ of $T^*(B_{Y^*})\subset C_0(L)^*\equiv M(L)$. We define the (finite regular) measure μ by $\mu=\sum_{i=1}^t|\mu_i|$. For each $1\leqslant i\leqslant t$, we have that $\mu_i\ll\mu$, hence the Radon-Nikodým theorem allows us to find a function $g_i\in L_1(\mu)$ such that $\mu_i=g_i\mu$. Since the set of simple functions is dense in $L_1(\mu)$, we may choose a set of simple functions $\{s_i: i=1,\ldots,t\}$ such that $\|g_i-s_i\|_1<\frac{\varepsilon}{12}$ for every $1\leqslant i\leqslant t$. Next, consider a finite family $(A_j)_{j=1}^{m_0}$ of pairwise disjoint measurable sets such that for every $1\leqslant i\leqslant t$, there is a family $(\alpha_j^i)_{j=1}^{m_0}$ of scalars such that $s_i=\sum\limits_{j=1}^{m_0}\alpha_j^i\chi_{A_j}$. Let M be a positive real number satisfying $M\geqslant\max\big\{|\alpha_j^i|: 1\leqslant i\leqslant t, 1\leqslant j\leqslant m_0\big\}$. Since μ is regular, for each $1\leqslant j\leqslant m_0$ we find a compact set $C_j\subset A_j$ such that $\mu(A_j\setminus C_j)<\frac{\varepsilon}{12m_0M}$. As f_0 is continuous and each C_j is compact, we may divide each C_j into a family of Borel sets $(B_j^p)_{p=1}^{n_j}$ such that

$$\mathrm{Osc}(f_0,B_j^p)<\varepsilon\qquad\forall 1\leqslant j\leqslant m_0, 1\leqslant p\leqslant n_j.$$

Applying the regularity again, for each j and p, there is a compact set $K_j^p \subset B_j^p$ such that $\mu(B_j^p \setminus K_j^p) < \frac{\varepsilon}{12 m_0 n_j M}$. Finally, choose suitable $m \in \mathbb{N}$, a rearrangement $(K_j)_{j=1}^m$ of the family $\{K_j^p : 1 \leqslant j \leqslant m_0,\ 1 \leqslant p \leqslant n_j,\ \mu(K_j^p) > 0\}$ and scalars (β_j^i) for $j \leqslant m$ and $i \leqslant t$ such that

$$\sum_{j=1}^{m} \beta_j^i \chi_{K_j} = \sum_{j=1}^{m_0} \alpha_j^i \left(\sum_{p=1}^{n_j} \chi_{K_j^p} \right).$$

Using Urysohn's lemma, we may choose a family $(\varphi_j)_{j=1}^m$ in $C_0(L)$ satisfying that $0 \leqslant \varphi_j \leqslant 1$, $\varphi_j \equiv 1$ on K_j for each $j \leqslant m$ and $\operatorname{supp} \varphi_i \cap \operatorname{supp} \varphi_j = \emptyset$ for every $i \neq j$.

To finish the proof, we only have to check (4). Indeed, for $i = 1, \ldots, t$, we write $\nu_i = \sum_{j=1}^m \beta_j^i \chi_{K_j} \mu \in M(L) = C_0(L)^*$. By defining the operator P as in condition (4), it is easy to check that P is a norm one projection onto the linear span of $\{\varphi_1, \ldots, \varphi_m\}$ and $P^* \nu_i = \nu_i$ for each $1 \leqslant i \leqslant t$. Therefore, for every $1 \leqslant i \leqslant t$ we have that

$$\begin{aligned}
\|\mu_i - P^*\nu_i\| &= \|g_i\mu - \nu_i\| \leqslant \|g_i\mu - s_i\mu\| + \|s_i\mu - \nu_i\| \\
&\leqslant \|g_i - s_i\|_1 + \left\| s_i\mu - \sum_{j=1}^{m_0} \alpha_j^i \chi_{C_j}\mu \right\| + \left\| \sum_{j=1}^{m_0} \alpha_j^i \chi_{C_j}\mu - \sum_{j=1}^{m} \beta_j^i \chi_{K_j}\mu \right\| \\
&< \frac{\varepsilon}{12} + \sum_{j=1}^{m_0} |\alpha_j^i| \mu(A_j \setminus C_j) + \left\| \sum_{j=1}^{m_0} \left(\alpha_j^i \sum_{p=1}^{n_j} \chi_{B_j^p} \right)\mu - \sum_{j=1}^{m_0} \left(\alpha_j^i \sum_{p=1}^{n_j} \chi_{K_j^p} \right)\mu \right\| \\
&< \frac{\varepsilon}{12} + \frac{\varepsilon}{12} + \sum_{j=1}^{m_0} |\alpha_j^i| \sum_{p=1}^{n_j} \mu(B_j^p \setminus K_j^p) \\
&< \frac{\varepsilon}{12} + \frac{\varepsilon}{12} + \frac{\varepsilon}{12} = \frac{\varepsilon}{4}.
\end{aligned}$$

Since $\{\mu_1, \ldots, \mu_t\}$ is a $\frac{\varepsilon}{4}$-net of $T^*(B_{Y^*})$, the above inequality shows that $\{\nu_1, \ldots, \nu_t\}$ is a $\frac{\varepsilon}{2}$-net of $T^*(B_{Y^*})$. Now, given $y^* \in B_{Y^*}$, we can choose $i \leqslant t$ satisfying $\|\nu_i - T^*y^*\| < \frac{\varepsilon}{2}$ and observe that

$$\begin{aligned}
\|T^*y^* - P^*T^*y^*\| &\leqslant \|T^*y^* - \nu_i\| + \|\nu_i - P^*T^*y^*\| \\
&= \|T^*y^* - \nu_i\| + \|P^*\nu_i - P^*T^*y^*\| \leqslant 2\|T^*y^* - \nu_i\| < \varepsilon.
\end{aligned}$$

Hence, we have $\|T - TP\| = \|T^* - P^*T^*\| < \varepsilon$, as desired. $\square$

The following result shows that $\mathcal{K}(C_0(L), Y)$ satisfies the Bishop-Phelps-Bollobás property for every locally compact Hausdorff topological space L and every uniformly convex space Y, and that the function $\eta(\varepsilon)$ involved in the definition of the property does not depend on L.

Theorem 3.3. *Let Y be a uniformly convex Banach space. For every $0 < \varepsilon < 1$ there is $0 < \eta(\varepsilon) < 1$ such that for any locally compact Hausdorff topological space L, if $T \in S_{\mathcal{K}(C_0(L),Y)}$ and $f_0 \in S_{C_0(L)}$ satisfy $\|Tf_0\| > 1 - \eta(\varepsilon)$, there exist $S \in S_{\mathcal{K}(C_0(L),Y)}$ and $g_0 \in S_{C_0(L)}$ such that*

$$\|Sg_0\| = 1, \quad \|S - T\| < \varepsilon \quad \text{and} \quad \|g_0 - f_0\| < \varepsilon.$$

Proof. Given $0 < \varepsilon < 1$, we choose $0 < \delta < \frac{\varepsilon}{4}$ such that $0 < \gamma(\delta) < \frac{\varepsilon}{4}$, where $\gamma(\delta)$ satisfies the statement of Lemma 3.1. We also consider α such that $0 < \alpha < \min\{\delta, \frac{\gamma(\delta)}{2}\}$ and $\eta(\varepsilon) := \alpha > 0$.

Fix $0 < \varepsilon < 1$, $T \in S_{\mathcal{K}(C_0(L),Y)}$ and $f_0 \in S_{C_0(L)}$ with $\|Tf_0\| > 1 - \eta(\varepsilon) = 1 - \alpha$. Applying Proposition 3.2, we get a positive regular Borel measure μ on L, a non-negative integer m, pairwise disjoint compact subsets K_j of L and $\varphi_j \in C_0(L)$ $(1 \leqslant j \leqslant m)$ such that

(1) $\operatorname{Osc}(f_0, K_j) < \alpha$,
(2) For every $1 \leqslant j \leqslant m$, $0 \leqslant \varphi_j \leqslant 1$ and $\varphi_j \equiv 1$ on K_j,
(3) $\operatorname{supp} \varphi_i \cap \operatorname{supp} \varphi_j = \emptyset$ for $i \neq j$,
(4) $\|T - TP\| < \alpha$,

where $P \in \mathcal{L}(C_0(L))$ is given by

$$P(f) := \sum_{j=1}^{m} \frac{1}{\mu(K_j)} \Big(\int_{K_j} f \, d\mu \Big) \varphi_j \qquad \big(f \in C_0(L)\big),$$

and it is a norm-one projection onto the linear span of $\{\varphi_1, \ldots, \varphi_m\}$.

Now, if $t \in K_j$ for some $j = 1, \ldots, m$, we obtain that

$$\begin{aligned} \big|[P(f_0)](t) - f_0(t)\big| &= \left| \frac{1}{\mu(K_j)} \int_{K_j} \big(f_0(s) - f_0(t)\big) \, d\mu(s) \right| \\ &\leqslant \frac{1}{\mu(K_j)} \int_{K_j} \big|f_0(s) - f_0(t)\big| \, d\mu(s) \leqslant \alpha. \end{aligned}$$

Hence

$$\max \Big\{ \big|[P(f_0) - f_0](t)\big| \, : \, t \in \bigcup\nolimits_{j=1}^{m} K_j \Big\} \leqslant \alpha. \tag{6}$$

We also have that

$$\|TP(f_0)\| \geqslant \|T(f_0)\| - \|T - TP\| > 1 - 2\alpha > 1 - \gamma(\delta), \tag{7}$$

and this implies that

$$1 - 2\alpha \leqslant \|TP\| \leqslant 1 \quad \text{and} \quad 1 - 2\alpha \leqslant \|P(f_0)\|.$$

Since the functions $\{\varphi_j \, : \, 1 \leqslant j \leqslant m\}$ have pairwise disjoint support, the linear operator $\Phi : \operatorname{lin}\{\varphi_1, \ldots, \varphi_m\} \longrightarrow \ell_\infty^m$ satisfying $\Phi(\varphi_j) = e_j$ for every $j = 1, \ldots, m$ is an onto linear isometry (where $\{e_1, \ldots, e_m\}$ is the natural basis of ℓ_∞^m). We define $U_1 := T \circ \Phi^{-1} : \ell_\infty^m \longrightarrow Y$ and observe that, clearly, $\|U_1\| \leqslant 1$. On the other hand, the element $x^0 := \Phi(P(f_0)) \in B_{\ell_\infty^m}$ satisfies that $U_1(x^0) = TP(f_0)$ so, in view of (7),

$$\big\|U_1(x^0)\big\| > 1 - \gamma(\delta) \quad \text{and so} \quad \|U_1\|, \|x^0\| > 1 - \gamma(\delta) > 0.$$

We consider $U = \frac{U_1}{\|U_1\|}$ and apply Lemma 3.1 to the pair $\Big(U, \frac{x^0}{\|x^0\|}\Big)$, to get an operator $V : \ell_\infty^m \longrightarrow Y$ with $\|V\| = 1$ and $x^1 \in S_{\ell_\infty^m}$ with

$$\|V - U\| < \delta, \quad \left\| x^1 - \frac{x^0}{\|x^0\|} \right\| < \delta, \quad \text{and} \quad \|V\big(x^1\big)\| = 1\ . \tag{8}$$

We clearly have that

$$\|U - U_1\| = \left\| \frac{U_1}{\|U_1\|} - U_1 \right\| = \big|1 - \|U_1\|\big| \leqslant \gamma(\delta) \tag{9}$$

and, also,

$$\|x^1 - x^0\| \leqslant \left\| x^1 - \frac{x^0}{\|x^0\|} \right\| + \left\| \frac{x^0}{\|x^0\|} - x^0 \right\| \leqslant \delta + 1 - \|x^0\| \leqslant \delta + \gamma(\delta). \tag{10}$$

As a consequence,

$$\|V - U_1\| \leqslant \|V - U\| + \|U - U_1\| \leqslant \delta + \gamma(\delta).$$

Finally, we define the operator $S : C_0(L) \longrightarrow Y$ given by $S(f) := V(\Phi(P(f)))$ for every $f \in C_0(L)$, which is clearly a compact operator and satisfies $\|S\| \leqslant 1$. Consider the element $f_1 = \sum_{j=1}^m x^1(j)\varphi_j \in B_{C_0(L)}$. It is clear that $P(f_1) = f_1$, $\Phi(f_1) = x^1$ and that

$$\|S\| \geqslant \|S(f_1)\| = \|V\Phi P(f_1)\| = \left\|V\left(x^1\right)\right\| = 1 \ .$$

We deduce that $f_1 \in S_{C_0(L)}$ and that $\|S\| = \|S(f_1)\| = 1$. This implies that S attains its norm at f_1.

Next, we estimate the distance between S and T as follows

$$\begin{aligned}
\|S - T\| &= \|V\Phi P - T\| \leqslant \|V\Phi P - TP\| + \|TP - T\| \\
&\leqslant \|V\Phi P - U\Phi P\| + \|U\Phi P - TP\| + \alpha \\
&\leqslant \|V - U\| + \|U\Phi P - T\Phi^{-1}\Phi P\| + \alpha \\
&< \delta + \|U - T\Phi^{-1}\| + \alpha \qquad \text{(by (8))} \\
&\leqslant 2\delta + \|U - U_1\| \leqslant 2\delta + \gamma(\delta) < \varepsilon \qquad \text{(by (9))}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\max\left\{\left|[f_1 - f_0](t)\right| \,:\, t \in \bigcup\nolimits_{j=1}^m K_j\right\} &= \max_{1\leqslant j\leqslant m}\max_{t\in K_j}\left|x^1(j) - f_0(t)\right| \\
&\leqslant \max_{1\leqslant j\leqslant m}\left\{\left|x^1(j) - x^0(j)\right| + \max_{t\in K_j}\left|x^0(j) - f_0(t)\right|\right\} \\
&\leqslant \|x^1 - x^0\| \ + \max_{1\leqslant j\leqslant m}\max_{t\in K_j}\left|[P(f_0) - f_0](t)\right| \\
&\leqslant \delta + \gamma(\delta) + \alpha < 2\delta + \gamma(\delta) \qquad \text{(by (10) and (6))}.
\end{aligned}$$

Hence, there exists an open set $O \subset L$ such that

$$\bigcup\nolimits_{j=1}^m K_j \subset O, \quad \left|[f_1 - f_0](t)\right| < 3\delta + \gamma(\delta) \qquad \big(t \in O\big). \tag{11}$$

By Urysohn's Lemma again, there is $\psi \in C_0(L)$ such that $0 \leqslant \psi \leqslant 1$, $\psi \equiv 1$ on $\bigcup_{k=1}^m K_j$ and $\operatorname{supp}\psi \subset O$. We write $g_0 := \psi f_1 + (1-\psi)f_0 \in B_{C_0(L)}$ and we claim that S attains its norm at g_0 and that $\|f - g_0\| < \varepsilon$, which finishes the proof. Indeed, on the one hand, it is clear that the restriction of g_0 to $\bigcup_{k=1}^m K_j$ coincides with f_1. It follows that $P(g_0) = P(f_1)$ and so $S(g_0) = S(f_1)$ and S attains its norm at g_0. On the other hand, for $t \in L \setminus O$ we have $g(t) = f_0(t)$. If, otherwise, $t \in O$, condition (11) gives that

$$\left|g_0(t) - f_0(t)\right| = \left|\psi(t)\big(f_1(t) - f_0(t)\big)\right| < 3\delta + \gamma(\delta) < \varepsilon. \qquad \square$$

4. Compact operators into a predual of an $L_1(\mu)$-space

Our goal is to show that the space of compact operators from an arbitrary Banach space into an isometric predual of an L_1-space has the Bishop-Phelps-Bollobás property in both the real and the complex case.

We need a preliminary result which follows easily from the Bishop-Phelps-Bollobás theorem. It is also a very particular case of [1, Theorem 2.2].

Lemma 4.1. *For every $0 < \varepsilon < 1$, there is $0 < \eta'(\varepsilon) < 1$ such that for every positive integer n and every Banach space X, the pair (X, ℓ_∞^n) has the BPBp for operators with this function $\eta'(\varepsilon)$. More concretely, given an operator $U \in S_{\mathcal{L}(X,\ell_\infty^n)}$ and an element $x_0 \in S_X$ such that $\|U(x_0)\| > 1 - \eta'(\varepsilon)$, there exist $V \in S_{\mathcal{L}(X,\ell_n^\infty)}$ and $z_0 \in S_X$ satisfying*

$$\|Vz_0\| = 1, \qquad \|z_0 - x_0\| < \varepsilon \quad \textit{and} \quad \|V - U\| < \varepsilon.$$

Theorem 4.2. *For every $0 < \varepsilon < 1$ there is $\eta(\varepsilon) > 0$ such that if X is any Banach space, Y is a predual of an L_1-space, $T \in S_{\mathcal{K}(X,Y)}$ and $x_0 \in S_X$ satisfy $\|Tx_0\| > 1 - \eta(\varepsilon)$, then there exist $S \in S_{\mathcal{F}(X,Y)}$ and $z_0 \in S_X$ with*

$$\|Sz_0\| = 1, \qquad \|z_0 - x_0\| < \varepsilon \quad \textit{and} \quad \|S - T\| < \varepsilon.$$

Proof. For any $0 < \varepsilon < 1$ we take $\eta(\varepsilon) = \min\{\frac{\varepsilon}{4}, \eta'(\varepsilon/2)\}$, where η' is the function provided by the previous lemma.

Fix $0 < \varepsilon < 1$, $T \in S_{\mathcal{K}(X,Y)}$ and $x_0 \in S_X$ satisfying $\|Tx_0\| > 1-\eta(\varepsilon)$. Let us choose a positive number δ with $\delta < \frac{1}{4}\min\{\frac{\varepsilon}{4}, \|T(x_0)\| - 1 + \eta'(\frac{\varepsilon}{2})\}$ and let $\{y_1, \ldots, y_n\}$ be a δ-net of $T(B_X)$. In view of [17, Theorem 3.1] and [18, Theorem 1.3], there is a subspace $E \subset Y$ isometric to ℓ_∞^m for some natural number m and such that dist $(y_i, E) < \delta$ for every $i \leqslant n$. Let $P : Y \longrightarrow Y$ be a norm one projection onto E. We will check that $\|PT - T\| < 4\delta$. In order to show that we fix any element $x \in B_X$ and so $\|Tx - y_i\| < \delta$ for some $i \leqslant n$. Let $e \in E$ be any element satisfying $\|e - y_i\| < \delta$. Then we have

$$\begin{aligned}\|T(x) - PT(x)\| &\leqslant \|T(x) - y_i\| + \|y_i - e\| + \|e - PT(x)\| \\ &\leqslant 2\delta + \|P(e) - PT(x)\| \leqslant 2\delta + \|e - T(x)\| \\ &\leqslant 2\delta + \|e - y_i\| + \|y_i - T(x)\| < 4\delta.\end{aligned}$$

So $\|PT\| > \|T\| - 4\delta = 1 - 4\delta > 0$. As a consequence we also obtain that

$$\|PT(x_0)\| > \|T(x_0)\| - 4\delta > 1 - \eta'\left(\frac{\varepsilon}{2}\right).$$

Hence the operator $R = \frac{PT}{\|PT\|}$ satisfies $\|R(x_0)\| > 1 - \eta'(\frac{\varepsilon}{2})$. Since E is isometric to ℓ_∞^m, by Lemma 4.1 there exist an operator $S \in \mathcal{L}(X, E) \subset \mathcal{L}(X, Y)$ with $\|S\| = 1$ and $z_0 \in S_X$ satisfying that

$$\|S - R\| < \frac{\varepsilon}{2}, \qquad \|z_0 - x_0\| < \frac{\varepsilon}{2}, \quad \text{and} \quad \|Sz_0\| = 1.$$

Finally, we have that

$$\begin{aligned}\|S - T\| &\leqslant \|S - R\| + \|R - PT\| + \|PT - T\| \\ &< \frac{\varepsilon}{2} + 1 - \|PT\| + 4\delta \\ &< \frac{\varepsilon}{2} + 8\delta < \varepsilon.\end{aligned}$$

□

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