LUSHNESS, NUMERICAL INDEX 1 AND THE DAUGAVET PROPERTY IN REARRANGEMENT INVARIANT SPACES

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Abstract. We show that for spaces with 1-unconditional bases lushness, the alternative Daugavet property and numerical index 1 are equivalent. In the class of rearrangement invariant (r.i.) sequence spaces the only examples of spaces with these properties are c_0, ℓ_1 and ℓ_∞. The only lush r.i. separable function space on $[0,1]$ is $L_1[0,1]$; the same space is the only r.i. separable function space on $[0,1]$ with the Daugavet property over the reals.

1. Introduction

The Daugavet property of a Banach space X can be defined by requiring that $\|\mathrm{Id}+T\| = 1+\|T\|$ for all compact operators $T\colon X \to X$. (See Section 2 for a more detailed discussion.) This is an isometric property of the particular norm of X and it is not invariant under equivalent norms. In the setting of the classical function spaces this property seems to be closely linked to the sup-norm or the L_1-norm since for example $C[0,1]$, $L_1[0,1]$, the disc algebra and H^∞ (in their natural norms) have the Daugavet property whereas $L_p[0,1]$ fails it for $1 < p < \infty$. Nevertheless, there are very different examples of spaces with the Daugavet property, for instance, the space of Lipschitz functions on a metric space, cf. [9], which is in general not even an $\mathcal{L}_\infty$-space; or some more exotic spaces such as Talagrand's space [15, 21] and Bourgain-Rosenthal's space [5, 16]. One of the main results in the present paper (Corollary 4.9) is that in the class of real separable rearrangement invariant Köthe function spaces on a finite measure space there is, however, isometrically only one space with the Daugavet property, namely $L_1[0,1]$. (A somewhat weaker statement was previously proved in [1].)

We also study relatives of the Daugavet property like the alternative Daugavet property, lushness and having numerical index 1. These properties will be recalled in the next section. In Section 3 we prove, building on results from [3], that these three properties are equivalent for spaces with a 1-unconditional basis and that they characterise c_0, ℓ_1 or ℓ_∞ among the symmetric sequence spaces.

Let us briefly indicate the structure of the paper. Section 2 contains pertinent definitions and background material. In Section 3 we study symmetric sequence

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spaces and prove the results just mentioned. Finally Section 4 deals with lushness, the Daugavet property and the almost Daugavet property for rearrangement invariant Köthe function spaces.

We finish this introduction with some notation. We write $\mathbb{T}$ to denote the set of (real or complex) scalars of modulus one. By $\mathrm{Re}(\cdot)$ we denote the real part if we are in the complex case and just the identity if we are in the real case. Given a Banach space X and a subset $A \subset X$, we write $\overline{\mathrm{conv}}(A)$ to denote the closed convex hull of A and $\overline{\mathrm{aconv}}(A)$ for the closed absolutely convex hull of A (i.e., $\overline{\mathrm{aconv}}(A) = \overline{\mathrm{conv}}(\mathbb{T}\, A)$). A *slice* of a convex subset $B \subset X$ is a non-empty set which is formed by the intersection of B with an open real half-space. Every slice of B has the form

$$S(B, x^*, \alpha) := \{x \in B \colon\ \mathrm{Re}\, x^*(x) > \sup \mathrm{Re}\, x^*(B) - \alpha\}$$

for suitable $x^* \in X^*$ and $\alpha > 0$. Further, B_X stands for the closed unit ball of X and S_X for the unit sphere.

2. Basic definitions

In this section we recall some basic definitions and facts about real or complex rearrangement invariant spaces and the properties we are going to investigate. For background on rearrangement invariant spaces (and on Köthe spaces in general) we refer the reader to the classical book by J. Lindenstrauss and L. Tzafriri [17] for the real case, and to [20] for the complex case. In the sequel we follow the notation of [17]. Let (Ω, Σ, μ) be a complete σ-finite measure space. A real or complex Banach space X consisting of equivalence classes, modulo equality almost everywhere, of locally integrable scalar valued functions on Ω is a *Köthe function space* if the following conditions hold.

(1) X is *solid*, i.e., if $|f| \leqslant |g|$ a.e. on Ω with f measurable and $g \in X$, then $f \in X$ and $\|f\| \leqslant \|g\|$.
(2) For every $A \in \Sigma$ with $\mu(A) < \infty$ the characteristic function $\mathbf{1}_A$ of A belongs to X.

Let us comment that the definition of a Köthe space is usually given in the real case (this is the case of [17]), but it extends to the complex case in an obvious way. Most of the basic properties we are going to use are known in the real case but their proofs extend without many problems to the complex case.

If X is a Köthe function space, then every measurable function g on Ω so that $gf \in L_1(\mu)$ for every $f \in X$ defines an element x_g^* in X^* by $x_g^*(f) = \int_\Omega fg\, d\mu$. Any functional on X of the form x_g^* is called an *integral* and the linear space of all integrals is denoted by X'. In the norm induced on X' by X^*, this space is also a Köthe function space on (Ω, Σ, μ). The space X is *order continuous* if whenever $\{f_n\}$ is a decreasing sequence of positive functions which converges to 0 a.e., then $\{f_n\}$ converges to 0 in norm. (We note that for general Banach lattices, the above defines σ-order continuity, which is weaker than order continuity in this more general context.) If X is order continuous, then every continuous linear functional on X is an integral, i.e., $X^* = X'$.

Let (Ω, Σ, μ) be one of the measure spaces $\mathbb{N}$ or $[0, 1]$. A Köthe function space is a *rearrangement invariant* (*r.i. space*) or *symmetric space* if the following conditions hold.

(1) If $\tau\colon \Omega \to \Omega$ is an automorphism, i.e., a measure-preserving bijection, and f is a measurable function on Ω, then $f \in X$ if and only if $f \circ \tau \in X$, and in this case $\|f\| = \|f \circ \tau\|$.

(2) X' is a norming subspace of X^* and thus X is isometric to a subspace of X''. As a subspace of X'', either $X = X''$, or X is the closed linear span of the simple integrable functions of X''.

(3) a. If $\Omega = \mathbb{N}$ then, as sets,

$$\ell_1 \subset X \subset \ell_\infty$$

and the inclusion maps are of norm one, i.e., if $f \in \ell_1$ then $\|f\|_X \leqslant \|f\|_1$, and if $f \in X$ then $\|f\|_\infty \leqslant \|f\|_X$.

b. If $\Omega = [0, 1]$ then, as sets,

$$L_\infty[0, 1] \subset X \subset L_1[0, 1]$$

and the inclusion maps are of norm one, i.e., if $f \in L_\infty[0, 1]$ then $\|f\|_X \leqslant \|f\|_\infty$, and if $f \in X$ then $\|f\|_1 \leqslant \|f\|_X$.

Let us emphasise some results on r.i. spaces which we will use throughout the paper.

Remarks 2.1.

(a) An r.i. space X is order continuous if and only if it is separable (cf. [17, p. 118]). In this case, all bounded linear functionals on X are integrals (i.e., $X^* = X'$).

(b) When $\Omega = \mathbb{N}$ we will denote $e_n = \mathbf{1}_{\{n\}} \in X$ and $e'_n = \mathbf{1}_{\{n\}} \in X'$ for $n \in \mathbb{N}$. For $x \in X$, one has that

$$\|x\| = \lim_n \Big\| \sum_{k=1}^{n} x_k e_k \Big\|.$$

This can easily be deduced from the fact that X' is norming for X and the monotone convergence theorem (see [17, Proposition 1.b.18]).

We now discuss the isometrical Banach space properties in which we are interested in this paper.

A Banach space X has the *Daugavet property* if the following identity

$$\|\mathrm{Id} + T\| = 1 + \|T\|, \tag{2.1}$$

called *the Daugavet equation*, holds true for every rank-one operator $T\colon X \to X$, i.e., $T = f \otimes e$, where $e \in X$ and $f \in X^*$. This notion was introduced in [15] where it was shown that then weakly compact operators also satisfy (2.1). It was also shown in [15, Lemma 2.2] that this property is equivalent to the following slice condition:

For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$ there is a $y \in S_X$ such that $\operatorname{Re} x^*(y) > 1 - \varepsilon$ and $\|x + y\| > 2 - \varepsilon$.

A weakening of the Daugavet property was introduced in [19]. If every rank-one operator $T \in L(X)$ satisfies the norm equality

$$\max_{\theta \in \mathbb{T}} \|\mathrm{Id} + \theta T\| = 1 + \|T\|, \tag{2.2}$$

X is said to have the *alternative Daugavet property* (ADP for short). In this case again all weakly compact operators on X also satisfy (2.2). A slice characterisation similar to the above one holds for the ADP as well [19, Proposition 2.1]:

For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$ there is a $y \in S_X$ such that $\operatorname{Re} x^*(y) > 1-\varepsilon$ and $\max_{\theta\in\mathbb{T}} \|\theta x + y\| > 2-\varepsilon$.

This notion is strongly linked to the theory of numerical ranges. For every $T \in L(X)$ the quantity

$$v(T) = \sup\bigl\{|x^*(Tx)| : x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1\bigr\}$$

is called the *numerical radius* of T. A Banach space is said to have *numerical index* 1 [7] if every $T \in L(X)$ satisfies the condition $v(T) = \|T\|$. It is known [7] that

$$v(T) = \|T\| \qquad \Longleftrightarrow \qquad T \text{ satisfies (2.2)}.$$

Thus, X has numerical index 1 if and only if every $T \in L(X)$ satisfies (2.2). Evidently, both the Daugavet property and numerical index 1 imply the ADP. On the other hand, the space $C([0,1],\ell_2) \oplus_\infty c_0$ has the ADP, but it has neither the Daugavet property nor numerical index 1 [19, Example 3.2].

A Banach space X is said to be *lush* [6] if for every $x, y \in S_X$ and every $\varepsilon > 0$, there is $x^* \in S_{X^*}$ such that

$$x \in S := S(B_X, x^*, \varepsilon) \qquad \text{and} \qquad \operatorname{dist}(y, \overline{\operatorname{aconv}}(S)) < \varepsilon.$$

Lush spaces have numerical index 1 [6, Proposition 2.2], but it has very recently been shown that the converse result is not true in general [12, Remark 4.2.a].

3. Sequence spaces

In this section we demonstrate that in the class of r.i. sequence spaces ADP, lushness and numerical index 1 are equivalent properties, and that apart from the classical examples c_0, ℓ_1 and ℓ_∞ there are no r.i. sequence spaces with these properties. We remark that r.i. sequence spaces do not have the Daugavet property since they admit rank-one unconditional projections.

3.1. **ADP, lushness and numerical index 1 are equivalent for spaces with 1-unconditional bases.** Let X be a Banach space and let A be a convex bounded subset of X. According to [2], [3] (see also [14]) a countable family $\{V_n : n \in \mathbb{N}\}$ of subsets of A is called *determining* for A if $A \subset \overline{\operatorname{conv}}(B)$ for every $B \subset A$ intersecting all the sets V_n. Equivalently, $\{V_n : n \in \mathbb{N}\}$ is determining for A if every slice of A contains one of the V_n.

A convex bounded subset A of a Banach space X is *slicely countably determined* (*SCD set* for short) if there is a determining sequence of slices of A. Equivalently [3, Proposition 2.18], A is SCD if there is a determining sequence of relatively weakly open subsets of A.

The next theorem gives in particular a positive answer to Question 7.4(b) of [3].

Theorem 3.1. *Let X be a space with a 1-unconditional basis $(e_n)_{n\in\mathbb{N}}$. Then B_X is an SCD set.*

Proof. Fix a countable dense subset $D \subset B_X$ consisting of vectors with finite supports, and for every $a \in D$, $a = \sum_{k=1}^n a_k e_k$, select the corresponding relative weak neighbourhoods

$$U(a,m) = \Bigl\{x = \sum_{k=1}^\infty x_k e_k \in B_X :\ \max_{j\leqslant n} |a_j - x_j| < \frac{1}{m}\Bigr\}.$$

Let us show that $\{U(a,m)\colon a\in D,\ m\in\mathbb{N}\}$ is a determining collection of weak neighbourhoods. Let $V\subset B_X$ be a closed convex set that intersects all the $U(a,m)$. For fixed $f\in S_{X^*}$ and $\varepsilon>0$, we have to show that the slice

$$S(B_X,f,\varepsilon)=\{u\in B_X\colon\ \operatorname{Re} f(u)>1-\varepsilon\}$$

intersects V. To do so, take $a=\sum_{k=1}^n a_ke_k\in D\cap S(B_X,f,\varepsilon/4)$ and observe that there is an element $x=\sum_{k=1}^\infty x_ke_k\in V$ whose first n coordinates are arbitrarily close to the corresponding a_k so that we can assume $\|a-\sum_{k=1}^n x_ke_k\|<\varepsilon/4$. Therefore, we have

$$\operatorname{Re} f\Big(\sum_{k=1}^n x_ke_k\Big)>\operatorname{Re} f(a)-\frac{\varepsilon}{4}>1-\frac{\varepsilon}{2}\,.$$

Besides, by 1-unconditionality it is clear that $\|\sum_{k=1}^n x_ke_k-\sum_{k=n+1}^\infty x_ke_k\|=\|x\|\leqslant 1$. Hence, we can finally write

$$\operatorname{Re} f(x)=2\operatorname{Re} f\Big(\sum_{k=1}^n x_ke_k\Big)-\operatorname{Re} f\Big(\sum_{k=1}^n x_ke_k-\sum_{k=n+1}^\infty x_ke_k\Big)>1-\varepsilon$$

which gives $x\in V\cap S(B_X,f,\varepsilon)$, finishing the proof. □

It is known that every Banach space X with the alternative Daugavet property whose unit ball is an SCD set is lush [3, Theorem 4.4], so we have the following corollary.

Corollary 3.2. *In the class of spaces with* 1*-unconditional bases the three properties ADP, lushness and numerical index* 1 *are equivalent.*

3.2. **The only separable r.i. sequence spaces with numerical index 1 are c_0 and ℓ_1.** First note that separable r.i. sequence spaces are nothing but Banach spaces with 1-symmetric bases. So the results of the previous subsection are applicable to this kind of spaces. We start with the separable case.

Theorem 3.3. *Let X be a separable r.i. space on $\mathbb{N}$. If X is lush, then X is c_0 or ℓ_1.*

For the proof of this result we need two easy lemmas. The second one, which we state here for the readers' convenience, appears in [18] for the wider class of Banach spaces with 1-unconditional bases.

Lemma 3.4. *Let X be a Banach space and let $x^*\in S_{X^*}$ be such that $|x^{**}(x^*)|=1$ for every $x^{**}\in\operatorname{ext}(B_{X^{**}})$. If $J\colon X^*\to X^*$ is an onto isometry then $|x^{**}(Jx^*)|=1$ for every $x^{**}\in\operatorname{ext}(B_{X^{**}})$.*

Proof. Just note that J^* is an onto isometry on X^{**} and thus $J^*(x^{**})\in\operatorname{ext}(B_{X^{**}})$ for every $x^{**}\in\operatorname{ext}(B_{X^{**}})$. □

Lemma 3.5. [18, Lemma 3.2] *Let X be an r.i. space on $\mathbb{N}$ and let $x^*\in S_{X^*}$ be such that $|x^{**}(x^*)|=1$ for every $x^{**}\in\operatorname{ext}(B_{X^{**}})$. Then,*

$$|x^*(n)|\in\{0,1\}\qquad \textit{for every } n\in\mathbb{N}.$$

Proof of Theorem 3.3. Since X is a separable lush space, Theorem 4.3 in [11] tells us that the set

$$\mathcal{A}=\{x^*\in S_{X^*}:|x^{**}(x^*)|=1 \text{ for every } x^{**}\in\operatorname{ext}(B_{X^{**}})\}$$

is norming for X. By Lemma 3.5 one has $|a^*(n)| \in \{0,1\}$ for every $a^* \in \mathcal{A}$ and every $n \in \mathbb{N}$. Therefore, we can split the proof into the following two cases:

(1) *There is an $a_0^* \in \mathcal{A}$ such that the set $I = \{n \in \mathbb{N} \colon |a_0^*(n)| = 1\}$ is infinite.* In this case X is isometrically isomorphic to ℓ_1. Indeed, for fixed $x \in B_X$ and $N \in \mathbb{N}$, consider $\omega_n \in \mathbb{T}$ such that $\omega_n x(n) = |x(n)|$ for every $n = 1, \ldots, N$ and define $x^* = \sum_{n=1}^N \omega_n e_n'$. Next, take a bijection $\tau \colon \mathbb{N} \to \mathbb{N}$ such that $\tau(\{1, \ldots, N\}) \subset I$ and consider the onto isometry $J \in L(X)$ given by $Jx = x\tau^{-1}$ $(x \in X)$. Then, $J^* \in L(X^*)$ is an onto isometry such that

$$\{1, \ldots, N\} \subset \{n \in \mathbb{N} \colon |(J^* a_0^*)(n)| = 1\}.$$

Therefore, we can write

$$1 \leqslant \|x^*\| \leqslant \|J^* a_0^*\| = 1$$

and so $\|x^*\| = 1$. Finally, it suffices to observe that

$$\sum_{n=1}^N |x(n)| = |x^*(x)| \leqslant \|x\|$$

which tells us that $x \in \ell_1$ and $\|x\|_1 \leqslant \|x\|$ (the reversed inequality is always true).

(2) *For every $a^* \in \mathcal{A}$ the set $\{n \in \mathbb{N} \colon |a^*(n)| = 1\}$ is finite.* In this case we will show that X is isometrically isomorphic to c_0. For fixed $a^* \in \mathcal{A}$ we are first going to show that $\#\operatorname{supp}(a^*) = 1$. Using Lemma 3.4 we can assume, up to isometry, that $a^*(n) = 1$ for every $n \in \operatorname{supp}(a^*)$. Take $x^{**} \in \operatorname{ext}(B_{X^{**}})$ satisfying $x^{**}(a^*) = 1$ and observe that $x^{**}(e_n') \geqslant 0$ for every $n \in \operatorname{supp}(a^*)$. We claim that $\{x^{**}(e_n')\}_{n\in\mathbb{N}}$ is constant on the support of a^*. Indeed, fix $j, k \in \operatorname{supp}(a^*)$ and $m \notin \operatorname{supp}(a^*)$, take $\omega \in \mathbb{T}$ satisfying $\omega x^{**}(e_m') = |x^{**}(e_m')|$, and define $a_j^*, a_k^* \in \mathcal{A}$ by

$$a_j^* = \omega e_m' + \sum_{n \in \operatorname{supp}(a^*)\setminus\{j\}} e_n' \qquad \text{and} \qquad a_k^* = \omega e_m' + \sum_{n \in \operatorname{supp}(a^*)\setminus\{k\}} e_n'$$

(observe that $a_j^*, a_k^* \in \mathcal{A}$ by Lemma 3.4). Then we can write

$$\begin{aligned} |x^{**}(e_m')| + \sum_{n \in \operatorname{supp}(a^*)\setminus\{j\}} x^{**}(e_n') &= |x^{**}(a_j^*)| = 1 = |x^{**}(a_k^*)| \\ &= |x^{**}(e_m')| + \sum_{n \in \operatorname{supp}(a^*)\setminus\{k\}} x^{**}(e_n') \end{aligned}$$

and, therefore, $x^{**}(e_j') = x^{**}(e_k')$. So $x^{**}(e_n') = \frac{1}{\#\operatorname{supp}(a^*)}$ for every $n \in \operatorname{supp}(a^*)$. Now it is clear that $\#\operatorname{supp}(a^*) = 1$: otherwise there are $j \neq k$ in $\operatorname{supp}(a^*)$; we define $\tilde{a}^* \in \mathcal{A}$ by

$$\tilde{a}^* = e_k' - \sum_{n \in \operatorname{supp}(a^*)\setminus\{k\}} e_n'$$

and we observe that

$$1 = |x^{**}(\tilde{a}^*)| = \Big| x^{**}(e_k') - \sum_{n \in \operatorname{supp}(a^*)\setminus\{k\}} x^{**}(e_n') \Big| = \frac{\#\operatorname{supp}(a^*) - 2}{\#\operatorname{supp}(a^*)},$$

which is impossible.

Finally, since $\mathcal{A}$ is norming for X we have that

$$B_{X^*} = \overline{\operatorname{aconv}}^{w^*}(\mathcal{A}) = \overline{\operatorname{aconv}}^{w^*}(\{e_n' : n \in \mathbb{N}\})$$

and thus, $\|x\| = \sup\{|x(n)| \colon n \in \mathbb{N}\}$ for every $x \in X$. Since X is the closed linear span of $\{e_n : n \in \mathbb{N}\}$ we deduce that X is isometric to c_0, finishing the proof. $\square$

The last theorem together with Corollary 3.2 gives the result announced in the title of the subsection.

Corollary 3.6. *The only separable r.i. sequence spaces with numerical index* 1 *are* c_0 *and* ℓ_1*. The same spaces are the only examples of separable r.i. sequence spaces with the ADP.*

3.3. The only non-separable r.i. sequence space with numerical index 1 is ℓ_∞.

Theorem 3.7. *Let* X *be a non-separable r.i. space on* $\mathbb{N}$*. If* X *has the ADP, then* X *is* ℓ_∞*.*

We need a preliminary result whose proof is borrowed from [1, Theorem 1.1].

Lemma 3.8. *Let* X *be an r.i. space on* $\mathbb{N}$*. Denote by* E *the closed linear span of the set of canonical basis vectors* e_n*,* $n \in \mathbb{N}$*. If* X *has the ADP, then* E *also has the ADP.*

Proof. Fix $x \in S_E$, $f \in S_{E^*}$ and $\varepsilon > 0$. Our goal is to find a $y \in B_E$ with

$$|f(y)| > 1 - \varepsilon \quad \text{and} \quad \max_{\theta \in \mathbb{T}} \|\theta x + y\| > 2 - \varepsilon. \tag{3.1}$$

First remark that f can be considered as a sequence of scalars $f = (f_1, f_2, \dots)$ that acts on arbitrary $z = (z_1, z_2, \dots)$ by

$$f(z) = \sum_{k=1}^{\infty} f_k z_k. \tag{3.2}$$

By the same formula (3.2), f defines a linear functional on X with $\|f\|_{X^*} = \|f\|_{E^*} = 1$. Since X has the ADP, there is a $z = (z_1, z_2, \dots) \in B_X$ with

$$|f(z)| > 1 - \varepsilon \quad \text{and} \quad \max_{\theta \in \mathbb{T}} \|\theta x + z\| > 2 - \varepsilon.$$

One may now select $n \in \mathbb{N}$ big enough to fulfill $\max_{\theta \in \mathbb{T}} \| \sum_{k=1}^n z_k e_k + \theta x\| > 2 - \varepsilon$ and $| \sum_{k=1}^n z_k f_k| > 1-\varepsilon$. Then $y := \sum_{k=1}^n z_k e_k \in E$ fulfills the conditions (3.1). □

Proof of Theorem 3.7. According to Lemma 3.8, the subspace $E \subset X$ spanned by the canonical basis vectors e_n has the ADP. Since E is a separable r.i. space on $\mathbb{N}$, E must be either c_0 or ℓ_1 by Corollary 3.6. When $E = \ell_1$, for fixed $x \in X$ one has that

$$\|x\| = \lim_n \Big\| \sum_{k=1}^n x_k e_k \Big\| = \lim_n \sum_{k=1}^n |x_k|$$

by Remark 2.1.b, and so $X \subset \ell_1$ contradicting the non-separability of X.

When $E = c_0$, we fix $x \in X$ and use again Remark 2.1.b to deduce that

$$\|x\| = \sup\{|x_n| : n \in \mathbb{N}\}$$

and then $X \subset \ell_\infty$ isometrically. So it remains to show that every element of ℓ_∞ lies in X. Since X is solid, it suffices to check that the element $(1, 1, 1, \dots)$ lies in X. Let $x = (x_1, x_2, \dots) \in X \setminus c_0$ be such that $x_n \geqslant 0$ for every $n \in \mathbb{N}$ and $\limsup x_n > 1$. Using this and rearranging x we may suppose without loss of generality that $x_{2n} > 1$ for every $n \in \mathbb{N}$. Therefore, using again the solidity, we get that $(0, 1, 0, 1, \dots) \in X$ and, by symmetry, that $(1, 0, 1, 0, \dots) \in X$. Finally, we can write

$$(1, 1, 1, 1, \dots) = (0, 1, 0, 1, \dots) + (1, 0, 1, 0, \dots) \in X$$

which finishes the proof. □

4. Symmetric function spaces on [0,1]

4.1. **Lushness in separable r.i. function spaces.** Our next goal is to prove a result similar to Theorem 3.3 for rearrangement invariant function spaces on $[0,1]$. To do so, we need the following easy lemma.

Lemma 4.1. *Let X be a Köthe function space and $f \in S_X$ with $f \geqslant 0$. If $x^* \in S_{X^*}$ satisfies $x^*(f) = 1$ then x^* is positive on the subspace $X_f = \{g \in X : |g| \leqslant cf$ for some $c > 0\}$.*

Proof. Let $0 \leqslant g \in X_f$; multiplying by a suitable constant we can assume without loss of generality that $0 \leqslant g \leqslant f$. Therefore, we have for all scalars $\theta \in \mathbb{T}$

$$|g + \theta(f - g)| \leqslant |g| + |f - g| = f$$

and consequently $\|g + \theta(f - g)\| \leqslant \|f\| = 1$. Therefore

$$|x^*(g) + \theta(1 - x^*(g))| \leqslant 1$$

for all $\theta \in \mathbb{T}$, and it follows that

$$|x^*(g)| + |1 - x^*(g)| \leqslant 1$$

which means that $x^*(g)$ is a real number in the interval $[0,1]$. □

We can now present the promised result.

Theorem 4.2. *Let X be a separable r.i. space on $[0,1]$. If X is lush, then $X = L_1[0,1]$.*

Proof. Since X is a separable lush space, Theorem 4.3 in [11] tells us that the set

$$\mathcal{A} = \{f \in S_{X^*} : |x^{**}(f)| = 1 \text{ for every } x^{**} \in \operatorname{ext}(B_{X^{**}})\}$$

is norming for X. To finish the proof it suffices to show that $|f| = \mathbf{1}$ for every $f \in \mathcal{A}$ (recall that $X^* = X'$ by separability). Indeed, given $x \in X$ we can then write

$$\|x\|_1 \leqslant \|x\| \leqslant \sup\left\{\left|\int_0^1 f(t)x(t)\,dt\right| : f \in \mathcal{A}\right\} \leqslant \|x\|_1,$$

and taking into account that all simple functions are in X we obtain that $X = L_1[0,1]$.

For fixed $f \in \mathcal{A}$, there is an onto isometry on X^* sending f to $|f|$. Then Lemma 3.4 tells us that $|f| \in \mathcal{A}$ and so we can assume without loss of generality that $f \geqslant 0$. Since $f \in X \subset L_1[0,1]$, there exist positive numbers α and Δ such that

$$\mu\left(\{t \in [0,1] : \alpha \leqslant f(t)\}\right) \geqslant \Delta.$$

Claim: For every $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and every $\delta > 0$ there is an interval $I \subset [0,1]$ with $\mu(I) \leqslant \delta$ such that $x^{**}(g\mathbf{1}_{[0,1]\setminus I}) = 0$ for every $g \in L_\infty[0,1]$.
Proof of the Claim. We may suppose that $0 < \delta < \Delta/2$. Now, consider a partition of $[0,1] = I_1 \cup \cdots \cup I_n$ into disjoint intervals with $\mu(I_k) \leqslant \delta$ for $k = 1, \ldots, n$ and find for fixed $j, k \in \{1, \ldots, n\}$ a rearrangement $\tilde{f}$ of f such that

$$\alpha\mathbf{1}_{I_j} \leqslant \tilde{f}\mathbf{1}_{I_j} \qquad \text{and} \qquad \alpha\mathbf{1}_{I_k} \leqslant \tilde{f}\mathbf{1}_{I_k}. \tag{4.1}$$

Take $\omega \in \mathbb{T}$ such that $\omega x^{**}(\tilde{f}) = 1$ and use Lemma 4.1 to obtain that the functional ωx^{**} is positive on the subspace $X_{\tilde{f}} = \{g \in X : |g| \leqslant c\tilde{f} \text{ for some } c > 0\}$. Now observe that

$$\tilde{f}_\theta = \tilde{f}\mathbf{1}_{I_j} + \theta\tilde{f}\mathbf{1}_{[0,1]\setminus I_j} \in \mathcal{A}$$

for every $\theta \in \mathbb{T}$. Therefore, there are $\theta_1, \theta_2 \in \mathbb{T}$ satisfying

$$\begin{aligned} 1 = |x^{**}(\tilde{f}_{\theta_1})| &= |x^{**}(\tilde{f}\mathbf{1}_{I_j})| + |x^{**}(\tilde{f}\mathbf{1}_{[0,1]\setminus I_j})| \qquad \text{and} \\ 1 = |x^{**}(\tilde{f}_{\theta_2})| &= \Big| |x^{**}(\tilde{f}\mathbf{1}_{I_j})| - |x^{**}(\tilde{f}\mathbf{1}_{[0,1]\setminus I_j})| \Big| \end{aligned}$$

which clearly implies $\big\{|x^{**}(\tilde{f}\mathbf{1}_{I_j})|, |x^{**}(\tilde{f}\mathbf{1}_{[0,1]\setminus I_j})|\big\} = \{0,1\}$. Suppose first that $|x^{**}(\tilde{f}\mathbf{1}_{I_j})| = 0$ and use (4.1) to observe that $g\mathbf{1}_{I_j} \in X_{\tilde{f}}$ for every $g \in L_\infty[0,1]$. Thus we can write

$$|\omega x^{**}(g\mathbf{1}_{I_j})| \leqslant \omega x^{**}(|g|\mathbf{1}_{I_j}) \leqslant \omega x^{**}(\tilde{f}\mathbf{1}_{I_j}) = 0$$

and, therefore, $x^{**}(g\mathbf{1}_{I_j}) = 0$. If otherwise $|x^{**}(\tilde{f}\mathbf{1}_{[0,1]\setminus I_j})| = 0$, then

$$|\omega x^{**}(g\mathbf{1}_{I_k})| \leqslant \omega x^{**}(|g|\mathbf{1}_{I_k}) \leqslant \omega x^{**}(\tilde{f}\mathbf{1}_{[0,1]\setminus I_j}) = 0.$$

Hence we have shown that either $x^{**}(g\mathbf{1}_{I_j}) = 0$ for every $g \in L_\infty[0,1]$ or $x^{**}(g\mathbf{1}_{I_k}) = 0$ for every $g \in L_\infty[0,1]$. Finally, the arbitrariness of $j,k \in \{1,\dots,n\}$ finishes the proof of the Claim.

We continue the proof showing that f is necessarily bounded (see (a) below) and even constant (see (b) below).

(a) Suppose for contradiction that f is unbounded and define

$$B = \{t \in [0,1] : f(t) \geqslant 6\}$$

which has positive measure and satisfies $\mu(B) \leqslant 1/6$. Fix $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and use the Claim to find an interval $I \subset [0,1]$ with $\mu(I) \leqslant \mu(B)$ and such that $x^{**}(g\mathbf{1}_{[0,1]\setminus I}) = 0$ for every $g \in L_\infty[0,1]$. Up to rearrangement of f, we can assume without loss of generality that $I \subset B$. For every $n \in \mathbb{N}$ consider the set

$$B_n = \{t \in [0,1] : 5 + n \leqslant f(t) < 6 + n\},$$

split it into two sets of equal measure $B_n = B_{n,1} \cup B_{n,2}$, and define

$$B_1 = \bigcup_{n=1}^{\infty} B_{n,1}\,, \qquad B_2 = \bigcup_{n=1}^{\infty} B_{n,2}\,, \qquad f_1 = f\mathbf{1}_{B_1}\,, \qquad \text{and} \qquad f_2 = f\mathbf{1}_{B_2}\,.$$

Take an automorphism τ of $[0,1]$ which fixes $[0,1]\setminus B$ and sends $B_{n,1}$ to $B_{n,2}$ and $B_{n,2}$ to $B_{n,1}$, for every $n \in \mathbb{N}$. Setting $\widehat{f_i} = f_i \circ \tau$, observe that

$$f_1 \leqslant \frac{7}{6}\widehat{f_2}, \qquad f_2 \leqslant \frac{7}{6}\widehat{f_1}. \tag{4.2}$$

Besides, we can write, using the Claim with the bounded function $g = f\mathbf{1}_{[0,1]\setminus B}$,

$$1 = |x^{**}(f)| = |x^{**}(f_1) + x^{**}(f_2) + x^{**}(f\mathbf{1}_{[0,1]\setminus B})| = |x^{**}(f_1) + x^{**}(f_2)|$$

and by means of an argument as in the proof of the Claim one can easily deduce that $|x^{**}(f_1)| = 1$ or $|x^{**}(f_2)| = 1$. For the sake of notation let us assume without loss of generality that

$$|x^{**}(f_1)| = 1.$$

Since $\mu(\{t \in [0,1] \colon f(t) \leqslant 2\}) \geqslant \frac{1}{2}$ we can now take an automorphism π such that we have for $\widetilde{f} = f \circ \pi$ and $\widetilde{B} = \{t \in [0,1] \colon \widetilde{f}(t) \geqslant 6\} = \pi^{-1}(B)$

$$\widetilde{B} \cap B = \emptyset \qquad \text{and} \qquad I \subset \{t \in [0,1] \colon \widetilde{f}(t) \leqslant 2\}.$$

Using this and $I \subset B$, observe that $\frac{1}{2}|\widetilde{f}\mathbf{1}_I| \leqslant 1 \leqslant \frac{1}{6}|f\mathbf{1}_I|$ and, therefore, $\|\widetilde{f}\mathbf{1}_I\| \leqslant \frac{1}{3}$. Let $\widetilde{B}_i = \pi^{-1}(B_i)$ and $\widetilde{f}_i = f_i \circ \pi$ be the corresponding rearrangements of B_i and f_i associated to π. Next use that $\widetilde{f}\mathbf{1}_{[0,1]\setminus\widetilde{B}} \in L_\infty[0,1]$ to deduce that $x^{**}(\widetilde{f}\mathbf{1}_{[0,1]\setminus(\widetilde{B}\cup I)}) = 0$. Hence, one can write

$$\begin{aligned} 1 = |x^{**}(\widetilde{f})| &= |x^{**}(\widetilde{f}_1) + x^{**}(\widetilde{f}_2) + x^{**}(\widetilde{f}\mathbf{1}_I) + x^{**}(\widetilde{f}\mathbf{1}_{[0,1]\setminus(\widetilde{B}\cup I)})| \\ &\leqslant |x^{**}(\widetilde{f}_1)| + |x^{**}(\widetilde{f}_2)| + \frac{1}{3}\,. \end{aligned}$$

Therefore, there is $k \in \{1,2\}$ such that $|x^{**}(\widetilde{f}_k)| \geqslant \frac{1}{3}$.

Finally, for each $\theta \in \mathbb{T}$ define $h_\theta = f_1 + \theta\widetilde{f}_k$ and take $\theta_0 \in \mathbb{T}$ such that $|x^{**}(h_{\theta_0})| = |x^{**}(f_1)| + |x^{**}(\widetilde{f}_k)|$ and hence

$$\frac{4}{3} = 1 + \frac{1}{3} \leqslant |x^{**}(f_1)| + |x^{**}(\widetilde{f}_k)| = |x^{**}(h_{\theta_0})| \leqslant \|h_{\theta_0}\|.$$

If $k = 2$, then $f_1 + \widetilde{f}_2$, which dominates $|h_{\theta_0}|$, is a rearrangement of $f_1 + f_2$, which is $\leqslant f$, and hence $\|h_{\theta_0}\| \leqslant \|f\| = 1$, a contradiction. On the other hand, if $k = 1$, we can use (4.2) to deduce that

$$|h_{\theta_0}| \leqslant f_1 + \widetilde{f}_1 \leqslant \frac{7}{6}(\widehat{f}_2 + \widetilde{f}_1).$$

Therefore, we have likewise

$$\|h_{\theta_0}\| \leqslant \frac{7}{6}\|\widehat{f}_2 + \widetilde{f}_1\| \leqslant \frac{7}{6}\|f\| < \frac{4}{3},$$

which again gives us a contradiction.

Hence, we have that $f \in L_\infty[0,1]$ and indeed that $f \leqslant 6$ a.e.

(b) Suppose for contradiction that f is non-constant. Then, there are numbers $0 \leqslant c < d$ and sets

$$C = \{t \in [0,1] \colon f(t) \leqslant c\} \qquad \text{and} \qquad D = \{t \in [0,1] \colon f(t) \geqslant d\}$$

such that $\mu(C) > 0$ and $\mu(D) > 0$. Now fix $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and use the Claim to find an interval $I \subset [0,1]$ with $\mu(I) \leqslant \min\{\mu(C), \mu(D)\}$ and such that $x^{**}(g\mathbf{1}_{[0,1]\setminus I}) = 0$ for every $g \in L_\infty[0,1]$. Take rearrangements $f_1, f_2 \in \mathcal{A}$ of f such that

$$f_1(t) \leqslant c \qquad \text{and} \qquad f_2(t) \geqslant d \qquad \text{for every } t \in I;$$

hence $\|f_1\mathbf{1}_I\| \leqslant \frac{c}{d}\|f_2\mathbf{1}_I\|$. Then, using the fact that $f_1 \in L_\infty[0,1]$, we can write

$$1 = |x^{**}(f_1)| = |x^{**}(f_1\mathbf{1}_I) + x^{**}(f_1\mathbf{1}_{[0,1]\setminus I})| = |x^{**}(f_1\mathbf{1}_I)|$$

and, therefore,

$$1 \leqslant \|f_1\mathbf{1}_I\| \leqslant \frac{c}{d}\|f_2\mathbf{1}_I\| \leqslant \frac{c}{d} < 1,$$

which is the desired contradiction.

Hence, $f \in L_\infty[0,1]$ and it is constant. Now it is immediate to deduce that $f = \mathbf{1}$, finishing the proof. $\square$

4.2. **The Daugavet property in separable r.i. function spaces.** In their paper [1], M. D. Acosta, A. Kamińska, and M. Mastyło proved in Proposition 1.6 that if a separable real r.i. function space on $[0,1]$ with the Fatou property has the Daugavet property, then as a set of functions X coincides with $L_1[0,1]$, but they left open the question whether the norm on X is necessarily the same as the standard L_1-norm.

In this section we answer the above question in the positive even if we remove the assumption of the Fatou property; i.e., we show that the only separable real r.i. function space on $[0,1]$ with the Daugavet property is $L_1[0,1]$ endowed with its canonical norm.

Below X is a separable (hence order continuous) real r.i. function space on $[0,1]$. We remark that order continuity implies that the subspace of simple functions and the subspace of continuous functions are dense in X. Denote by ϕ the *fundamental function* of X, that is $\phi(t) = \|\mathbf{1}_{[0,t]}\|_X$. Let us list here some known properties of ϕ:

(a) ϕ is non-decreasing,
(b) $t \leqslant \phi(t) \leqslant 1$,
(c) $\phi(t+\tau) \leqslant \phi(t) + \phi(\tau)$,
(d) $\lim_{t\to 0} \phi(t) = 0$ (see [4, Chapter 2, Theorem 5.5], for instance).

We need several preliminary results. The first one is certainly known, but we haven't been able to locate a reference. It characterises $L_1[0,1]$ among separable r.i. function spaces on $[0,1]$.

Lemma 4.3. *Let X be a separable r.i. function space on $[0,1]$ and let ϕ be its fundamental function. If $\liminf_{\tau\to 0} \phi(\tau)/\tau = 1$, then $X = L_1[0,1]$ endowed with its canonical norm.*

Proof. It is sufficient to prove that $\phi(t) = t$ for all $t \in [0,1)$. Indeed, in this case for every simple function $f = \sum_{k=1}^n a_k \mathbf{1}_{A_k}$ we have that $\|f\|_{L_1} \leqslant \|f\|_X \leqslant \sum_{k=1}^n |a_k| \phi(\mu(A_k)) = \|f\|_{L_1}$. So fix $t \in [0,1)$ and select a sequence of $\tau_n > 0$, $\tau_n \to 0$, such that $\phi(\tau_n)/\tau_n \to 1$. Denote $m(n)$ the smallest positive integer such that $m(n)\tau_n \geqslant t$ and observe that $t \leqslant m(n)\tau_n < t + \tau_n$. Then

$$t \leqslant \phi(t) \leqslant \phi(m(n)\tau_n) \leqslant m(n)\phi(\tau_n) = \tau_n m(n)\phi(\tau_n)/\tau_n \to t$$

as $n \to \infty$. □

The following lemma prepares an elementary proof of Corollary 4.5, which also follows from the contractivity of conditional expectations in r.i. spaces [17, Theorem 2.a.4].

Lemma 4.4. *Let $\Delta = [0,a] \subset [0,1]$ be a subinterval. Define for every $\tau \in \Delta$ the Δ-circling shift operator $T_\tau\colon (T_\tau f)(t) = f(t)$ for $t > a$, $(T_\tau f)(t) = f(t+\tau)$ for $0 \leqslant t \leqslant a-\tau$, and $(T_\tau f)(t) = f(t-a+\tau)$ for $a-\tau < t \leqslant a$. Then for every $f \in X$ the map $\tau \mapsto T_\tau f$ is continuous in the norm topology of X and hence is Riemann integrable. Moreover,*

$$\frac{1}{a}\int_0^a T_\tau f\, d\tau = \left(\frac{1}{a}\int_0^a f(t)\, dt\right)\mathbf{1}_\Delta + f\mathbf{1}_{[0,1]\setminus\Delta}.$$

Proof. The fact is evident when f is continuous and fulfills $f(0) = f(a)$. Since, as remarked above, such functions form a dense subset of X, we are done. □

Corollary 4.5. *Let $[0,1]$ be split into a disjoint union of measurable subsets Δ_1 and Δ_2. Then for every $g \in X$*

$$\left\| |g|\mathbf{1}_{\Delta_1} + \left(\frac{1}{\mu(\Delta_2)}\int_{\Delta_2} g(t)\,dt\right)\mathbf{1}_{\Delta_2}\right\|_X \leqslant \|g\|_X.$$

Proof. We can assume without loss of generality $\Delta_2 = [0,a]$ and apply the previous lemma. □

Corollary 4.6. *Let $g \in X$. Then for every $t \geqslant \mu(\operatorname{supp} g)$*

$$\frac{1}{t}\phi(t)\|g\|_1 \leqslant \|g\|_X$$

Proof. We may assume without loss of generality that $\Delta_2 := [0,t] \supset \operatorname{supp} g$ and apply the previous corollary. □

Lemma 4.7. *Let $g \in L_\infty[0,1]$. Then for every $\alpha > 0$*

$$\|g\|_X \leqslant \alpha + \|g\|_\infty \phi(\alpha^{-1}\|g\|_1).$$

In particular, if $f_n \in L_\infty[0,1]$, $\sup_n \|f_n\|_\infty < \infty$ and $\lim_{n\to\infty}\|f_n\|_1 = 0$, then $\lim_{n\to\infty}\|f_n\|_X = 0$.

Proof. Remark that

$$|g| \leqslant \alpha + \|g\|_\infty \mathbf{1}_{\{\tau\in[0,1]\colon\, |g(\tau)|>\alpha\}},$$

and that $\mu(\{\tau \in [0,1] \colon |g(\tau)| > \alpha\}) \leqslant \alpha^{-1}\|g\|_1$. □

Theorem 4.8. *Let X be a separable real r.i. space on $[0,1]$ with the following property: for every $\varepsilon > 0$ there is an $f = f_\varepsilon \in X$ such that*

(a) $\|f\|_X = 1$
(b) $\int_0^1 f(t)\,dt < -1 + \varepsilon$
(c) $\|f + \mathbf{1}\|_X \geqslant 2 - \varepsilon$.

Then $X = L_1[0,1]$ (endowed with its canonical norm).

Before giving the proof, we first record the main result of this subsection as an immediate consequence.

Corollary 4.9. *The only separable real r.i. function space on $[0,1]$ with the Daugavet property is $L_1[0,1]$ in its canonical norm.*

Indeed, the characterisation of the Daugavet property in terms of slices ([15, Lemma 2.2]) that was cited in Section 2 allows us to deduce this corollary from Theorem 4.8 by putting $x = \mathbf{1}$, $x^* = -\mathbf{1}$ and taking as f the corresponding y.

Proof of Theorem 4.8. Fix $\varepsilon > 0$ and $f = f_\varepsilon \in X$ with the properties (a), (b) and (c). Consider the following partition:

$$[0,1] = A \cup B = A_1 \cup A_2 \cup B_1 \cup B_2,$$

where

$$\begin{aligned} A &= \{t \in [0,1] \colon f(t) \leqslant 0\}, & B &= \{t \in [0,1] \colon f(t) > 0\},\\ A_1 &= \{t \in A \colon |f(t)| \leqslant 2\}, & A_2 &= \{t \in A \colon |f(t)| > 2\},\\ B_1 &= \{t \in B \colon |f(t)| \leqslant 2\}, & B_2 &= \{t \in B \colon |f(t)| > 2\} \end{aligned}$$

(all these sets depend on ε).

Remark first that (a) and (b) imply

$$\int_B f\,d\mu < \varepsilon, \tag{4.3}$$

otherwise $g = f\mathbf{1}_A - f\mathbf{1}_B$ would be a norm-one function with $|\int_0^1 g(t)\,dt| > 1$. In particular

$$\int_{B_1} f\,d\mu < \varepsilon,$$

and Lemma 4.7 says that

$$u(\varepsilon) := \|f\mathbf{1}_{B_1}\|_X \to 0 \tag{4.4}$$

as $\varepsilon \to 0$. Since also $\int_{B_2} f\,d\mu < \varepsilon$ and $f \geqslant 2$ on B_2 we have

$$\mu(B_2) < \frac{\varepsilon}{2}. \tag{4.5}$$

From $\|f\|_1 \leqslant \|f\|_X = 1$ we deduce

$$\mu(A_2 \cup B_2) < \frac{1}{2}. \tag{4.6}$$

Now, using the facts $|\mathbf{1}_{A_1} + f\mathbf{1}_{A_1}| \leqslant \mathbf{1}_{A_1}$ and $|\mathbf{1}_{A_2} + f\mathbf{1}_{A_2}| \leqslant |f|\mathbf{1}_{A_2}$, it is easy to check that

$$|\mathbf{1} + f| \leqslant \mathbf{1}_{A_1} + |f|\mathbf{1}_{A_2} + \mathbf{1}_{B_1} + |f|\mathbf{1}_{B_2} + |f|\mathbf{1}_{B_1} + \mathbf{1}_{B_2}$$

and, therefore, one can write

$$\begin{aligned}2 - \varepsilon \leqslant \|\mathbf{1} + f\| &\leqslant \|\mathbf{1}_{A_1} + |f|\mathbf{1}_{A_2} + \mathbf{1}_{B_1} + |f|\mathbf{1}_{B_2}\| + \||f|\mathbf{1}_{B_1}\| + \|\mathbf{1}_{B_2}\| \\ &\leqslant \||f|\mathbf{1}_{A_2\cup B_2} + \mathbf{1}_{A_1\cup B_1}\| + u(\varepsilon) + \phi(\varepsilon)\end{aligned}$$

by (4.4) and (4.5). An application of Corollary 4.5 with $\Delta_1 = A_2 \cup B_2$, $\Delta_2 = A_1 \cup B_1$ and

$$g = |f| + \left(1 - \frac{1}{\mu(\Delta_2)} \int_{\Delta_2} |f|\,d\mu\right) \mathbf{1}_{\Delta_2}$$

implies

$$\begin{aligned}2 - \varepsilon &\leqslant \|g\|_X + u(\varepsilon) + \phi(\varepsilon) \\ &\leqslant 1 + \phi(\mu(\Delta_2)) \left(1 - \frac{1}{\mu(\Delta_2)} \int_{\Delta_2} |f|\,d\mu\right) + u(\varepsilon) + \phi(\varepsilon).\end{aligned} \tag{4.7}$$

Since we have by (4.6) $\mu(\Delta_2) \geqslant \frac{1}{2}$ for all values of ε, the last inequality implies $\lim_{\varepsilon\to 0} \int_{\Delta_2} |f|\,d\mu = 0$. Together with (4.3) this means that

$$\lim_{\varepsilon\to 0} \int_{A_2} |f|\,d\mu = 1. \tag{4.8}$$

Condition (4.7) also implies that

$$\lim_{\varepsilon\to 0} \phi(\mu(\Delta_2)) = 1. \tag{4.9}$$

Since $\mu(A_2) \leqslant \mu(\Delta_1) \leqslant \mu(\Delta_2)$ we can apply Corollary 4.6 for $g = |f|\mathbf{1}_{A_2}$ and $t = \mu(\Delta_2)$. Then,

$$1 \geqslant \frac{1}{\mu(\Delta_2)} \phi(\mu(\Delta_2)) \int_{A_2} |f|\,d\mu. \tag{4.10}$$

By (4.8), (4.9) and (4.10) this implies $\mu(\Delta_2) \to 1$ and consequently $\mu(A_2) \to 0$ as $\varepsilon \to 0$. Now we can apply again the same Corollary 4.6 but for $t = \mu(A_2)$ and

$g = |f|\mathbf{1}_{A_2}$. This gives us that $\liminf_{\varepsilon\to 0} \phi(t)/t = 1$, and since $t \to 0$ as $\varepsilon \to 0$ an application of Lemma 4.3 completes the proof. $\square$

Remark 4.10. Theorem 4.8 also implies that $L_1[0,1]$ is the only separable real r.i. space on $[0,1]$ with "bad projections" (defined in [8] to mean that $\|\mathrm{Id} - P\| \geqslant 2$ for every rank-one projection) and the only separable real r.i. space on $[0,1]$ with the property that $\|\mathrm{Id} + T\| = \|\mathrm{Id} - T\|$ for every rank-one operator T (the last property appeared in [10]). This is so since the latter property is stronger than the former one, and since spaces with "bad projections" fit the conditions of Theorem 4.8 by using a characterization of this property in terms of slices from [8]: X is a space with "bad projections" if and only if for every $x^* \in S_{X^*}$, every $\varepsilon > 0$ and every $x \in S_X$ with $\operatorname{Re} x^*(x) > 1 - \varepsilon$, there is $y \in S_X$ such that $\|x - y\| > 2 - \varepsilon$ and $\operatorname{Re} x^*(y) > 1 - \varepsilon$.

Remark 4.11. In order to extend the results in this subsection to the complex case, one would have to replace (b) of Theorem 4.8 by $\int_0^1 \operatorname{Re} f(t)\, dt < -1 + \varepsilon$. Unfortunately we haven't succeeded in proving this.

4.3. **The almost Daugavet property in separable r.i. function spaces.** In this subsection we will deal with another weakening of the Daugavet property. Let Y be a closed linear subspace of X^*. According to [13], X has the Daugavet property with respect to Y if the Daugavet equation (2.1) holds true for every rank-one operator $T\colon X \to X$ of the form $T = f \otimes e$, where $e \in X$ and $f \in Y$. X is said to be an *almost Daugavet space* if there is a norming subspace $Y \subset X^*$ such that X has the Daugavet property with respect to Y. A separable space is an almost Daugavet space [13, Theorem 1.1] if and only if there is a sequence $(v_n) \subset B_X$ such that for every $x \in X$

$$\lim_{n\to\infty} \|x + v_n\| = \|x\| + 1.$$

We will now show that there are r.i. renormings X of $L_1[0,1]$ with the almost Daugavet property that are different from $L_1[0,1]$; in fact, the Banach-Mazur distance $\operatorname{dist}(X, L_1[0,1])$ can be arbitrarily large.

Theorem 4.12. *For every $\alpha \in (0,1)$ denote X_α the linear space $L_1[0,1]$ equipped with the norm given by*

$$p_\alpha(f) = \frac{1}{\alpha} \sup\left\{ \int_A |f|\, d\mu \colon A \in \Sigma,\ \mu(A) \leqslant \alpha \right\} \qquad (f \in X).$$

Then, the following hold:

1. *X_α is an almost Daugavet r.i. space;*
2. *$\operatorname{dist}(X_\alpha, L_1[0,1]) \to \infty$ as $\alpha \to 0$.*

Proof. By construction, X_α is rearrangement invariant. Denote $v_n = \frac{\alpha}{n}\mathbf{1}_{[0,1/n]}$. Evidently, $p_\alpha(v_n) = 1$ for all $n > \frac{1}{\alpha}$. If we show that $\lim_{n\to\infty} p_\alpha(f + v_n) = p_\alpha(f) + 1$ for every $f \in X_\alpha$, then the almost Daugavet property of X_α will be proved. Indeed, fix an $f \in X_\alpha$. By the definition of p_α there is a sequence of $A_n \in \Sigma$ such that $\mu(A_n) \leqslant \alpha$ and $\frac{1}{\alpha}\int_{A_n} |f|\, d\mu \to p_\alpha(f)$. By the absolute continuity of the Lebesgue integral one can modify A_n in order to fulfill additionally the conditions $\mu(A_n) \leqslant$

$\alpha - \frac{1}{n}$, $A_n \cap [0, 1/n] = \emptyset$. Then

$$\begin{aligned} p_\alpha(f + v_n) &\geqslant \frac{1}{\alpha} \int_{A_n \cup [0,1/n]} |f + v_n| \, d\mu \\ &= \frac{1}{\alpha} \int_{A_n} |f| \, d\mu + \frac{1}{\alpha} \int_{[0,1/n]} |f + v_n| \, d\mu \\ &\geqslant \frac{1}{\alpha} \int_{A_n} |f| \, d\mu + 1 - \frac{1}{\alpha} \int_{[0,1/n]} |f| \, d\mu \\ &\to p_\alpha(f) + 1. \end{aligned}$$

So, (1) is proved. To prove (2) it is enough to remark that X_α contains a subspace isometric to $\ell_\infty^{(m)}$, where m is the entire part of $1/\alpha$. This subspace is spanned by the functions $\mathbf{1}_{[0,\alpha]}, \mathbf{1}_{[\alpha,2\alpha]}, \ldots, \mathbf{1}_{[(m-1)\alpha,m\alpha]}$. □

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