# NUMERICAL INDEX OF ABSOLUTE SUMS OF BANACH SPACES

MIGUEL MARTÍN, JAVIER MERÍ, MIKHAIL POPOV, AND BEATA RANDRIANANTOANINA

Abstract. We study the numerical index of absolute sums of Banach spaces, giving general conditions which imply that the numerical index of the sum is less or equal than the infimum of the numerical indices of the summands and we provide some examples where the equality holds covering the already known case of $c_0$-, $\ell_1$- and $\ell_\infty$-sums and giving as a new result the case of $E$-sums where $E$ has the RNP and $n(E) = 1$ (in particular for finite-dimensional $E$ with $n(E) = 1$). We also show that the numerical index of a Banach space $Z$ which contains a dense increasing union of one-complemented subspaces is greater or equal than the limit superior of the numerical indices of those subspaces. Using these results, we give a detailed short proof of the already known fact that the numerical indices of all infinite-dimensional $L_p(\mu)$-spaces coincide.

## 1. Introduction

Given a Banach space $X$, we write $B_X$, $S_X$ and $X^*$ to denote its closed unit ball, its unit sphere and its topological dual and define

$$\Pi(X) := \big\{(x, x^*) \in S_X \times S_{X^*} \ : \ x^*(x) = 1\big\},$$

and denote the Banach algebra of all (bounded linear) operators on $X$ by $L(X)$. For an operator $T \in L(X)$, its *numerical radius* is defined as

$$v(T) := \sup\{|x^*(Tx)| \ : \ (x, x^*) \in \Pi(X)\},$$

which is a seminorm on $L(X)$ smaller than the operator norm. The *numerical index* of $X$ is the constant given by

$$n(X) := \inf\{v(T) \ : \ T \in L(X),\ \|T\| = 1\} = \max\{k \geqslant 0 \ : \ k\,\|T\| \leqslant v(T)\ \forall T \in L(X)\}.$$

The numerical radius of bounded linear operators on Banach spaces was introduced, independently, by F. Bauer and G. Lumer in the 1960's extending the Hilbert space case from the 1910's. The definition of numerical index appeared for the first time in the 1970 paper [6], where the authors attributed the authorship of the concept to G. Lumer. Classical references here are the monographs by F. Bonsall and J. Duncan [2, 3] from the 1970's. The reader will find the state-of-the-art on numerical indices in the survey paper [11] and references therein. We refer to all these references for background. Only newer results which are not covered there will be explicitly referenced in this introduction.

Let us present here the context necessary for the paper. First, real and complex Banach spaces do not behave in the same way with respect to numerical indices. In the real case, all values in $[0, 1]$ are possible for the numerical index. In the complex case, $1/\mathrm{e} \leqslant n(X) \leqslant 1$ and all of these values are possible. There are some classical Banach spaces for which the numerical index has been calculated. For instance, the numerical index of $L_1(\mu)$ is 1, and this property is shared by any of its isometric preduals. In particular, $n\big(C(K)\big) = 1$ for every compact $K$. Also, $n(Y) = 1$ for every finite-codimensional subspace $Y$ of $C[0, 1]$. If $H$ is a Hilbert space of dimension greater than one then

*Date*: March 16th, 2010. Revised August 19th, 2010.

2010 *Mathematics Subject Classification.* Primary 46B04. Secondary 46B20, 46E30, 47A12.

*Key words and phrases.* Banach space; numerical index; $L_p$-space; absolute sum; Köthe space.

First and second authors partially supported by Spanish MICINN and FEDER project no. MTM2009-07498 and Junta de Andalucía and FEDER grants G09-FQM-185, P06-FQM-01438 and P09-FQM-4911. Third author supported by Junta de Andalucía and FEDER grant P06-FQM-01438 and by Ukr. Derzh. Tema N 0103Y001103. Fourth author participant, NSF Workshop in Linear Analysis and Probability, Texas A&M University.

$n(H) = 0$ in the real case and $n(H) = 1/2$ in the complex case. The exact value of the numerical indices of $L_p(\mu)$ spaces is still unknown when $1 < p < \infty$ and $p \neq 2$, but it is is known [7, 8] that all infinite-dimensional $L_p(\mu)$ spaces have the same numerical index, which coincides with the infimum of the numerical indices of finite-dimensional $L_p(\mu)$ spaces, and the result has been extended to vector-valued $L_p$ spaces [9]. It has been shown very recently [18] that every real $L_p(\mu)$ space has positive numerical index for $p \neq 2$. Some known results about absolute sums of Banach spaces and about vector-valued function spaces are the following. The numerical index of the $c_0$-, $\ell_1$- or $\ell_\infty$-sum of a family of Banach spaces coincides with the infimum of the numerical indices of the elements of the family, while the numerical index of the $\ell_p$-sum is only smaller or equal than the infimum. For a Banach space $X$, it is known that, among others, the following spaces have the same numerical index as $X$: $C(K, X)$, $L_1(\mu, X)$, $L_\infty(\mu, X)$.

Our main goal in this paper is to study the numerical index of absolute sums of Banach spaces. Given a nonempty set $\Lambda$ and a linear subspace $E$ of $\mathbb{R}^\Lambda$ with absolute norm (see Section 2 for the exact definition), we may define the $E$-sum of a family of Banach spaces indexed in $\Lambda$. We give very general conditions on the space $E$ to assure that the numerical index of an $E$-sum of a family of Banach spaces is smaller or equal than the infimum of the numerical indices of the elements of the family. It covers the already known case of $\ell_p$-sums ($1 \leqslant p \leqslant \infty$) and also the case when $E$ is a Banach space with a one-unconditional basis. On the other hand, we give a condition on $E$ to get that the numerical index of an $E$-sum of a family of Banach spaces is equal to the infimum of the numerical indices of the elements of the family. As a consequence, we obtain the already known result for $c_0$-, $\ell_1$- and $\ell_\infty$-sums with a unified approach and also the case of $E$-sums when $E$ has the Radon-Nikodým property (RNP in short) and $n(E) = 1$. In particular, the numerical index of a finite $E$-sum of Banach spaces is equal to the minimum of the numerical indices of the summands when $n(E) = 1$.

Besides of the above results, we discuss in Section 3 when the numerical index of a Banach space is smaller than the numerical index of its one-complemented subspaces, and we give examples showing that this is not always the case for unconditional subspaces. We show in Section 4 sufficient conditions on a Köthe space $E$ to ensure that $n\big(E(X)\big) \leqslant n(X)$ for every Banach space $X$. These conditions cover the already known cases of $E = L_p(\mu)$ ($1 \leqslant p \leqslant \infty$) with a unified approach but they also give the case of an order-continuous Köthe space $E$. In Section 5 it is shown that the numerical index of a Banach space which contains a dense increasing union of one-complemented subspaces is greater or equal than the limit superior of the numerical indices of those subspaces. As a consequence, if a Banach space has a monotone basis, its numerical index is greater or equal than the limit superior of the numerical indices of the ranges of the projections associated to the basis.

Finally, in Section 6 we deduce from the results of the previous sections the already known result [7, 8, 9] that for every positive measure $\mu$ such that $L_p(\mu)$ is infinite-dimensional and every Banach space $X$, $n\big(L_p(\mu, X)\big) = n\big(\ell_p(X)\big) = \inf_{m\in\mathbb{N}} n\big(\ell_p^m(X)\big)$. In our opinion, the abstract vision we are developing in this paper allows to understand better the properties of $L_p$-spaces underlying the proofs: $\ell_p$-sums are absolute sums, $L_p$-norms are associative, every measure space can be decomposed into parts of finite measure, every finite measure algebra is isomorphic to the union of homogeneous measure algebras (Maharam's theorem) and, finally, the density of simple functions via the conditional expectation projections.

We recall that given a measure space $(\Omega, \Sigma, \mu)$ and a Banach space $X$, $L_p(\mu, X)$ denotes the Banach space of (equivalent classes of) Bochner-measurable functions from $\Omega$ into $X$. Let us observe that we may suppose that the measure $\mu$ is complete since every positive measure and its completion provide the same vector-valued $L_p$-spaces. When $\Omega$ has $m$ elements and $\mu$ is the counting measure, we write $\ell_p^m(X)$. When $\Omega$ is an infinite countable set and $\mu$ is the counting measure, we write $\ell_p(X)$. We will write $X \oplus_p Y$ to denote the $\ell_p$-sum of two Banach spaces $X$ and $Y$.

We finish the introduction with the following result from [2] which allows to calculate numerical radii of operators using a dense subset of the unit sphere and one supporting functional for each of the points of this dense subset, and which we will use along the paper.

**Lemma 1.1** ([2, Theorem 9.3]). *Let $X$ be a Banach space, let $\Gamma$ be subset of $\Pi(X)$ such that the projection on the first coordinate is dense in $S_X$. Then*

$$v(T) = \sup\{|x^*(Tx)| \ : \ (x, x^*) \in \Gamma\}$$

*for every $T \in L(X)$.*

## 2. Absolute sums of Banach spaces

Let $\Lambda$ be a nonempty set and let $E$ be a linear subspace of $\mathbb{R}^\Lambda$. An *absolute norm* on $E$ is a complete norm $\|\cdot\|_E$ satisfying

(a) Given $(a_\lambda), (b_\lambda) \in \mathbb{R}^\Lambda$ with $|a_\lambda| = |b_\lambda|$ for every $\lambda \in \Lambda$, if $(a_\lambda) \in E$, then $(b_\lambda) \in E$ with $\|(a_\lambda)\|_E = \|(b_\lambda)\|_E$.
(b) For every $\lambda \in \Lambda$, $\chi_{\{\lambda\}} \in E$ with $\|\chi_{\{\lambda\}}\|_E = 1$, where $\chi_{\{\lambda\}}$ is the characteristic function of the singleton $\{\lambda\}$.

The following results can be deduced from the definition above:

(c) Given $(x_\lambda), (y_\lambda) \in \mathbb{R}^\Lambda$ with $|y_\lambda| \leqslant |x_\lambda|$ for every $\lambda \in \Lambda$, if $(x_\lambda) \in E$, then $(y_\lambda) \in E$ with $\|(y_\lambda)\|_E \leqslant \|(x_\lambda)\|_E$.
(d) $\ell_1(\Lambda) \subseteq E \subseteq \ell_\infty(\Lambda)$ with contractive inclusions.

Observe that $E$ is a Banach lattice in the pointwise order (actually, $E$ can be viewed as a Köthe space on the measure space $(\Lambda, \mathcal{P}(\Lambda), \nu)$ where $\nu$ is the counting measure on $\Lambda$, which is non-necessarily $\sigma$-finite, see Section 4). The *Köthe dual* $E'$ of $E$ is the linear subspace of $\mathbb{R}^\Lambda$ defined by

$$E' = \left\{ (b_\lambda) \in \mathbb{R}^\Lambda \ : \ \|(b_\lambda)\|_{E'} := \sup_{(a_\lambda) \in B_E} \sum_{\lambda \in \Lambda} |b_\lambda||a_\lambda| < \infty \right\}.$$

The norm $\|\cdot\|_{E'}$ on $E'$ is an absolute norm. Every element $(b_\lambda) \in E'$ defines naturally a continuous linear functional on $E$ by the formula

$$(a_\lambda) \longmapsto \sum_{\lambda \in \Lambda} b_\lambda a_\lambda \qquad \big((a_\lambda) \in E\big),$$

so we have $E' \subseteq E^*$ and this inclusion is isometric. We say that $E$ is *order continuous* if $0 \leqslant x_\alpha \downarrow 0$ and $x_\alpha \in E$ imply that $\lim \|x_\alpha\| = 0$ (since $E$ is order complete, this is known to be equivalent to the fact that $E$ does not contain an isomorphic copy of $\ell_\infty$, see [16, p. 7]). If $E$ is order continuous, the set of those functions with finite support is dense in $E$ and the inclusion $E' \subseteq E^*$ is surjective (this is shown for Köthe spaces defined on a $\sigma$-finite space by using the Radon-Nikodým theorem; in the case we are studying here, the measure spaces are not necessarily $\sigma$-finite, but since they are discrete the proof of the fact that $E' = E^*$ is straightforward).

Given an arbitrary family $\{X_\lambda \,:\, \lambda \in \Lambda\}$ of Banach spaces, the *$E$-sum* of the family is the space

$$\Big[\bigoplus_{\lambda \in \Lambda} X_\lambda\Big]_E := \Big\{(x_\lambda) \,:\, x_\lambda \in X_\lambda \ \forall \lambda \in \Lambda, \ (\|x_\lambda\|) \in E\Big\}$$
$$= \Big\{(a_\lambda x_\lambda) \,:\, a_\lambda \in \mathbb{R}_0^+, \ x_\lambda \in S_{X_\lambda} \ \forall \lambda \in \Lambda, \ (a_\lambda) \in E\Big\}$$

endowed with the complete norm $\|(x_\lambda)\| = \|(\|x_\lambda\|)\|_E$. We will use the name *absolute sum* when the space $E$ is clear from the context. Write $X = \Big[\bigoplus_{\lambda \in \Lambda} X_\lambda\Big]_E$. For every $\kappa \in \Lambda$, we consider the natural inclusion $I_\kappa : X_\kappa \longrightarrow X$ given by $I_\kappa(x) = x\,\chi_{\{\kappa\}}$ for every $x \in X_\kappa$, which is an isometric embedding, and the natural projection $P_\kappa : X \longrightarrow X_\kappa$ given by $P_\kappa\big((x_\lambda)\big) = x_\kappa$ for every $(x_\lambda) \in X$, which is contractive. Clearly, $P_\kappa I_\kappa = \mathrm{Id}_{X_\kappa}$. We write

$$X' = \Big[\bigoplus_{\lambda \in \Lambda} X_\lambda^*\Big]_{E'}$$

and observe that every element in $(x_\lambda^*) \in X'$ defines naturally a continuous linear functional on $X$ by the formula

$$(x_\lambda) \longmapsto \sum_{\lambda\in\Lambda} x_\lambda^*(x_\lambda) \qquad \big((x_\lambda)\in E\big),$$

so we have $X' \subseteq X^*$ and this inclusion is isometric.

Examples of absolute sums are $c_0$-sums, $\ell_p$-sums for $1 \leqslant p \leqslant \infty$, i.e. given a nonempty set $\Lambda$, we are considering $E = c_0(\Lambda)$ or $E = \ell_p(\Lambda)$. More examples are the absolute sums produced using a Banach space $E$ with a one-unconditional basis, finite (i.e. $E$ is $\mathbb{R}^m$ endowed with an absolute norm) or infinite (i.e. $E$ is a Banach space with an one-unconditional basis viewed as a linear subspace of $\mathbb{R}^{\mathbb{N}}$ via the basis).

Our first main result gives an inequality between the numerical index of an $E$-sum of Banach spaces and the infimum of the numerical index of the summands, provided that $E'$ contains sufficiently many norm-attaining functionals.

**Theorem 2.1.** *Let $\Lambda$ be a non-empty set and let $E$ be a linear subspace of $\mathbb{R}^\Lambda$ endowed with an absolute norm. Suppose there is a dense subset $A \subseteq S_E$ such that for every $(a_\lambda) \in A$, there exists $(b_\lambda) \in S_{E'}$ satisfying $\sum_{\lambda\in\Lambda} b_\lambda a_\lambda = 1$. Then, given an arbitrary family $\{X_\lambda : \lambda\in\Lambda\}$ of Banach spaces,*

$$n\left(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_E\right) \leqslant \inf\big\{n(X_\lambda) \,:\, \lambda\in\Lambda\big\}.$$

*Proof.* Write $X = \Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_E$ and $X' = \Big[\bigoplus_{\lambda\in\Lambda} X_\lambda^*\Big]_{E'} \subseteq X^*$. Fix $\kappa\in\Lambda$. For every $S \in L(X_\kappa)$, we define $T \in L(X)$ by $T = I_\kappa S P_\kappa$. It then follows that $\|T\| \leqslant \|S\|$. Since $S = P_\kappa T I_\kappa$, $\|S\| \leqslant \|T\|$ and so $\|T\| = \|S\|$.

We claim that $v(T) \leqslant v(S)$. Indeed, we consider the set $\mathcal{A} \subseteq S_X$ given by

$$\mathcal{A} = \big\{(a_\lambda x_\lambda) \,:\, a_\lambda \in \mathbb{R}_0^+,\, x_\lambda \in S_{X_\lambda}\, \forall\lambda\in\Lambda,\ (a_\lambda)\in A\big\}$$

and for every $a = (a_\lambda x_\lambda) \in \mathcal{A}$, we write

$$\Upsilon(a) = (b_\lambda x_\lambda^*) \in S_{X'} \subseteq S_{X^*}$$

where $x_\lambda^* \in S_{X_\lambda^*}$ satisfies $x_\lambda^*(x_\lambda) = 1$ and $(b_\lambda) \in S_{E'}$ satisfies $\sum_{\lambda\in\Lambda} b_\lambda a_\lambda = 1$. The set $\mathcal{A}$ is dense in $S_X$ and $[\Upsilon(a)](a) = 1$ for every $a \in \mathcal{A}$. It then follows from Lemma 1.1 that

$$v(T) = \sup\left\{|[\Upsilon(a)](T(a))| \,:\, a\in\mathcal{A}\right\}.$$

For every $a \in \mathcal{A}$, we have

$$|[\Upsilon(a)](T(a))| = |[\Upsilon(a)](I_\kappa(S(a_\kappa x_\kappa)))| = |b_\kappa a_\kappa|\,|x_\kappa^*(S(x_\kappa))| \leqslant |x_\kappa^*(S(x_\kappa))| \leqslant v(S),$$

where the last inequality follows from the fact that $(x_\kappa, x_\kappa^*) \in \Pi(X_\kappa)$. Taking supremum with $a \in \mathcal{A}$, we get $v(T) \leqslant v(S)$ as desired.

Now, we observe that

$$v(S) \geqslant v(T) \geqslant n(X)\|T\| \geqslant n(X)\|S\|,$$

and the arbitrariness of $S \in L(X_\kappa)$ gives us that $n(X_\kappa) \geqslant n(X)$. □

Let us list here the main consequences of the above result.

Let $E$ be a linear subspace of $\mathbb{R}^\Lambda$ with an absolute norm. If $E$ is order continuous, the hypotheses of Theorem 2.1 are trivially satisfied (since $E^* = E'$). Again in this case, $E'$ is a linear subspace of $\mathbb{R}^\Lambda$ with an absolute norm and the hypotheses of Theorem 2.1 are satisfied for $E'$ thanks to the Bishop-Phelps theorem (the set of those norm-one elements in $E' = E^*$ attaining the norm on $E \subset E''$ is dense in the unit sphere of $E'$). Therefore, the following result follows.

**Corollary 2.2.** *Let $\Lambda$ be a nonempty set and let $E$ be a linear subspace of $\mathbb{R}^\Lambda$ with an absolute norm which is order continuous. Then, given an arbitrary family $\{X_\lambda \,:\, \lambda\in\Lambda\}$ of Banach spaces,*

$$n\left(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_E\right) \leqslant \inf\big\{n(X_\lambda)\,:\,\lambda\in\Lambda\big\}, \qquad \textit{and} \qquad n\left(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{E'}\right) \leqslant \inf\big\{n(X_\lambda)\,:\,\lambda\in\Lambda\big\}.$$

The spaces $E=c_0(\Lambda)$ and $E=\ell_p(\Lambda)$ for $1\leqslant p<\infty$ are order continuous. For $E=\ell_1(\Lambda)$ we have $E^*=E'=\ell_\infty(\Lambda)$. Therefore, the following corollary follows from the above result. It appeared in [19, Proposition 1 and Remark 2.a].

**Corollary 2.3.** *Let $\Lambda$ be a non-empty set and let $\{X_\lambda \,:\, \lambda\in\Lambda\}$ be a family of Banach spaces. Let $X$ denote the $c_0$-sum or $\ell_p$-sum of the family ($1\leqslant p\leqslant\infty$). Then*

$$n(X) \leqslant \inf\big\{n(X_\lambda)\,:\,\lambda\in\Lambda\big\}.$$

A particular case of the above corollary is the absolute sums associated to a Banach space with one-unconditional basis (finite or infinite). Related to infinite bases, let us comment that order continuous linear subspaces of $\mathbb{R}^\mathbb{N}$ with absolute norm have one-unconditional basis and, reciprocally, if a Banach space has a one-unconditional basis it can be viewed (via the basis) as an order continuous linear subspace of $\mathbb{R}^\mathbb{N}$ with absolute norm.

**Corollary 2.4.**

(a) *Let $E$ be $\mathbb{R}^m$ endowed with an absolute norm and let $X_1,\ldots,X_m$ be Banach spaces. Then*

$$n\left(\big[X_1\oplus\cdots\oplus X_m\big]_E\right) \leqslant \min\big\{n(X_1),\ldots,n(X_m)\big\}.$$

(b) *Let $E$ be a Banach space with a one-unconditional (infinite) basis and let $\{X_j \,:\, j\in\mathbb{N}\}$ be a sequence of Banach spaces. Then*

$$n\left(\Big[\bigoplus_{j\in\mathbb{N}} X_j\Big]_E\right) \leqslant \inf\big\{n(X_j)\,:\,j\in\mathbb{N}\big\}.$$

Our goal in the rest of the section is to present some cases in which we may get the reversed inequality to the one given in Theorem 2.1. When both theorems are applied, we get an exact formula for the numerical index of some absolute sums. The more general result we are able to prove is the following.

**Theorem 2.5.** *Let $\Lambda$ be a non-empty set and let $E$ be a subspace or $\mathbb{R}^\Lambda$ endowed with an absolute norm. Suppose that there are a subset $A\subseteq S_E$ with $\overline{\mathrm{conv}}(A)=B_E$ and a subset $B\subseteq S_{E'}$ norming for $E$ such that for every $(a_\lambda)\in A$ and every $(b_\lambda)\in B$, there is $\kappa\in\Lambda$ such that*

$$a_\lambda b_\lambda = 0 \quad \textit{if } \lambda\neq\kappa \quad \textit{and} \quad |a_\kappa b_\kappa|=1.$$

*Then, given an arbitrary family $\{X_\lambda \,:\, \lambda\in\Lambda\}$ of Banach spaces,*

$$n\left(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_E\right) \geqslant \inf\big\{n(X_\lambda)\,:\,\lambda\in\Lambda\big\}.$$

*Proof.* Write $X=\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_E$ and $X'=\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda^*\Big]_{E'}\subseteq X^*$. Consider the sets

$$\mathcal{A}=\big\{(a_\lambda x_\lambda)\ :\ a_\lambda\in\mathbb{R}_0^+,\, x_\lambda\in S_{X_\lambda}\,\forall\lambda\in\Lambda,\ (a_\lambda)\in A\big\}\subset S_X,$$
$$\mathcal{B}=\big\{(b_\lambda x_\lambda^*)\ :\ b_\lambda\in\mathbb{R}_0^+,\, x_\lambda^*\in S_{X_\lambda^*}\,\forall\lambda\in\Lambda,\ (b_\lambda)\in B\big\}\subset S_{X'}.$$

Then, it is clear that $\overline{\mathrm{conv}}(\mathcal{A})=B_X$ and that $\mathcal{B}$ is norming for $X$.

Fix $T \in L(X)$ and $\varepsilon > 0$, and write $T = (T_\lambda)$ where $T_\lambda = P_\lambda T \in L(X, X_\lambda)$. We may find $x = (a_\lambda x_\lambda) \in \mathcal{A}$ and $x^* = (b_\lambda x_\lambda^*) \in \mathcal{B}$ such that

$$\|T\| - \varepsilon < |x^*(Tx)| = \left| \sum_{\lambda \in \Lambda} b_\lambda x_\lambda^* \big(T_\lambda\big((a_\lambda x_\lambda)_{\lambda \in \Lambda}\big)\big) \right|.$$

By hypothesis and the definition of $\mathcal{A}$ and $\mathcal{B}$, there is $\kappa \in \Lambda$ such that

$$a_\lambda b_\lambda = 0 \text{ if } \lambda \neq \kappa \quad \text{and} \quad a_\kappa b_\kappa = 1, \tag{1}$$

and using the Bishop-Phelps theorem, we may and do suppose that $x_\kappa^* \in S_{X_\kappa^*}$ attains its norm on an element $\widetilde{x}_\kappa \in S_{X_\kappa}$. We also take $y_\kappa^* \in S_{X_\kappa^*}$ such that $y_\kappa^*(x_\kappa) = 1$. For every $z \in X_\kappa$, we define $\Phi(z) \in X$ by

$$[\Phi(z)]_\lambda = a_\lambda \, y_\kappa^*(z) \, x_\lambda \ \text{ if } \lambda \neq \kappa, \quad \text{and} \quad [\Phi(z)]_\kappa = a_\kappa \, z,$$

which is well-defined since the norm of $E$ is absolute, satisfies $\|\Phi(z)\| \leqslant \|z\|$ for every $z \in X_\kappa$ and that $\Phi(z)$ is linear in $z$. Also, it is clear that $\Phi(x_\kappa) = x$.

We consider the operator $S \in L(X_\kappa)$ given by

$$S(z) = \left[ \sum_{\lambda \neq \kappa} b_\lambda x_\lambda^* \big(T_\lambda(\Phi(z))\big) \right] \widetilde{x}_\kappa \ + \ b_\kappa T_\kappa(\Phi(z)) \qquad \big(z \in X_\kappa\big)$$

and observe that

$$|x_\kappa^*(Sx_\kappa)| = \left| \sum_{\lambda \neq \kappa} b_\lambda x_\lambda^* \big(T_\lambda(\Phi(x_\kappa))\big) + b_\kappa x_\kappa^* \big(T_\kappa(\Phi(x_\kappa))\big) \right| = |x^*(Tx)| > \|T\| - \varepsilon,$$

so $\|S\| > \|T\| - \varepsilon$. It follows that $v(S) > n(X_\kappa)(\|T\| - \varepsilon)$ and so there is $(\zeta, \zeta^*) \in \Pi(X_\kappa)$ such that

$$|\zeta^*(S\zeta)| \geqslant n(X_\kappa)(\|T\| - \varepsilon). \tag{2}$$

Now, we consider $\Psi(\zeta^*) \in X' \subset X^*$ given by

$$[\Psi(\zeta^*)]_\lambda = b_\lambda \, \zeta^*(\widetilde{x}_\kappa) \, x_\lambda^* \ \text{ if } \lambda \neq \kappa \quad \text{and} \quad [\Psi(\zeta^*)]_\kappa = b_\kappa \zeta^*,$$

which is well-defined since $E'$ has absolute norm, and satisfies $\|\Psi(\zeta^*)\|_{X^*} \leqslant 1$. We observe that, by (1),

$$[\Psi(\zeta^*)](\Phi(\zeta)) = \sum_{\lambda \neq \kappa} b_\lambda a_\lambda \zeta^*(\widetilde{x}_\kappa) y_\kappa^*(\zeta) x_\lambda^*(x_\lambda) + b_\kappa a_\kappa \zeta^*(\zeta) = b_\kappa a_\kappa \zeta^*(\zeta) = \zeta^*(\zeta) = 1$$

and that

$$[\Psi(\zeta^*)]\big(T(\Phi(\zeta))\big) = \left| \left[ \sum_{\lambda \neq \kappa} b_\lambda \, x_\lambda^* \big(T_\lambda(\Phi(\zeta))\big) \right] \zeta^*(\widetilde{x}_\kappa) + b_\kappa \zeta^* \big(T_\kappa(\Phi(\zeta))\big) \right| = |\zeta^*(S\zeta)|.$$

It then follows from (2) that

$$v(T) \geqslant n(X_\kappa)(\|T\| - \varepsilon) \geqslant \inf\{n(X_\lambda) \,:\, \lambda \in \Lambda\}(\|T\| - \varepsilon).$$

Letting $\varepsilon \downarrow 0$ and considering all $T \in L(X)$, we get $n(X) \geqslant \inf\{n(X_\lambda) \,:\, \lambda \in \Lambda\}$. □

Let us list the main consequences of the above theorem. The first one gives a formula for the numerical index of $\ell_1$-sums and $\ell_\infty$-sums. This result appeared in [19, Proposition 1].

**Corollary 2.6.** *Let $\Lambda$ be a nonempty set and let $\{X_\lambda \,:\, \lambda \in \Lambda\}$ be an arbitrary family of Banach spaces. Then*

$$n\left( \Big[ \bigoplus_{\lambda \in \Lambda} X_\lambda \Big]_{\ell_1} \right) = n\left( \Big[ \bigoplus_{\lambda \in \Lambda} X_\lambda \Big]_{\ell_\infty} \right) = \inf\big\{n(X_\lambda) \,:\, \lambda \in \Lambda\big\}.$$

*Proof.* One inequality was proved in Corollary 2.3. To get the reversed inequality, we just show that Theorem 2.5 is applicable. For $E = \ell_1(\Lambda)$, we consider

$$A = \{\chi_{\{\lambda\}} \ : \ \lambda \in \Lambda\} \subset S_E \quad \text{and} \quad B = \{(b_\lambda) \ : \ |b_\lambda| = 1 \ \forall \lambda \in \Lambda\} \subset S_{E'}.$$

Then it is immediate that $\overline{\operatorname{conv}}(A) = S_E$, that $B$ is norming for $E$ and that given $a \in A$ and $b \in B$, there is $\kappa \in \Lambda$ such that

$$a_\lambda b_\lambda = 0 \quad \text{if } \lambda \neq \kappa \quad \text{and} \quad |a_\kappa b_\kappa| = 1.$$

For $E = \ell_\infty(\Lambda)$, we interchange the roles of the sets above and consider

$$A = \{(a_\lambda) \ : \ |a_\lambda| = 1 \ \forall \lambda \in \Lambda\} \subset S_E \quad \text{and} \quad B = \{\chi_{\{\lambda\}} \ : \ \lambda \in \Lambda\} \subset S_{\ell_1(\Lambda)} \subset S_{E'}.$$

Now, $B$ is trivially norming for $\ell_\infty(\Lambda)$ and $\overline{\operatorname{conv}}(A) = B_E$ (indeed, the set of those norm-one functions which have finitely many values is dense in $S_E$ by Lebesgue theorem and the fact that a function taking finitely many values belongs to $\operatorname{conv}(A)$ is easily proved by induction on the number of values). Finally, it is immediate that given $a \in A$ and $b \in B$, there is $\kappa \in \Lambda$ such that

$$a_\lambda b_\lambda = 0 \quad \text{if } \lambda \neq \kappa \quad \text{and} \quad |a_\kappa b_\kappa| = 1. \qquad \square$$

**Remark 2.7.** For $c_0$-sums the result above is also true but it is not possible to prove it using Theorem 2.5 (it is not possible to find sets $A \subset S_E$ and $B \subset S_{E'}$ like that in $E = c_0(\Lambda)$ unless it is finite-dimensional). We have to wait until Section 5 to provide a proof.

Next result allows to calculate the numerical index of $E$-sums of Banach spaces when $E$ has the RNP and $n(E) = 1$. We will use very recent results of H.-J. Lee and the first and second named authors of this paper [14].

**Corollary 2.8.** *Let $\Lambda$ be a non-empty set and let $E$ be a subspace of $\mathbb{R}^\Lambda$ endowed with an absolute norm. Suppose $E$ has the RNP and $n(E) = 1$. Then, given an arbitrary family $\{X_\lambda \ : \ \lambda \in \Lambda\}$ of Banach spaces,*

$$n\left(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_E\right) = \inf\big\{n(X_\lambda) \ : \ \lambda \in \Lambda\big\}.$$

*Proof.* Since $E$ has the RNP, it is order continuous (it does not contain $\ell_\infty$) and so $E^* = E'$. Then inequality $\leqslant$ follows from Corollary 2.2. Let us prove the reversed inequality. Let $A$ be the set of denting points of $B_E$ and let $B$ be the set of extreme points of $B_{E'}$. It follows from the RNP that $\overline{\operatorname{conv}}(A) = B_E$ and Krein-Milman theorem gives that $B$ is norming. It is shown in [14] that, under these hypotheses, given $a = (a_\lambda) \in A$ and $b = (b_\lambda) \in B$, there is $\kappa \in \Lambda$ such that

$$a_\lambda b_\lambda = 0 \quad \text{if } \lambda \neq \kappa \quad \text{and} \quad |a_\kappa b_\kappa| = 1.$$

This allows us to use Theorem 2.5 to get the desired inequality. $\square$

**Remark 2.9.** It follows from the proof of the above result that the hypothesis of RNP is not needed in its full generality. Actually, only two facts are needed: that $n(E) = 1$ and that $\overline{\operatorname{conv}}(A) = B_E$ where $A$ is the set of denting points of $B_E$.

Let us particularize here the above result for Banach spaces with one-unconditional basis (finite or infinite).

**Corollary 2.10.**

(a) *Let $E$ be $\mathbb{R}^m$ endowed with an absolute norm such that $n(E) = 1$, and let $X_1, \ldots, X_m$ be Banach spaces. Then*

$$n\left(\big[X_1 \oplus \cdots \oplus X_m\big]_E\right) = \min\big\{n(X_1), \ldots, n(X_m)\big\}.$$

(b) *Let $E$ be a Banach space with one-unconditional basis, having the RNP and such that $n(E) = 1$. Then, given an arbitrary sequence $\{X_j : j \in \mathbb{N}\}$ of Banach spaces,*

$$n\left(\Big[\bigoplus_{j\in\mathbb{N}} X_j\Big]_E\right) = \inf\big\{n(X_j) : j \in \mathbb{N}\big\}.$$

Let us comment that the proof of Corollary 2.8 in the case of a finite-dimensional space can be done using results of S. Reisner [22] and so, in this case, the very recent reference [14] is not needed.

## 3. Numerical index and one-complemented subspaces

One may wonder if there is any general inequality between the numerical index of a Banach space and the numerical indices of its subspaces (or of some kind of subspaces). Since $n(C(K)) = 1$ and every Banach space contains (isometrically) a $C(K)$-space as subspace (maybe with dimension one) and it is contained (isometrically) in a $C(K)$ space (Banach-Mazur theorem), it is not possible to get any general inequality. If we restrict ourselves to special kind of subspaces, we may show a positive result. Indeed, item (a) of Corollary 2.4 for $m = 2$ shows that the numerical index of a Banach space which is the absolute sum of two subspaces is less or equal than the numerical index of the subspaces. Let us comment that in this case the absolute sum can be written in a different form. Indeed, suppose we have a Banach space $X$ and two subspaces $Y$ and $Z$ such that $X = Y \oplus Z$ and, for every $y \in Y$ and $z \in Z$, the norm of $y + z$ only depends on $\|y\|$ and $\|z\|$. In such a case, it is known that there exists an absolute norm $|\cdot|$ on $\mathbb{R}^2$ such that

$$\|x + z\| = |(\|x\|, \|z\|)| \qquad \big(x \in X,\ z \in Z\big),$$

i.e. $X \equiv [Y \oplus Z]_E$ for $E = (\mathbb{R}^2, |\cdot|)$ and so Corollary 2.4 applies. We refer the reader to [3, § 21] and [21] for background.

**Corollary 3.1.** *Let $X$ be a Banach space and let $Y$, $Z$ be closed subspaces of $X$ such that $X = Y \oplus Z$ and, for every $y \in Y$ and $z \in Z$, $\|y + z\|$ only depends on $\|y\|$ and $\|z\|$. Then*

$$n(X) \leqslant \min\big\{n(Y),\ n(Z)\big\}.$$

Let us comment that the above corollary already appeared in the PhD dissertation (2000) of the first named author and was published (in Spanish) in [17, Proposición 1]. Also, Corollary 2.3 follows from the above corollary since $c_0$-sums and $\ell_p$-sums are associative (i.e. the whole sum is the $c_0$-sum or $\ell_p$-sum of each summand and the sum of the rest of summands). This was the way in which this result was proved in [19]. As a matter of fact, let us comment that the unique associative absolute sums are $c_0$-sums and $\ell_p$-sums [1], and so Theorem 2.1 does not follow from the already known Corollary 3.1.

It is natural to ask whether it would be possible that the hypothesis of absoluteness in Corollary 3.1 can be weakened to general one-complemented subspaces, but we will show that it is not possible. Moreover, we will show that the numerical index of unconditional sums need not be smaller than the numerical indices of the summands, even for projections associated to a one-unconditional and one-symmetric norms in a three-dimensional space. We recall that a closed subspace $Y$ of a Banach space $X$ is said to be an *unconditional summand* of $X$ if there exists another closed subspace $Z$ such that $X = Y \oplus Z$ and $\|y + z\| = \|y + \theta z\|$ for every $y \in Y$, $z \in Z$ and $|\theta| = 1$. It is also said that $X$ is the *unconditional sum* of $Y$ and $Z$. When both $Y$ and $Z$ are one-dimensional, an unconditional sum is actually an absolute sum, but this is not true for higher dimensions.

**Example 3.2.** Let $X$ be the space $\mathbb{R}^3$ endowed with the norm

$$\|(x, y, z)\| = \max\left\{\sqrt{x^2 + y^2},\ \sqrt{x^2 + z^2},\ \sqrt{y^2 + z^2}\right\} \qquad (x, y, z) \in \mathbb{R}^3.$$

Then, the usual basis is one-unconditional and one-symmetric for $X$, $n(X) > 0$ but $n\big(P_2(X)\big) = 0$ ($P_2$ is the projection on the subspace of vectors supported on the first two coordinates).

*Proof.* It is clear that $P_2(X)$ is isometrically isomorphic to the two-dimensional Hilbert space, so $n\big(P_2(X)\big) = 0$. Since $X$ is finite-dimensional, to prove that $n(X) > 0$ it is enough to show that the unique operator $T \in L(X)$ with $v(T) = 0$ is $T = 0$. Let $T$ be an operator with $v(T) = 0$ represented by the matrix $(a_{ij})$. Consider the following norm-one elements in $X$ and $X^*$:

$$x_1 = (1,0,0), \quad x_2 = (0,1,0), \quad x_3 = (0,0,1), \quad x_4 = \frac{1}{\sqrt{2}}(1,1,0),$$
$$x_5 = \frac{1}{\sqrt{2}}(0,1,1), \quad x_6 = \frac{1}{\sqrt{2}}(1,0,1), \quad x_7 = \frac{1}{\sqrt{2}}(1,-1,1), \quad x_8 = \frac{1}{\sqrt{2}}(1,1,1)$$
$$x_1^* = (1,0,0), \quad x_2^* = (0,1,0), \quad x_3^* = (0,0,1),$$
$$x_4^* = \frac{1}{\sqrt{2}}(1,1,0), \quad x_5^* = \frac{1}{\sqrt{2}}(0,1,1), \quad x_6^* = \frac{1}{\sqrt{2}}(1,0,1), \quad x_7^* = \frac{1}{3\sqrt{2}}(1,-1,0) + \frac{2}{3\sqrt{2}}(0,-1,1).$$

Observe that $\|x_7^*\| \leqslant 1$ and $x_7^*(x_7) = 1$.

Since $x_i^*(x_i) = 1$ for $i = 1,2,3$ we have that $a_{ii} = x_i^*(Tx_i) = 0$ for $i = 1,2,3$. Analogously, using that

$$x_4^*(x_4) = 1 = x_5^*(x_5) = x_6^*(x_6) = x_4^*(x_8) = x_6^*(x_8) = x_7^*(x_7)$$

we obtain the following restraints on $a_{ij}$ :

$$\begin{aligned}
0 = x_4^*(Tx_4) &= \frac{1}{2}(a_{12} + a_{21}) \text{ which implies } a_{21} = -a_{12},\\
0 = x_5^*(Tx_5) &= \frac{1}{2}(a_{23} + a_{32}) \text{ which implies } a_{32} = -a_{23},\\
0 = x_6^*(Tx_6) &= \frac{1}{2}(a_{13} + a_{31}) \text{ which implies } a_{31} = -a_{13},\\
0 = x_5^*(Tx_8) &= \frac{1}{2}(-a_{12} - a_{13}) \text{ which implies } a_{13} = -a_{12},\\
0 = x_6^*(Tx_8) &= \frac{1}{2}(a_{12} - a_{23}) \text{ which implies } a_{23} = a_{12},\\
0 = x_7^*(Tx_7) &= \frac{1}{3}a_{12}.
\end{aligned}$$

Therefore, we have that $T = 0$. ☐

We do not know whether the example given above can be adapted to the complex case. Nevertheless, we are able to show a complex example with one-unconditional (not symmetric) norm.

**Example 3.3.** Let us consider the normed space $X$ to be $\mathbb{K}^5$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) endowed with the norm

$$\|(x_1,x_2,x_3,x_4,x_5)\| = \max\big\{|x_1| + |x_2|, |x_2| + |x_3| + |x_5|, |x_3| + |x_4|\big\} \qquad (x_1,x_2,x_3,x_4,x_5) \in \mathbb{K}^5.$$

Then, the usual basis is one-unconditional for $X$, $n(X) = 1$ and $n\big(P_4(X)\big) < 1$.

*Proof.* For the real case, it was proved in [22, §3] that $X$ is a so-called CL-space and that $P_4(X)$ is not. In the finite-dimensional case, this is equivalent to say that $n(X) = 1$ and $n\big(P_4(X)\big) < 1$ (see [11, §3], for instance). For the complex case, it was shown in [12, Proposition 4.3] that the (natural) complexification of a $n$-dimensional normed space with absolute norm (i.e. the usual basis of $\mathbb{R}^n$ is one-unconditional) is a CL-space if and only if the real version is a CL-space. Therefore, $X$ is a (complex) CL-space and $P_4(X)$ is not. As in the real case, this gives that $n(X) = 1$ and $n\big(P_4(X)\big) < 1$. ☐

Using the continuity of the numerical index with respect to the Banach-Mazur distance for equivalent norms [10], it is possible to obtain examples as in 3.2 and 3.3 which are uniformly convex and uniformly smooth.

**Examples 3.4.**

(a) REAL CASE: For $p \geqslant 1$, we consider $X_p = (\mathbb{R}^3, \|\cdot\|_{(p)})$ where

$$\|(x,y,z)\|_{(p)} = 2^{-\frac{1}{p}} \left( (x^2+y^2)^{\frac{p}{2}} + (x^2+z^2)^{\frac{p}{2}} + (y^2+z^2)^{\frac{p}{2}} \right)^{\frac{1}{p}} \qquad (x,y,z) \in \mathbb{R}^3.$$

Then, the usual basis is one-unconditional and one-symmetric for every $X_p$ and $P_2(X_p) \equiv (\mathbb{R}^2, \|\cdot\|_{(p)})$ where

$$\|(x,y)\|_{(p)} = 2^{-\frac{1}{p}} \left( (x^2+y^2)^{\frac{p}{2}} + |x|^p + |y|^p \right)^{\frac{1}{p}} \qquad (x,y) \in \mathbb{R}^2.$$

By the continuity of the numerical index with respect to the Banach-Mazur distance for equivalent norms [10, Proposition 2], one has that

$$\lim_{p\to\infty} n(X_p) = n(X) > 0 \qquad \text{and} \qquad \lim_{p\to\infty} n\big(P_2(X_p)\big) = n(H_2) = 0$$

where $X$ is the three-dimensional space constructed in Example 3.2 and $H_2$ is the two-dimensional real Hilbert space.

Therefore, for $p$ big enough, the uniformly convex and uniformly smooth space $X_p$ satisfies

$$n(X_p) > \frac{1}{2} n(X) \qquad \text{and} \qquad n\big(P_2(X_p)\big) < \frac{1}{2} n(X).$$

Let us comment that the spaces $X_p$ are actually Lorentz spaces.

(b) COMPLEX CASE: Using the same kind of tricks that the ones given above, it is possible to produce uniformly convex and uniformly smooth versions of Example 3.3. Therefore, there is a 5-dimensional complex uniformly convex and uniformly smooth normed space $X$ with one-unconditional basis such that $n(X) > n\big(P_4(X)\big)$.

## 4. KÖTHE-BOCHNER FUNCTION SPACES

Let $(\Omega, \Sigma, \mu)$ be a complete $\sigma$-finite measure space. We denote by $L_0(\mu)$ the space of all (equivalent classes modulo equality a.e. of) $\Sigma$-measurable locally integrable real-valued functions on $\Omega$. A *Köthe function space* is a linear subspace $E$ of $L_0(\mu)$ endowed with a complete norm $\|\cdot\|_E$ and satisfying the following conditions.

(i) If $|f| \leqslant |g|$ a.e. on $\Omega$, $g \in E$ and $f \in L_0$, then $f \in E$ and $\|f\|_E \leqslant \|g\|_E$.
(ii) For every $A \in \Sigma$ with $0 < \mu(A) < \infty$, the characteristic function $\chi_A$ belongs to $E$.

We refer the reader to the classical book by J. Lindenstrauss and L. Tzafriri [16] for more information and background on Köthe function spaces. Let us recall some useful facts about these spaces which we will use in the sequel. First, $E$ is a Banach lattice in the pointwise order. The *Köthe dual* $E'$ of $E$ is the function space defined as

$$E' = \left\{ g \in L_0(\mu) \ : \ \|g\|_{E'} := \sup_{f\in B_E} \int_\Omega |fg|\, d\mu < \infty \right\},$$

which is again a Köthe space on $(\Omega, \Sigma, \mu)$. Every element $g \in E'$ defines naturally a continuous linear functional on $E$ by the formula

$$f \longmapsto \int_\Omega fg\, d\mu \qquad (f \in E),$$

so we have $E' \subseteq E^*$ and this inclusion is isometric.

Let $E$ be a Köthe space on a complete $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$ and let $X$ be a real or complex Banach space. A function $f : \Omega \longrightarrow X$ is said to be *simple* if $f = \sum_{i=1}^n x_i\, \chi_{A_i}$ for some $x_1, \ldots, x_n \in X$ and some $A_1, \ldots, A_n \in \Sigma$. The function $f$ is said to be *strongly measurable* if there is a sequence of simple functions $\{f_n\}$ with $\lim \|f_n(t) - f(t)\|_X = 0$ for almost all $t \in \Omega$. We write $E(X)$ for the space of those strongly measurable functions $f : \Omega \longrightarrow X$ such that the function

$$t \longmapsto \|f(t)\|_X \qquad (t \in \Omega)$$

belongs to $E$, and we endow $E(X)$ with the norm

$$\|f\|_{E(X)} = \big\|t \longmapsto \|f(t)\|_X\big\|_E.$$

Then $E(X)$ is a real or complex (depending on $X$) Banach space and it is called a *Köthe-Bochner function space*. We refer the reader to the recent book by P.-K. Lin [15] for background. Let us introduce some notation and recall some useful facts which we will use in the sequel. For an element $f \in E(X)$ we write $|f| \in E$ for the function $|f|(\cdot) = \|f(\cdot)\|_X$, we consider a measurable function $\widetilde{f} : \Omega \longrightarrow S_X$ such that $f = |f|\,\widetilde{f}$ a.e. and we observe that $\|f\|_{E(X)} = \|\,|f|\,\|_E$.

We write $E'(X^*, w^*)$ to denote the space of $w^*$-scalarly measurable functions $\Phi : \Omega \longrightarrow X^*$ such that $\|\Phi(\cdot)\|_{X^*} \in E'$, which act on $E(X)$ as integral functionals:

$$\langle \Phi, f\rangle = \int_\Omega \langle \Phi(t), f(t)\rangle \, d\mu(t) \qquad \big(f \in E(X)\big).$$

For an integral functional $\Phi \in E'(X^*, w^*)$, we write $|\Phi| \in E'$ for the function $|\Phi|(\cdot) = \|\Phi(\cdot)\|_{X^*}$ and we consider a $w^*$-scalarly measurable function $\widetilde{\Phi} : \Omega \longrightarrow S_{X^*}$ such that $\Phi = |\Phi|\,\widetilde{\Phi}$.

Our first result gives an inequality between the numerical index of $E(X)$ and $n(X)$, provided that there are sufficiently many integral functionals.

**Theorem 4.1.** *Let $(\Omega, \Sigma, \mu)$ be a complete $\sigma$-finite measure space, let $E$ be a Köthe space on $(\Omega, \Sigma, \mu)$, and let $X$ be a Banach space. Suppose that there is a dense subset $\mathcal{A}$ of $S_{E(X)}$ such that for every $f \in \mathcal{A}$ there is $\Phi_f \in E'(X^*, w^*)$ satisfying $\|\,|\Phi_f|\,\|_{E'} = \langle \Phi_f\,,\, f\rangle = 1$. Then*

$$n\big(E(X)\big) \leqslant n(X).$$

*Proof.* We take an operator $S \in L(X)$ with $\|S\| = 1$, and define $T \in L(E(X))$ by

$$[T(f)](t) = S(f(t)) \,\Big[= |f|(t)\, S(\widetilde{f}(t))\Big] \qquad \big(t \in \Omega,\ f \in E(X)\big).$$

We claim that $T$ is well-defined and $\|T\| = 1$. Indeed, for $f \in E(X)$, $T(f)$ is strongly measurable and

$$\|[T(f)](t)\|_X = |f|(t)\, \|S(\widetilde{f}(t))\| \leqslant |f|(t) \qquad (t \in \Omega),$$

so $T(f) \in E(X)$ with $\|T(f)\|_{E(X)} \leqslant \|\,|f|\,\|_E = \|f\|_{E(X)}$. This gives $\|T\| \leqslant 1$. Conversely, we fix $A \in \Sigma$ such that $0 < \mu(A) < \infty$ and for each $x \in S_X$ consider $f = \|\chi_A\|_E^{-1}\, x\, \chi_A \in S_{E(X)}$. Then, $\|f\| = 1$ and

$$\|[T(f)](t)\|_X = \frac{\chi_A(t)\|S(x)\|_X}{\|\chi_A\|_E}, \qquad \text{so} \qquad \|T\| \geqslant \|T(f)\|_{E(X)} = \left\|\frac{\chi_A\, \|S(x)\|_X}{\|\chi_A\|_E}\right\|_E \geqslant \|S(x)\|_X.$$

By just taking supremum on $x \in S_X$, we get $\|T\| \geqslant \|S\| = 1$ as desired.

Next, we consider $f \in \mathcal{A}$ and observe that

$$\begin{aligned} 1 = \langle \Phi_f, f\rangle &= \int_\Omega \langle \Phi_f(t), f(t)\rangle \, d\mu(t) = \int_\Omega |\Phi_f|(t)\, |f|(t)\, \langle \widetilde{\Phi_f}(t), \widetilde{f}(t)\rangle \; d\mu(t) \\ &\leqslant \int_\Omega |\Phi_f|(t)\, |f|(t)\; d\mu(t) \leqslant \langle |\Phi_f|, |f|\rangle \leqslant \|\,|\Phi_f|\,\|_{E'}\, \|\,|f|\,\|_E = 1. \end{aligned}$$

It follows that

$$\langle \widetilde{\Phi_f}(t), \widetilde{f}(t)\rangle = 1 \ \text{ a.e.} \qquad \text{and} \qquad \int_\Omega |\Phi_f|(t)\, |f|(t)\; d\mu(t) = 1.$$

On the other hand,

$$|\langle \Phi_f, T(f)\rangle| = \left|\int_\Omega \langle \Phi_f(t), S(f(t))\rangle \, d\mu(t)\right| = \left|\int_\Omega |\Phi_f|(t)\, |f|(t)\, \langle \widetilde{\Phi_f}(t), S(\widetilde{f}(t))\rangle \, d\mu(t)\right|$$
$$\leqslant \int_\Omega |\Phi_f|(t)\, |f|(t)\, \left|\langle \widetilde{\Phi_f}(t), S(\widetilde{f}(t))\rangle\right| \, d\mu(t) \leqslant \int_\Omega |\Phi_f|(t)\, |f|(t)\, v(S)\, d\mu(t) = v(S).$$

Since $\mathcal{A}$ is dense in the unit sphere of $E(X)$, it follows from Lemma 1.1 that the above inequality implies that $v(T) \leqslant v(S)$. Therefore, $n\big(E(X)\big) \leqslant v(S)$. In view of the arbitrariness of $S \in L(X)$ with $\|S\| = 1$, we get $n\big(E(X)\big) \leqslant n(X)$, as desired. □

The main application of the above theorem concerns order continuous Köthe spaces. We say that a Köthe space $E$ is *order continuous* if $0 \leqslant x_\alpha \downarrow 0$ and $x_\alpha \in E$ imply that $\lim \|x_\alpha\| = 0$ (this is known to be equivalent to the fact that $E$ does not contain an isomorphic copy of $\ell_\infty$). If $E$ is order continuous, the inclusion $E' \subseteq E^*$ is surjective (so $E^*$ completely identifies with $E'$) and the set of those simple functions belonging to $E(X)$ is norm-dense in $E(X)$.

**Corollary 4.2.** *Let $(\Omega, \Sigma, \mu)$ be a complete $\sigma$-finite measure space and let $E$ be an order continuous Köthe space. Then, for every Banach space $X$,*

$$n\big(E(X)\big) \leqslant n(X).$$

*Proof.* We consider $\mathcal{A}$ to denote the set of norm-one simple functions belonging to $E(X)$. Consider $f \in \mathcal{A}$, which is of the form

$$f = \sum_{j=1}^{m} a_j\, x_j\, \chi_{A_j} \in E(X),$$

where $m \in \mathbb{N}$, $a_j \geqslant 0$, $x_j \in S_X$, $A_1, \ldots, A_m \in \Sigma$ are pairwise disjoint, and $|f| = \sum a_j \chi_{A_j} \in S_E$. Since $E$ is order-continuous, $E^* = E'$ and so we may find a positive function $\varphi \in E'$ with $\|\varphi\|_{E'} = 1$ such that

$$1 = \langle f, \varphi\rangle = \int_\Omega f\varphi \, d\mu = \sum_{j=1}^{m} \int_{A_j} \varphi(t) a_j \, d\mu(t).$$

Now, for $j = 1, \ldots, m$, we choose $x_j^* \in S_{X^*}$ such that $x_j^*(x_j) = 1$ and consider $\Phi_f \in E'(X^*, w^*)$ defined by

$$\langle \Phi_f, g\rangle = \sum_{j=1}^{m} \int_{A_j} \varphi(t) x_j^*\big(g(t)\big)\, d\mu(t) \qquad \big(g \in E(X)\big).$$

Then, we have $\|\, |\Phi|\, \|_{E'} = \|\varphi\|_{E'} = 1$ and also

$$\langle \Phi_f, f\rangle = \sum_{j=1}^{m} \int_{A_j} \varphi(t) a_j x_j^*(x_j)\, d\mu(t) = \sum_{j=1}^{m} \int_{A_j} \varphi(t) a_j \, d\mu(t) = 1.$$

Since $E$ is order continuous, $\mathcal{A}$ is dense in $S_{E(X)}$ and we may apply Theorem 4.1.

Alternatively, we can prove this corollary by using the deep result of the theory of Köthe-Bochner spaces that for an order continuous Köthe space $E$ and a Banach space $X$, the whole $E(X)^*$ identifies isometrically with $E'(X^*, w^*)$ (see [15, Theorem 3.2.4]) and, therefore, for every $f \in S_{E(X)}$ there is a norm-one element $\Phi_f$ of $E'(X^*, w^*)$ such that $\langle \Phi_f, f\rangle = 1$. □

Since $L_p(\mu)$ spaces are order continuous Köthe spaces for $1 \leqslant p < \infty$, as an immediate consequence of Corollary 4.2 we obtain the following corollary. For $p = 1$ it appeared in [19] and for $1 < p < \infty$ it appeared in [9].

**Corollary 4.3.** *Let $(\Omega, \Sigma, \mu)$ be a complete $\sigma$-finite measure space, $1 \leqslant p < \infty$, and let $X$ be a Banach space. Then*

$$n\big(L_p(\mu, X)\big) \leqslant n(X).$$

The Köthe space $L_\infty(\mu)$ is not order continuous in the infinite-dimensional case, so Corollary 4.2 does not cover this case. Anyway, we may apply directly Theorem 4.1 to get the corresponding result. The following statement was proved in [20, Theorem 2.3].

**Corollary 4.4.** *Let $(\Omega, \Sigma, \mu)$ be a complete $\sigma$-finite measure space and let $X$ be a Banach space. Then*
$$n\big(L_\infty(\mu, X)\big) \leqslant n(X).$$

*Proof.* Since every element of $L_\infty(\mu, X)$ has essentially separable range, it is possible to show that the subset
$$\mathcal{A} = \{x\,\chi_A + g\,\chi_{\Omega\setminus A}\ :\ x \in S_X,\ g \in B_{L_\infty(\mu,X)},\ A \in \Sigma \text{ with } 0 < \mu(A) < \infty\}$$
is dense in the unit sphere of $L_\infty(\mu, X)$. For every $f \in \mathcal{A}$, write $f = x\,\chi_A + g\chi_{\Omega\setminus A}$, pick $x^* \in S_{X^*}$ such that $x^*(x) = 1$ and observe that the function
$$\Phi_f = \frac{1}{\mu(A)}\, x^*\,\chi_A$$
belongs to $L_1(\mu, X^*) \subset [L_\infty(\mu)]'(X^*, w^*)$, has norm one and $\langle \Phi_f, f\rangle = 1$. Then, the hypotheses of Theorem 4.1 are satisfied and so $n\big(L_\infty(\mu, X)\big) \leqslant n(X)$. □

For $E = L_1(\mu)$ and $E = L_\infty(\mu)$, it is actually known that $n\big(E(X)\big) = n(X)$ for every Banach space $X$ [19, 20]. It would be interesting to study for which Köthe spaces $E$ the above equality also holds.

## 5. Banach spaces with a dense increasing family of one-complemented subspaces

Our goal in this section is to show that the numerical index of a Banach space which contains a dense increasing union of one-complemented subspaces is greater or equal than the limit superior of the numerical indices of those subspaces, and to provide some consequences of this fact. We need some notation. Recall that a *directed set* (or *filtered set*) is a set $I$ endowed with an partial order $\leqslant$ such that for every $i, j \in I$, there is $k \in I$ such that $i \leqslant k$ and $j \leqslant k$.

**Theorem 5.1.** *Let $Z$ be a Banach space, let $I$ be a directed set, and let $\{Z_i\ :\ i \in I\}$ be an increasing family of one-complemented closed subspaces such that $Z = \overline{\bigcup_{i\in I} Z_i}$. Then,*
$$n(Z) \geqslant \limsup_{i\in I} n(Z_i).$$

*Proof.* Let us denote by $P_i : Z \longrightarrow Z_i$ the norm-one projection from $Z$ onto $Z_i$ and by $J_i : Z_i \longrightarrow Z$ the natural inclusion. Then,
$$\|P_i\| = \|J_i\| = 1, \quad P_i \circ J_i = \mathrm{Id}_{Z_i} \qquad (i \in I).$$
We fix $\varepsilon > 0$ and take an operator $T \in L(Z)$ such that
$$\|T\| = 1 \qquad \text{and} \qquad v(T) \leqslant (1+\varepsilon)\, n(Z).$$
For each $i \in I$, consider the operator $S_i = P_i\, T\, J_i \in L(Z_i)$.

*Claim.* $v(S_i) \leqslant v(T)$.
Indeed, for $(y, y^*) \in \Pi(Z_i)$, one has
$$\begin{aligned} |\langle y^*, S_i(y)\rangle| &= |\langle y^*, P_i(T(J_i(y)))\rangle| \\ &= |\langle P_i^*(y^*), T(J_i(y))\rangle| \leqslant v(T), \end{aligned}$$
where the last inequality follows from the fact that $\|P_i^*(y^*)\| \leqslant 1$, $\|J_i(y)\| \leqslant 1$ and
$$\langle P_i^*(y^*), J_i(y)\rangle = \langle y^*, [P_i \circ J_i](y)\rangle = \langle y^*, y\rangle = 1.$$
*Claim.* $\lim_{i\in I} \|S_i\| = 1$.
Indeed, on the one hand,
$$\|S_i\| = \|P_i T J_i\| \leqslant \|P_i\|\|T\|\|J_i\| = 1 \quad (i \in I).$$

On the other hand, for $\delta > 0$ fixed, we take $x \in S_Z$ such that $\|Tx\| > 1 - \delta/4$. Since the family $\{Z_i \,:\, i \in I\}$ is increasing and its union is dense, we may find $i_0 \in I$ such that for $i \geqslant i_0$ there are $y, z \in Z_i$ such that

$$\|Tx - J_i(z)\| < \delta/4 \qquad \text{and} \qquad \|x - J_i(y)\| < \delta/4.$$

Then, we have that

$$\|z\| = \|J_i(z)\| \geqslant \|Tx\| - \|J_i(z) - Tx\| > 1 - \delta/4 - \delta/4 = 1 - \delta/2$$

and

$$\begin{aligned}\|S_i(y)\| &= \|[P_iTJ_i](y)\| \\ &= \big\|P_i(J_i(z)) - \big[P_i(J_i(z)) - P_i(Tx)\big] - \big[P_i(Tx) - P_i(T(J_i(y)))\big]\big\| \\ &\geqslant \|z\| - \|P_i\|\|J_i(z) - Tx\| - \|P_i\|\|T\|\|x - J_i(y)\| \\ &> 1 - \delta/2 - \delta/4 - \delta/4 = 1 - \delta.\end{aligned}$$

Since $\|y\| = \|J_i(y)\| \leqslant 1 + \delta/4$, it follows that

$$\|S_i\| \geqslant \frac{1-\delta}{1+\delta/4}.$$

This gives $\lim\limits_{i\in I} \|S_i\| = 1$, as claimed.

To finish the proof, we just observe that for every $i \in I$, we have that

$$(1+\varepsilon)n(Z) \geqslant v(T) \geqslant v(S_i) \geqslant n(Z_i)\,\|S_i\|$$

and, therefore,

$$(1+\varepsilon)n(Z) \geqslant \limsup_{i\in I}\big[n(Z_i)\,\|S_i\|\big] = \limsup_{i\in I} n(Z_i)\ \lim_{i\in I}\|S_i\| = \limsup_{i\in I} n(Z_i).$$

The result follows by just taking $\varepsilon \downarrow 0$. □

The easiest particular case of the above result is to Banach spaces with a monotone basis (i.e. a basis whose basic constant is 1).

**Corollary 5.2.** *Let $Z$ be a Banach space with a monotone basis $(e_m)$ and for each $m \in \mathbb{N}$, let $Z_m = \operatorname{span}\{e_k \,:\, 1 \leqslant k \leqslant m\}$. Then*

$$n(Z) \geqslant \limsup_{m\to\infty} n(Z_m).$$

It is known, see Section 6, that if $X = \ell_p$ then the inequality above is actually an equality, but we do not know whether the same is true in any other type of spaces.

**Problem 5.3.** Let $Z$ be a Banach space with a monotone (or even one-unconditional, one-symmetric) basis $\{e_m\}_{m\in\mathbb{N}}$, and let $Z_m = \operatorname{span}\{e_k \,:\, 1 \leqslant k \leqslant m\}$. Is it true that $n(Z) = \limsup\limits_{m\to\infty} n(Z_m)$?

Next examples show that the inequality in Theorem 5.1 may be strict.

**Examples 5.4.**

(a) Real case: Let $X$ be the three-dimensional real space given in Example 3.2 such that

$$n(X) > n\big(P_2(X)\big) = 0,$$

where $P_2(X)$ is the subspace of $X$ spanned by the two first coordinates. Now, consider the space $Z = \ell_1(X)$ and for each $m \in \mathbb{N}$, we consider the subspace

$$\begin{aligned} Z_m &= \big\{x = (x_k) \in Z \ :\ x_{m+1}(3) = 0,\ x_k = 0\ \forall k \geqslant m+2\big\} \\ &\equiv X \oplus_1 \overset{m}{\cdots} \oplus_1 X \oplus_1 P_2(X). \end{aligned}$$

Then, we have $n(Z) = n(X) > 0$ and $n(Z_m) = n\big(P_2(X)\big) = 0$ by Corollary 2.6. Observe that each $Z_m$ is one-complemented in $Z$, the sequence $\{Z_m\}$ is increasing and $Z = \overline{\bigcup_{m\in\mathbb{N}} Z_m}$. But

$n(Z) > \limsup_{m\to\infty} n(Z_m)$. Let us comment that the space $Z$ has a monotone basis and that the subspaces $Z_m$ are actually the range of some of the projections associated to the basis.

(b) COMPLEX CASE: If we use the five-dimensional complex space $X$ of Example 3.3, we can repeat the proof above to get the same kind of example in the complex case.

Let us present here applications of Theorem 5.1. The first one allows to calculate the numerical index of $L_1(\mu, X)$. This result appeared in [19, Theorem 8].

**Corollary 5.5.** *Let $(\Omega, \Sigma, \mu)$ be a complete positive measure space and let $X$ be a Banach space. Then $n\big(L_1(\mu, X)\big) = n(X)$.*

*Proof.* Set $Z = L_1(\mu, X)$. We write $I$ for the family of all finite collections of pairwise disjoint elements of $\Sigma$ with finite measure, ordered by $\pi_1 \leqslant \pi_2$ if and only if each element in $\pi_1$ is a union of elements in $\pi_2$. Then $I$ is a directed set. For $\pi \in I$, we write $Z_\pi$ for the subspace of $Z$ consisting of all simple functions supported in the elements of $\pi$. Now, for every $\pi \in I$, the subspace $Z_\pi$ is isometrically isomorphic to $\ell_1^m(X)$ ($m$ is the number of elements in $\pi$, see [5, Lemma II.2.1] for instance), it is one-complemented by the conditional expectation associated to the partition $\pi$ and, finally, the density of simple functions on $L_1(\mu, X)$ gives that $Z = \overline{\bigcup_{\pi\in I} Z_\pi}$. Then, Theorem 5.1 applies and it follows that

$$n\big(L_1(\mu, X)\big) \geqslant \limsup_{\pi\in I} n(Z_\pi) = \limsup_{\pi\in I} n\big(\ell_1^m(X)\big) = n(X),$$

where the last equality above follows from Corollary 2.6. To get the reversed inequality, we start by using that there is a family of finite measure spaces $\{\nu_j \,:\, j \in J\}$ such that

$$L_1(\mu, X) \equiv \Big[\bigoplus_{j\in J} L_1(\nu_j, X)\Big]_{\ell_1}$$

(for $X = \mathbb{K}$ there is a proof of this fact in [4, p. 501] which obviously extends to any arbitrary space $X$). Then pick any $j \in J$ and use Corollary 2.3 to get that $n\big(L_1(\mu, X)\big) \leqslant n\big(L_1(\nu_j, X)\big)$. Since $\nu_j$ is finite, we may now apply Corollary 4.3 to get $n\big(L_1(\nu_j, X)\big) \leqslant n(X)$. ☐

In Section 6 we will use arguments similar to the ones in the above proof (and some more) to get a result on $n\big(L_p(\mu, X)\big)$ for $1 < p < \infty$.

Next application of Theorem 5.1 is to absolute sums of Banach spaces. As a particular case, we will calculate the numerical index of a $c_0$-sum of Banach spaces. We need some notation. Given a nonempty set $\Lambda$, let $\mathcal{I}_\Lambda$ denote the collection of all finite subsets of $\Lambda$ ordered by inclusion. Given a linear subspace $E$ of $\mathbb{R}^\Lambda$ for every $A \in \mathcal{I}_\Lambda$ we consider $E_A$ to be the subspace of $E$ consisting of those elements of $E$ supported in $A$ and observe that $E_A$ is a linear subspace of $\mathbb{R}^A$ with an absolute norm (the restriction of the one in $E$).

**Corollary 5.6.** *Let $\Lambda$ be a nonempty set and let $E$ be a linear subspace of $\mathbb{R}^\Lambda$ with an absolute norm. Suppose that there is a subset $I$ of $\mathcal{I}_\Lambda$, which is still directed by the inclusion, satisfying that $\bigcup_{A\in I} E_A$ is dense in $E$ and that $n(E_A) = 1$ for every $A \in I$. Then, given an arbitrary family $\{X_\lambda \,:\, \lambda \in \Lambda\}$ of Banach spaces,*

$$n\left(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_E\right) = \inf\big\{n(X_\lambda) \,:\, \lambda \in \Lambda\big\}.$$

*Proof.* We write $Z = \Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_E$ and for each $A = \{\lambda_1, \ldots, \lambda_m\} \in I$ we write $Z_A$ for the $E$-sum of the family $\{Y_\lambda\}$ where $Y_\lambda = X_\lambda$ for $\lambda \in \{\lambda_1, \ldots, \lambda_m\}$ and $Y_\lambda = \{0\}$ otherwise (we will use the agreement that $n(\{0\}) = 1$). We observe that each $Z_A$ is one-complemented in $Z$ and that $Z = \overline{\bigcup_{A\in I} Z_A}$. Therefore, Theorem 5.1 applies and provides that

$$n(Z) \geqslant \limsup_{A\in I} n(Z_A).$$

Now, we take $A = \{\lambda_1, \dots, \lambda_m\} \in I$, observe that $E_A$ is isometrically isomorphic to $\mathbb{R}^m$ endowed with an absolute norm which we denote by $\widetilde{E_A}$, and also observe that $Z_A$ is isometrically isomorphic to the $\widetilde{E_A}$-sum of $X_{\lambda_1}, \dots, X_{\lambda_m}$. As $n(\widetilde{E_A}) = n(E_A) = 1$ by hypothesis, we may use Corollary 2.10.a to get that

$$n(Z_A) = n\left(\left[X_{\lambda_1} \oplus \cdots \oplus X_{\lambda_m}\right]_{\widetilde{E_A}}\right) = \min\{n(X_{\lambda_1}), \dots, n(X_{\lambda_m})\} \geqslant \inf\{n(X_\lambda) \,:\, \lambda \in \Lambda\}.$$

To get the reversed inequality, we will use Theorem 2.1. For this, we write $\mathcal{A} = \bigcup_{A\in I} S_{E_A}$ which is dense in $S_E$ since $\bigcup_{A\in I} E_A$ is dense in $E$. For every $A = \{\lambda_1, \dots, \lambda_m\} \in I$ and every $a = (a_\lambda) \in S_{E_A}$, we use that $E_A \equiv \widetilde{E_A}$ to write $\widetilde{a} = (a_{\lambda_1}, \dots, a_{\lambda_m}) \in S_{\widetilde{E_A}}$ and consider $\widetilde{b} = (\beta_1, \dots, \beta_m) \in S_{\widetilde{E_A}^*}$ such that $\sum_{j=1}^m \beta_j a_{\lambda_j} = 1$. Now, we define $(b_\lambda) \in E'$ by the formula

$$b_\lambda = \beta_j \ \text{ if } \lambda = \lambda_j \text{ for } j = 1, \dots, m \quad \text{and} \quad b_\lambda = 0 \ \text{ otherwise.}$$

Then, $(b_\lambda) \in S_{E'}$ and

$$\sum_{\lambda\in\Lambda} b_\lambda a_\lambda = \sum_{j=1}^m \beta_j a_{\lambda_j} = 1.$$

This shows that we may use Theorem 2.1 to get

$$n\left(\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_E\right) \leqslant \inf\{n(X_\lambda) \,:\, \lambda \in \Lambda\}. \qquad \square$$

The hypotheses of the above corollary are clearly satisfied by $E = c_0(\Lambda)$ and $E = \ell_1(\Lambda)$. For the second space, the thesis already appeared in our Corollary 2.6. We state the result for $E = c_0(\Lambda)$. It already appeared in [19, Proposition 1].

**Corollary 5.7.** *Let $\Lambda$ be a nonempty set and let $\{X_\lambda \,:\, \lambda \in \Lambda\}$ be an arbitrary family of Banach spaces. Then*

$$n\left(\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{c_0}\right) = \inf\{n(X_\lambda) \,:\, \lambda \in \Lambda\}.$$

Another particular case of Corollary 5.6 appears when we deal with Banach spaces with one-unconditional basis.

**Corollary 5.8.** *let $E$ be a Banach space with a one-unconditional basis, let $(j_m)$ be an increasing sequence of positive integers and let $E_m$ be the subspace of $E$ spanned by the $j_m$ first coordinates. Suppose $n(E_m) = 1$ for every $m \in \mathbb{N}$. Then, for every sequence $\{X_j \,:\, j \in \mathbb{N}\}$ of Banach spaces,*

$$n\left(\left[\bigoplus_{j\in\mathbb{N}} X_\lambda\right]_E\right) = \inf\{n(X_j) \,:\, j \in \mathbb{N}\}.$$

## 6. The numerical index of $L_p$-spaces

Our goal in this section is to deduce from the results of the previous sections that all infinite-dimensional $L_p(\mu)$ spaces have the same numerical index, independent of the measure $\mu$. Actually, we will give the result for vector-valued spaces. Let us say that all the results in this section are already known: they were proved using particular arguments of the $L_p$ spaces in the papers [7, 8, 9] (some of them with additional unnecessary hypotheses on the measure $\mu$). In our opinion, the abstract vision we are developing in this paper allows to understand better the properties of $L_p$-spaces underlying the proofs: $\ell_p$-sums are absolute sums, $L_p$-norms are associative, every measure space can be decomposed into parts of finite measure, every finite measure algebra is isomorphic to the union of homogeneous measure algebras (Maharam's theorem) and, finally, the density of simple functions via the conditional expectation projections.

**Proposition 6.1.** [7, 8, 9] *Let $1 < p < \infty$ and let $X$ be any Banach space.*

(a) *The sequence* $\{n(\ell_p^m(X))\}_{m\in\mathbb{N}}$ *is decreasing and*

$$n(\ell_p(X)) = \lim_{m\to\infty} n(\ell_p^m(X)) \ \Big[= \inf_{m\in\mathbb{N}} n(\ell_p^m(X))\Big].$$

(b) *If* $\mu$ *is a positive measure such that* $L_p(\mu)$ *is infinite-dimensional, then*

$$n\big(L_p(\mu, X)\big) = n\big(\ell_p(X)\big).$$

*In particular, all infinite-dimensional* $L_p(\mu)$ *spaces have the same numerical index.*

*Proof of Proposition 6.1.a.* Since $\ell_p^m(X)$ is an absolute summand of $\ell_p^{m+1}(X)$, the decrease of the sequence $\{n(\ell_p^m(X))\}_{m\in\mathbb{N}}$ follows from Corollary 3.1. By the same reason we have that $n(\ell_p(X)) \leqslant n(\ell_p^m(X))$ for every $m \in \mathbb{N}$, and so $n(\ell_p(X)) \leqslant \lim_{m\to\infty} n(\ell_p^m(X))$ (cf. also Corollary 2.3). For the reversed inequality, we just use Theorem 5.1 with $Z = \ell_p(X)$, $I = \mathbb{N}$ with its natural order, and

$$Z_m = \big\{x \in \ell_p(X) \ : \ x(k) = 0 \ \forall k > m\big\} \equiv \ell_p^m(X).$$

(Observe that not only $Z_m$ is one-complemented in $Z$ but, actually, the natural projection is absolute. In this case, some of the arguments in the proof of Theorem 5.1 are simpler.) □

To proof item (b) in Proposition 6.1 we need the following lemma which is well known to experts, but since we have not found any concrete reference in the literature, we include a sketch of its proof for the sake of completeness. We only need the (trivial) decomposition of every measure space into disjoint finite measure spaces and Maharam's result on the decomposition of finite measure algebras into disjoint homogeneous parts.

**Lemma 6.2.** *Let* $(\Omega, \Sigma, \mu)$ *be a measure space and* $1 < p < \infty$ *such that the space* $L_p(\mu)$ *is infinite-dimensional, and let* $X$ *be any non-null Banach space. Then, there is a* $\sigma$*-finite measure* $\tau$ *and a Banach space* $Z$ *such that* $L_p(\tau, X) \neq 0$,

$$L_p(\mu, X) \equiv L_p(\tau, X) \oplus_p Z \qquad \textit{and} \qquad L_p(\tau, X) \equiv \ell_p\big(L_p(\tau, X)\big) \equiv L_p(\tau, \ell_p(X)).$$

*Proof.* It is known that for every measure space $(\Omega, \Sigma, \mu)$, there is a family of finite measure spaces $\{\nu_j \ : \ j \in J\}$ such that

$$L_p(\mu, X) \equiv \Big[\bigoplus_{j\in J} L_p(\nu_j, X)\Big]_{\ell_p}.$$

Indeed, it is enough to consider a maximal family $\{A_j \ : \ j \in J\}$ of pairwise disjoint sets of positive finite measure and $\nu_j$ is just the restriction of $\mu$ to $A_j$ (for $p = 1$ and $X = \mathbb{K}$ there is a proof of this fact in [4, p. 501] which obviously extends to $1 < p < \infty$ and arbitrary space $X$). If for every $j \in J$, $L_p(\nu_j)$ is finite-dimensional, then $L_p(\mu, X) \equiv \ell_p(\Gamma, X)$ for some infinite set $\Gamma$, and the result clearly follows by just taking an infinite and countable subset $\Gamma'$ of $\Gamma$, $L_p(\tau, X) \equiv \ell_p(\Gamma', X)$ and $Z = \ell_p(\Gamma \setminus \Gamma', X)$. Otherwise, pick $\nu = \nu_{j_0}$ to be one of the finite measures $\nu_j$ such that $L_p(\nu)$ is infinite-dimensional and observe that $L_p(\mu, X) \equiv L_p(\nu, X) \oplus_p Z_1$ for some subspace $Z_1$ of $L_p(\mu, X)$.

Now, $\nu$ is a finite positive measure such that $L_p(\nu)$ is infinite-dimensional and we may use Maharam's theorem [13, Theorems 7 and 8] on measure algebras to deduce (as is done in [13, Chapter 5] for the scalar case) that

$$L_p(\nu, X) \equiv \ell_p(\Gamma, X) \oplus_p \Big[\bigoplus_{\alpha\in I} L_p\big([0,1]^{\omega_\alpha}, X\big)\Big]_{\ell_p}$$

for some sets $\Gamma$ and $I$ ($I$ is a set of ordinals). Since $L_p(\nu, X)$ is infinite-dimensional and $\nu$ is finite, $\Gamma$ is countably infinite or $I$ is nonempty. If $\Gamma$ is infinite, just consider a measure $\tau$ such that $L_p(\tau, X) \equiv \ell_p(\Gamma, X)$ and

$$Z \equiv \Big[\bigoplus_{\alpha\in I} L_p\big([0,1]^{\omega_\alpha}, X\big)\Big]_{\ell_p} \oplus_p Z_1$$

and the result clearly follows. Otherwise, $I \neq \emptyset$ and we may take $\alpha_0 \in I$ and consider a measure $\tau$ such that $L_p(\tau, X) \equiv L_p\big([0,1]^{\omega_{\alpha_0}}, X\big)$ and

$$Z \equiv \ell_p(\Gamma, X) \oplus_p \Big[ \bigoplus_{\alpha_0 \neq \alpha \in I} L_p\big([0,1]^{\omega_\alpha}, X\big)\Big]_{\ell_p} \oplus_p Z_1.$$

Then, $L_p(\mu, X) \equiv L_p(\tau, X) \oplus_p Z$ and

$$L_p(\tau, X) \equiv \ell_p\big(L_p(\tau, X)\big)$$

by the homogeneity of the measure $\tau$ (see [13, pp. 122 and 128]). Finally, it is straightforward to check that $\ell_p\big(L_p(\tau, X)\big) \equiv L_p(\tau, \ell_p(X))$ by just using the associativity of the $\ell_p$-norm. □

*Proof of Proposition 6.1.b.* We use Lemma 6.2 to write $L_p(\mu, X) \equiv L_p(\tau, X) \oplus_p Z$ where the measure $\tau$ is $\sigma$-finite and satisfies that $L_p(\tau, X) \equiv L_p(\tau, \ell_p(X))$. First, we deduce from Corollary 2.3 that $n\big(L_p(\mu, X)\big) \leqslant n\big(L_p(\tau, X)\big)$. On the other hand, Corollary 4.3 gives us that $n\big(L_p(\tau, \ell_p(X))\big) \leqslant n\big(\ell_p(X)\big)$. Summarizing, we get

$$n\big(L_p(\mu, X)\big) \leqslant n\big(L_p(\tau, X)\big) = n\big(L_p(\tau, \ell_p(X))\big) \leqslant n\big(\ell_p(X)\big).$$

To get the reversed inequality, we argue in the same way that we did in the proof of Corollary 5.5. Let $I$ be the family of all finite collections of pairwise disjoint elements of $\Sigma$ with finite measure, ordered by $\pi_1 \leqslant \pi_2$ if and only if each element in $\pi_1$ is a union of elements in $\pi_2$. Then $I$ is a directed set. For $\pi \in I$, we write $Z_\pi$ for the subspace of $Z = L_p(\mu, X)$ consisting of all simple functions supported in the elements of $\pi$. Now, for every $\pi \in I$, the subspace $Z_\pi$ is isometrically isomorphic to $\ell_p^m(X)$-space ($m$ is the number of elements in $\pi$, see [5, Lemma II.2.1] for the case $p = 1$ and a finite measure, but the proof is the same in the general case), it is one-complemented by the conditional expectation associated to the partition $\pi$ and, finally, the density of simple functions on $L_p(\mu, X)$ gives that $Z = \overline{\bigcup_{\pi \in I} Z_\pi}$. Then, Theorem 5.1 applies and it follows that

$$n\big(L_p(\mu, X)\big) \geqslant \limsup_{\pi \in I} n(Z_\pi) \geqslant \inf_{m \in \mathbb{N}} n(\ell_p^m(X)) = n(\ell_p(X)). \qquad \square$$

**Remark 6.3.** Let us comment that the proof of Proposition 6.1.b given here can be heavily simplified for concrete measure spaces for which Lemma 6.2 can be avoided. For instance, the fact that $n\big(L_p[0,1]\big) = n(\ell_p)$ follows immediately from Corollary 5.2 (using the Haar system as monotone basis of $L_p[0,1]$) and Corollary 4.3 (using that $L_p\big([0,1], \ell_p\big) \equiv L_p[0,1]$).

## References


[1] E. Behrends et al., *$L^p$-structure in real Banach spaces*, Lecture Notes in Math. **613**, Springer-Verlag, Berlin 1977. 3

[2] F. F. Bonsall and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*, London Math. Soc. Lecture Note Series **2**, Cambridge, 1971. 1, 1.1

[3] F. F. Bonsall and J. Duncan, *Numerical Ranges II*, London Math. Soc. Lecture Note Series **10**, Cambridge, 1973. 1, 3

[4] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland Math. Studies **176**, Amsterdam 1993. 5, 6

[5] J. Diestel and J. J. Uhl, *Vector Measures*, Math. Surveys **15**, AMS, Providence 1977. 5, 6

[6] J. Duncan, C. McGregor, J. Pryce, and A. White, The numerical index of a normed space, *J. London Math. Soc.* **2** (1970), 481–488. 1

[7] E. Ed-dari, On the numerical index of Banach spaces, *Linear Algebra Appl.* **403** (2005), 86–96. 1, 6, 6.1

[8] E. Ed-dari and M. Khamsi, The numerical index of the $L_p$ space, *Proc. Amer. Math. Soc.* **134** (2006), 2019–2025. 1, 6, 6.1

[9] E. Ed-dari, M. Khamsi, and A. Aksoy, On the numerical index of vector-valued function spaces, *Linear Mult. Algebra* **55** (2007), 507–513. 1, 4, 6, 6.1

[10] C. Finet, M. Martín, and R. Payá, Numerical index and renorming, *Proc. Amer. Math. Soc.* **131** (2003), 871–877. 3, 3.4

[11] V. Kadets, M. Martín, and R. Payá, Recent progress and open questions on the numerical index of Banach spaces, *Rev. R. Acad. Cien. Serie A. Mat.* **100** (2006), 155–182. 1, 3

[12] J. Kim and H. J. Lee, Strong peak points and strongly norm attaining points with applications to denseness and polynomial numerical indices, *J. Funct. Anal.* **257** (2009), 931–947. 3

[13] H. E. Lacey, *The Isometric Theory of Classical Banach Spaces*, Springer-Verlag, Berlin etc., 1974. 6

[14] H.-J. Lee, M. Martín, and J. Merí, Polynomial numerical index of Banach spaces with absolute norm, (in preparation). 2, 2, 2

[15] P.-K. Lin, *Köthe-Bochner Function Spaces*, Birkhäuser, Boston, 2004. 4, 4

[16] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces. II*, Springer–Verlag, Berlin–Heidelberg–New York, 1979. 2, 4

[17] M. Martín, Índice numérico y subespacios (Spanish), *Proceedings of the Meeting of Andalusian Mathematicians (Sevilla, 2000)*, Vol. II, pp. 641–648, Colecc. Abierta **52**, Univ. Sevilla Secr. Publ., Sevilla, 2001. 3

[18] M. Martín, J. Merí and M. Popov, On the numerical index of real $L_p(\mu)$-spaces, *Israel J. Math.* (to appear). Available at `http://arxiv.org/abs/0903.2704` 1

[19] M. Martín and R. Payá, Numerical index of vector-valued function spaces, *Studia Math.* **142** (2000), 269–280. 2, 2, 3, 4, 4, 5, 5

[20] M. Martín and A. R. Villena, Numerical index and Daugavet property for $L_\infty(\mu, X)$, *Proc. Edinburgh Math. Soc.* **46** (2003), 415–420. 4, 4

[21] J. F. Mena, R. Payá, A. Rodríguez-Palacios, and D. Yost, Absolutely proximinal subspaces of Banach spaces, *J. Aprox. Theory* **65** (1991), 46–72. 3

[22] S. Reisner, Certain Banach spaces associated with graphs and CL-spaces with 1-unconditonal bases, *J. London Math. Soc* **43** (1991), 137–148. 2, 3

(Martín & Merí) Departamento de Análisis Matematico, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain

*E-mail address*: `mmartins@ugr.es` `jmeri@ugr.es`

(Popov) Department of Mathematics, Chernivtsi National University, str. Kotsjubyn'skogo 2, Chernivtsi, 58012 Ukraine

*E-mail address*: `misham.popov@gmail.com`

(Randrianantoanina) Department of Mathematics, Miami University, Oxford, OH 45056, USA

*E-mail address*: `randrib@muohio.edu`