EXTREMELY NON-COMPLEX BANACH SPACES

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Abstract. A Banach space X is said to be an extremely non-complex space if the norm equality $\|\operatorname{Id}+T^2\| = 1 + \|T^2\|$ holds for every bounded linear operator T on X. We show that every extremely non-complex Banach space has positive numerical index, it does not have an unconditional basis and that the infimum of the diameters of the slices of its unit ball is positive.

For a real Banach space X, we write S_X for its unit sphere and B_X for its unit ball. The dual space of X is denoted by X^* and $L(X)$ is the space of all (bounded linear) operators from X to X.

A Banach space X is said to be *extremely non-complex* if the norm equality

$$\|\operatorname{Id}+T^2\| = 1 + \|T^2\|$$

holds for every $T \in L(X)$. This concept was introduced in [7], where several different examples of $C(K)$ spaces are shown to be extremely non-complex, answering a question possed in [4, Question 4.11]. For instance, this is the case for some perfect compact spaces K constructed by P. Koszmider [6] such that $C(K)$ has few operators (in the sense that every operator is a weak multiplier). In [8] examples of extremely non-complex Banach spaces which are not $C(K)$ spaces were provided and they were used to provide an example of a Banach space whose group of surjective isometries reduces to $\pm\operatorname{Id}$, but the group of surjective isometries of its dual contains the group of isometries of a separable infinite-dimensional Hilbert space as a subgroup.

This notion is closely related to the so-called Daugavet equation. We recall that an operator S defined on a Banach space X satisfies the *Daugavet equation* if

$$\|\operatorname{Id}+S\| = 1 + \|S\|$$

and that the space X has the *Daugavet property* [5] if the Daugavet equation holds for every rank-one operator on X. It is known that a Banach space with the Daugavet property does not have unconditional basis and that every slice of its unit ball has diameter two.

Another somehow related notion is that of numerical index of Banach spaces. The *numerical index* of a Banach space X [1] is given by

$$n(X) = \inf\{v(T) \ : \ T \in L(X),\ \|T\| = 1\}$$

where $v(T)$ stands for the *numerical radius* of the operator T, i.e.

$$v(T) = \sup\{|\lambda| \ : \ \lambda \in V(T)\}$$

and $V(T)$ is the *numerical range* of T, i.e.

$$V(T) = \{x^*(T(x)) \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\}.$$

It is known that a Banach space X satisfies $n(X) = 1$ if and only if for every $T \in L(X)$, T or $-T$ satisfies the Daugavet equation [1] (see also [9, Lemma 2.3]).

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The results of this paper are the following. Let X be an extremely non-complex Banach space. Then, $n(X) > 0$ (actually, there is an uniform lower bound valid for all extremely non-complex spaces), X does not have an unconditional basis, and the infimum of the diameters of the slices of B_X is positive (actually, there is an uniform lower bound valid for all extremely non-complex spaces).

Of course, all these results would be evident if the following two questions had positive answer. Is it true that every extremely non-complex Banach space has numerical index 1? Is it true that every extremely non-complex Banach space satisfies the Daugavet property? We do not know the answer to these questions, which is positive for all known examples of extremely non-complex Banach spaces from [7, 8].

The results

Let us first show that the numerical index of any extremely non-complex Banach space is positive.

Theorem 1. *Let X be an extremely non-complex Banach space. Then,*

$$n(X) \geqslant \sqrt{1+\mathrm{e}^{-2}} - 1.$$

Proof. We claim that for every $T \in L(X)$ with $\|T\| \leqslant 1$, the operators $\mathrm{Id}+T$ and $\mathrm{Id}-T$ satisfy the Daugavet equation.

Indeed, we first observe that if $\|T\| < 1$, then $\mathrm{Id}+T$ and $\mathrm{Id}-T$ have square roots. This is probably well known, but we include a short proof for completeness. Let $\sum_{n\geqslant 0} a_n t^n$ be the power series expansion centered at the origin of the function $t \mapsto (1+t)^{1/2}$ with $t \in]-1,1[$. This power series is absolutely convergent on $]-1,1[$. Thus, the operators given by

$$S_1 = \sum_{n=0}^{+\infty} a_n T^n \qquad \text{and} \qquad S_2 = \sum_{n=0}^{+\infty} a_n (-T)^n$$

are well defined and satisfy $S_1^2 = \mathrm{Id}+T$ and $S_2^2 = \mathrm{Id}-T$ respectively. As X is extremely non-complex, this implies that $\mathrm{Id}+T$ and $\mathrm{Id}-T$ satisfy the Daugavet equation. Finally, using the fact that the set of operators satisfying the Daugavet equation is closed, one obtains the same result for every $T \in L(X)$ with $\|T\| \leqslant 1$.

Now, fixed $T \in L(X)$ with $\|T\| = 1$, one can use [1, Remark in p. 483] or [9, Lemma 2.3] to obtain

$$\|\mathrm{Id}+T\| = \sup V(\mathrm{Id}+T) \leqslant 1 + v(T) \qquad \text{and} \qquad \|\mathrm{Id}-T\| = \sup V(\mathrm{Id}-T) \leqslant 1 + v(T). \tag{1}$$

As $L(X)$ is a Banach algebra, [3, Theorem 2.1] gives us that

$$\max_{\pm} \|\mathrm{Id} \pm T\| - 1 \geqslant \sqrt{1+e^{-2}} - 1$$

which, together with (1) provides

$$v(T) \geqslant \sqrt{1+\mathrm{e}^{-2}} - 1.$$

Taking supremum on $T \in L(X)$ with $\|T\| = 1$, we get $n(X) \geqslant \sqrt{1+\mathrm{e}^{-2}} - 1$, as desired. □

The second result deals with the diameters of the slices of the unit ball of an extremely non-complex space. We need some notation. Let X be a real Banach space. A *slice* of B_X is a nonempty intersection of B_X with an open half-space, i.e. a subset of the form

$$S(B_X, x^*, \alpha) := \{x \in B_X \ : \ x^*(x) > 1 - \alpha\}$$

where $x^* \in S_{X^*}$ and $0 < \alpha \leqslant 1$.

Theorem 2. *Let X be an extremely non-complex Banach space of dimension greater than one. Then, the infimum of the diameter of the slices of B_X is positive.*

Proof. Let $x^* \in S_{X^*}$, $0 < \alpha \leqslant 1$, $S = S(B_X, x^*, \alpha)$, and let d be the diameter of S. We can assume without loss of generality that $d < 1$. Fixed $0 < \varepsilon < \min\{1-d, \alpha\}$, pick $y_0 \in S_X$ such that $x^*(y_0) > 1 - \varepsilon$. Then, the following

$$\|x + ty_0\| > (1 - \varepsilon - d)(\|x\| + |t|) \tag{2}$$

holds for every $x \in \ker x^*$ and $t \in \mathbb{R}$. Indeed, given $x \in S_{\ker x^*}$, use [4, Proposition 4.9] to find $y \in S$ such that $\|x + y\| > 2 - \varepsilon$. Therefore, using that $y, y_0 \in S$, one can write

$$\|x + y_0\| \geqslant \|x + y\| - \|y - y_0\| > 2 - \varepsilon - d.$$

From this it is easy to deduce (2) using some ideas of [5, Lemma 2.8]. We can assume that $x \neq 0$ since otherwise the estimate is trivial. Observe that by the symmetry of $\ker x^*$ it suffices to show the estimate for $t \geqslant 0$. Next, if $\|x\| \geqslant t$, we can write

$$\begin{aligned}
\|x + ty_0\| &= \left\| \frac{x}{\|x\|}\|x\| + ty_0 \right\| \geqslant \|x\| \left\| \frac{x}{\|x\|} + y_0 \right\| - (\|x\| - t)\|y_0\| \\
&\geqslant \|x\|(2 - \varepsilon - d) - (\|x\| - t) \geqslant (1 - \varepsilon - d)(\|x\| + t)
\end{aligned}$$

and an analogous argument gives this estimate if $\|x\| < t$.

Fixed $x_0 \in S_{\ker x^*}$, we use (2) and Hahn's Theorem (see [10, Theorem 1.9.10], for instance) to find $x_0^* \in X^*$ satisfying

$$x_0^*(x_0) = 1, \qquad x_0^*(y_0) = 0, \qquad \text{and} \qquad \|x_0^*\| \leqslant \frac{1}{1 - \varepsilon - d}.$$

Call $a = \frac{1}{\sqrt{2}}$ and define $z_0 = ax_0 - (1-a)y_0$ and $z_0^* = \frac{1-a}{a}x_0^* + x^*$ which satisfy

$$\|z_0\| \geqslant 1 - \varepsilon - d, \qquad \|z_0^*\| \geqslant 1 - \varepsilon, \qquad \text{and} \qquad z_0^*(z_0) \geqslant 0.$$

Therefore, using again [4, Proposition 4.9] we obtain that

$$\|\operatorname{Id} + z_0^* \otimes z_0\| = 1 + \|z_0^*\|\|z_0\| \geqslant 1 + (1 - \varepsilon)(1 - \varepsilon - d). \tag{3}$$

On the other hand, given $x \in \ker x^*$ and $t \in \mathbb{R}$, we can write

$$\begin{aligned}
&\big\|(\operatorname{Id} + z_0^* \otimes z_0)(x + ty_0)\big\| = \\
&\qquad = \left\| x + ty_0 + \left(\frac{1-a}{a} x_0^*(x) + tx^*(y_0) \right) (ax_0 - (1-a)y_0) \right\| \\
&\qquad = \left\| x + (1-a)x_0^*(x)x_0 - \frac{(1-a)^2}{a} x_0^*(x)y_0 + atx^*(y_0)x_0 + t\big(1 - (1-a)x^*(y_0)\big)y_0 \right\| \\
&\qquad \leqslant \|x\| + (1-a)\|x_0^*\|\|x\| + \frac{(1-a)^2}{a}\|x_0^*\|\|x\| + a|t| + |t|\big|1 - (1-a)x^*(y_0)\big| \\
&\qquad \leqslant \left(1 + \frac{1-a}{a}\|x_0^*\| \right) \|x\| + |t|(2a + \varepsilon) \\
&\qquad \leqslant \left(1 + \frac{1-a}{a}\|x_0^*\| + \varepsilon \right) (\|x\| + |t|) \\
&\qquad \leqslant \left(1 + \frac{1-a}{a} \frac{1}{1 - \varepsilon - d} + \varepsilon \right) \frac{1}{1 - \varepsilon - d} \|x + ty_0\|
\end{aligned}$$

where we used (2) in the last inequality. Hence we deduce that

$$\|\operatorname{Id} + z_0^* \otimes z_0\| \leqslant \left(1 + \frac{1-a}{a} \frac{1}{1 - \varepsilon - d} + \varepsilon \right) \frac{1}{1 - \varepsilon - d}$$

which, together with (3) tells us that

$$1 + (1-\varepsilon)(1-\varepsilon-d) \leqslant \left(1 + \frac{1-a}{a}\frac{1}{1-\varepsilon-d} + \varepsilon\right)\frac{1}{1-\varepsilon-d}.$$

Finally, letting $\varepsilon \to 0$, we obtain that

$$2-d \leqslant \left(1 + \frac{\sqrt{2}-1}{1-d}\right)\frac{1}{1-d}$$

which obviously implies that d cannot be arbitrarily close to 0. ☐

Let us note that in the above proof we only use the fact that the norm equality $\|\operatorname{Id}+T^2\| = 1 + \|T^2\|$ holds for every rank-one operator on the space. Therefore, *for every Banach space X satisfying this condition, the infimum of the diameter of the slices of B_X is positive.* Such spaces where studied in [11], where it is shown that their unit balls do not have strongly exposed points. Since the existence of a strongly exposed point of the unit ball gives slices of arbitrary small diameter, our result is an improvement of that.

It is proved in [2, Theorem 2.1] that Banach spaces with the Daugavet property cannot have an unconditional basis. Using the same ideas it is possible to show that extremely non-complex Banach spaces also lack an unconditional basis.

Theorem 3. *Let X be an extremely non-complex Banach space of dimension greater than one. Then, X does not have an unconditional basis.*

Proof. Suppose for the contrary that $\{e_n\}_{n\in\mathbb{N}}$ is an unconditional basis of X with unconditional base constant $K \geqslant 1$. Denote by $\{x_n^*\}_{n\in\mathbb{N}}$ the set of biorthogonal functionals associated to the basis, define for each $n\in\mathbb{N}$ the rank-two operator $T_n \in L(X)$ given by

$$T_n = x_{2n}^* \otimes e_{2n-1} - x_{2n-1}^* \otimes e_{2n},$$

and observe that

$$\begin{aligned} T_n^2(x) &= \big(x_{2n}^* \otimes e_{2n-1} - x_{2n-1}^* \otimes e_{2n}\big)\big(x_{2n}^*(x)e_{2n-1} - x_{2n-1}^*(x)e_{2n}\big) \\ &= -x_{2n-1}^*(x)e_{2n-1} - x_{2n}^*(x)e_{2n} \end{aligned}$$

for every $x \in X$. Therefore, one has

$$-\operatorname{Id} = \sum_{n=1}^{\infty} T_n^2$$

pointwise. Besides, since

$$\sup\left\{\left\|\sum_{i\in A} T_i^2\right\| \ : \ A\subseteq\mathbb{N},\ A \text{ finite}\right\} \leqslant K < \infty$$

it is possible to find a finite set $A_0 \subseteq \mathbb{N}$ such that

$$\sup\left\{\left\|\sum_{i\in A} T_i^2\right\| \ : \ A\subseteq\mathbb{N},\ A \text{ finite}\right\} < \left\|\sum_{i\in A_0} T_i^2\right\| + 1. \tag{4}$$

Taking into account that $\sum_{i\in A_0} T_i^2 = \big(\sum_{i\in A_0} T_i\big)^2$ and using that X is extremely non-complex, one can write

$$\left\|\operatorname{Id} + \sum_{i\in A_0} T_i^2\right\| = 1 + \left\|\sum_{i\in A_0} T_i^2\right\|.$$

Now, let $\{A_n\}_{n\in\mathbb{N}}$ be an increasing sequence of finite subsets of $\mathbb{N}$ satisfying $\cup_{n=1}^{\infty} A_n = \mathbb{N}\setminus A_0$. So, the following holds pointwise:

$$\lim_{n\to\infty} \sum_{i\in A_n} T_i^2 = -\operatorname{Id} - \sum_{i\in A_0} T_i^2$$

and, therefore,

$$\sup_{n\in\mathbb{N}}\left\|\sum_{i\in A_n} T_i^2\right\| \geqslant \left\|\mathrm{Id}+\sum_{i\in A_0} T_i^2\right\| = 1+\left\|\sum_{i\in A_0} T_i^2\right\|$$

which contradicts (4). □

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