POLYNOMIAL NUMERICAL INDICES OF BANACH SPACES WITH 1-UNCONDITIONAL BASES

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Abstract. The only infinite-dimensional complex space with 1-unconditional basis which has polynomial numerical index of order 2 equal to 1 is c_0. In the real case, there is no space of this type. We also show that, in the complex case, if X is an infinite-dimensional Banach sequence space with absolute norm whose dual is norming and has polynomial numerical index of order 2 equal to 1, then $c_0 \subset X \subset \ell_\infty$. In the real case, again there is no space of this type.

1. Introduction

The aim of this paper is to show that among Banach sequence spaces with absolute norm there are not too many spaces which have polynomial numerical index of order 2 equal to 1, improving, for sequence spaces, our previous results with J. Merí [17]. Let us present the relevant definitions.

Let X be a Banach space over a scalar field $\mathbb{K}$ ($=\mathbb{R}$ or $=\mathbb{C}$) and write B_X for the closed unit ball, S_X for the unit sphere and X^* for the dual space. $\mathbb{T}$ stands for the set of modulus-one scalars and given a set $A \subset X$, $\overline{\operatorname{conv}}(A)$ stands for the closed convex hull of A and, so, $\overline{\operatorname{conv}}(\mathbb{T}A)$ is the absolutely closed convex hull of the set A. The set A is *rounded* if $\mathbb{T}A = A$. Finally, given $x^* \in S_{X^*}$ which attains its norm, the *face* generated by x^* is $F(x^*) := \{x \in B_X \ : \ x^*(x) = 1\}$. Let X and Y be Banach spaces. For $k \in \mathbb{N}$, a bounded *k-homogeneous polynomial* $P : X \longrightarrow Y$ is $P(x) = L(x, \ldots, x)$ for all $x \in X$, where $L : X \times \cdots \times X \longrightarrow Y$ is a continuous k-multilinear map. We denote by $\mathcal{P}\left({}^k X; Y\right)$ the space of all bounded k-homogeneous polynomials from X into Y. A *polynomial* on X is just a linear combination of homogeneous polynomial and we write $\mathcal{P}(X; Y)$ for the space of all polynomials from X into Y, which is endowed with the norm

$$\|P\| = \sup\{\|P(x)\| \ : \ x \in B_X\}.$$

Then $\mathcal{P}\left({}^k X; Y\right)$ becomes a Banach space when considered as a subspace of $\mathcal{P}(X; Y)$. Given a polynomial $P \in \mathcal{P}(X; X)$, the *numerical radius* of P is

$$v(P) = \sup\big\{|x^*(Px)| \ : \ (x, x^*) \in S_X \times S_{X^*}, \ x^*(x) = 1\big\}.$$

In 2006, Y. S. Choi, D. García, S. G. Kim and M. Maestre [4] introduced the *polynomial numerical index of order k* of a Banach space X as the constant $n^{(k)}(X)$ defined by

$$\begin{aligned} n^{(k)}(X) &= \max\left\{c \geqslant 0 \ : \ c\,\|P\| \leqslant v(P) \ \ \forall\, P \in \mathcal{P}\left({}^k X; X\right)\right\} \\ &= \inf\left\{v(P) \ : \ P \in \mathcal{P}\left({}^k X; X\right), \ \|P\| = 1\right\} \end{aligned}$$

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for every $k \in \mathbb{N}$. For $k = 1$, this number is known as the *numerical index* of X and it was first suggested by G. Lumer in 1968 (see [8]). For more information and background, we refer the reader to the survey paper [10] and to [4, 5, 6, 9, 14, 15, 16] and references therein. Some examples of spaces whose polynomial numerical indices are known are the following. In the complex case, $n^{(k)}(C(K)) = n^{(k)}(c_0) = 1$ for every $k \in \mathbb{N}$ and $n^{(2)}(\ell_1) \leqslant \frac{1}{2}$. In the real case, $n^{(k)}(\mathbb{R}) = 1$ and

$$n^{(2)}(\ell_1^m) = n^{(2)}(\ell_\infty^m) = n^{(2)}(c_0) = n^{(2)}(\ell_1) = n^{(2)}(\ell_\infty) = 1/2.$$

We write ℓ_∞^m and ℓ_1^m for the m-dimensional ℓ_∞-space and ℓ_1-space respectively.

A *Banach sequence space with absolute norm* is a $\mathbb{K}$-linear subspace X of $\mathbb{K}^{\mathbb{N}}$ (the space of all sequences) endowed with a complete norm $\|\cdot\|_X$ satisfying

(a) Given $x, y \in \mathbb{K}^{\mathbb{N}}$ with $|y(n)| \leqslant |x(n)|$ for every $n \in \mathbb{N}$, if $x \in X$, then $y \in X$ with $\|y\|_X \leqslant \|x\|_X$.
(b) For every $n \in \mathbb{N}$, the sequence $e_n \in \mathbb{K}^{\mathbb{N}}$ given by $e_n(n) = 1$, $e_n(k) = 0$ if $k \neq n$, belongs to X with $\|e_n\|_X = 1$.

It can be easily deduced from the definition that

$$\ell_1 \subseteq X \subseteq \ell_\infty$$

with contractive inclusions or, equivalently, that

$$\sup\{|x(n)| : n \in \mathbb{N}\} \leqslant \|x\|_X \leqslant \sum_{n=1}^{\infty} |x(n)| \qquad (x \in X).$$

We will write $\|\cdot\| = \|\cdot\|_X$ when the space X is clear from the context.

A Banach sequence space with absolute norm X is actually a Köthe space on the counting measure on $\mathbb{N}$. We refer to [19] for background on Köthe spaces. From this book we take the following standard terminology. The *Köthe dual* X' of X is the collection of all sequences $y \in \mathbb{K}^{\mathbb{N}}$ such that

$$\|y\|_{X'} := \sup\left\{\sum_{n=1}^{\infty} |y(n)||x(n)| : x \in B_X\right\} < \infty.$$

It is clear that $(X', \|\cdot\|_{X'})$ is a Banach sequence space with absolute norm. Every element $y \in X'$ defines naturally a continuous linear functional on X by the formula

$$x \longmapsto \sum_{n=1}^{\infty} y(n)\, x(n) \qquad \big(x \in X\big),$$

so we have $X' \subseteq X^*$ with isometric inclusion. For $n \in \mathbb{N}$, we will write e_n' for the functional $e_n'(x) = x(n)$ for every $x \in X$ (i.e., e_n' is the sequence $e_n \in X'$ viewed as an element of X^*). We say that X is *order continuous* if $0 \leqslant x_\alpha \downarrow 0$ and $x_\alpha \in X$ imply that $\lim_\alpha \|x_\alpha\| = 0$. For Banach sequence spaces, this is known to be equivalent to separability and to the fact that $X' = X^*$. If X is order continuous (i.e. separable), the set of those sequences with finite support is dense in X. Moreover, $\{e_n\}$ is a 1-unconditional basis of X. Reversely, if X is any infinite-dimensional Banach space with 1-unconditional basis $\{u_n\}$, we may define an operator $\Phi : X \longrightarrow \mathbb{K}^{\mathbb{N}}$ by $\Phi(x) = (a_n)_{n\in\mathbb{N}}$ for $x = \sum_{n=1}^{\infty} a_n\, u_n/\|u_n\| \in X$. Then $\Phi(X)$ endowed with the norm inherited from X is an order continuous Banach sequence space with absolute norm which is completely identified with X. See [19] for more details.

In the recent paper [17], some restrictions for a Banach space with absolute norm (a more general concept than the one of Banach sequence space with absolute norm) to have polynomial numerical index of order 2 equal to 1 were presented. In particular, it is shown that for a complex Banach sequence space with absolute norm X such that $n^{(2)}(X) = 1$, if X has the RNP then $X = \ell_\infty^m$ for some $m \in \mathbb{N}$; if X is Asplund, then $X = c_0$. For real spaces, it is shown that a Banach sequence space X with absolute norm satisfying $n^{(2)}(X) = 1$ is one-dimensional if it has the RNP or it is Asplund. These results generalize previous results of the first author of this paper [14, 16] to the infinite-dimensional setting.

In the present paper, we are able to remove the RNP and Asplund assumptions. Indeed, it is shown that the only infinite-dimensional complex separable Banach sequence space with absolute norm X which satisfies $n^{(2)}(X) = 1$ is $X = c_0$; in the real case, we prove that for every infinite-dimensional separable Banach sequence space with absolute norm X, one has $n^{(2)}(X) < 1$. This is equivalent to say that c_0 is the only complex infinite-dimensional Banach space with 1-unconditional basis which has polynomial numerical index of order 2 equal to 1 and that for every infinite-dimensional real Banach space with 1-unconditional basis, the polynomial numerical index of order 2 cannot be equal to 1. We are also able to extend the above result to the nonseparable case when the Köthe dual is norming. We show that a complex infinite-dimensional Banach sequence space with absolute norm X such that X' is norming for X satisfies $n^{(2)}(X) = 1$ if and only if $c_0 \subset X \subset \ell_\infty$. In the real case, we prove that for every infinite-dimensional Banach sequence space with absolute norm such that X' is norming for X, one has $n^{(2)}(X) < 1$.

The outline of the paper is the following. Section 2 is devoted to prove some needed preliminary results on lush spaces and spaces with the alternative Daugavet property (definitions are there). Then we present in section 3 the announced results on Banach sequence spaces with polynomial numerical index one.

2. Preliminaries on lushness and on the alternative Daugavet property

A Banach space X is said to be *lush* [3] if for every $x, y \in S_X$ and every $\varepsilon > 0$, there is a slice $S = \{x \in B_X : \operatorname{Re} x^*(x) > 1 - \varepsilon\}$ with $x^* \in S_{X^*}$ such that $x \in S$ and $\operatorname{dist}(y, \overline{\operatorname{conv}}(\mathbb{T}S)) < \varepsilon$. Lush spaces have numerical index 1 [3, Proposition 2.2], but it has been very recently shown that the converse result is not true [12]. We refer to [2, 3, 11] for background.

A Banach space X has the *alternative Daugavet property* [20] if every rank-one operator $T \in L(X)$ satisfies the norm equality

$$\max_{\theta \in \mathbb{T}} \| \operatorname{Id} + \theta\, T\| = 1 + \|T\| \tag{aDE}$$

and then all weakly compact operators on X also satisfy (aDE). Equivalently, X has the alternative Daugavet property if and only if $v(T) = \|T\|$ for every rank-one (equivalently, for every weakly compact) operator $T \in L(X)$ [20, Theorem 2.2 and Lemma 2.3]. Therefore, if a Banach space has numerical index 1, then it has the alternative Daugavet property, being the reverse implication false in general [20, Example 3.2].

Summarizing, lushness implies numerical index 1 which implies the alternative Daugavet property, and none of these two implications reverse.

We present now the preliminary results that we will use in the next section.

Proposition 2.1. *Let X be a separable lush space. Then there exists a rounded subset $C \subset S_{X^*}$ which is norming for X and satisfies*

(a) $|x^{**}(x^*)| = 1$ *for every* $x^* \in C$ *and every* $x^{**} \in \operatorname{ext}(B_{X^{**}})$.
(b) $B_X = \overline{\operatorname{conv}}\big(\mathbb{T}\, F(x^*)\big)$ *for every* $x^* \in C$.
(c) *The set* $B = \bigcup_{x^* \in C} F(x^*)$ *is dense in* S_X.

The proof of this proposition relied on [11, Theorem 4.3] and [1, Corollary 3.5]. We state them here for the sake of completeness.

Lemma 2.2. [11, Theorem 4.3] *Let X be a separable lush space. Then there exists a rounded subset $C \subset S_{X^*}$ which is norming for X and satisfies that $|x^{**}(x^*)| = 1$ for every $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and every $x^* \in C$.*

Lemma 2.3. [1, Corollary 3.5] *Let X be a Banach space and let $x^* \in S_{X^*}$ such that $|x^{**}(x^*)| = 1$ for every $x^{**} \in \operatorname{ext}(B_{X^{**}})$. Then $F(x^*) \neq \emptyset$ and, moreover, $B_X = \overline{\operatorname{conv}}\big(\mathbb{T}\, F(x^*)\big)$.*

Proof of Proposition 2.1. We consider the set $C \subset S_{X^*}$ given by Lemma 2.2 which satisfies (a) and we observe that Lemma 2.3 gives (b).

Let us prove (c). Fix $x_0 \in S_X$ and $\varepsilon > 0$, and we look for $x \in B$ such that $\|x_0 - x\| < \varepsilon$. Consider $0 < \delta < 1$ such that $\sqrt{2\delta} + 2\delta + \delta^2/2 < \varepsilon$ and take $x_0^* \in C$ such that

$$\operatorname{Re} x_0^*(x_0) > 1 - \delta^2/2.$$

As we have $B_X = \overline{\operatorname{conv}}\big(\mathbb{T}\, F(x_0^*)\big)$ by (b), we may find $y \in B_X$ such that

$$\|x_0 - y\| < \delta^2/2 \qquad \text{and} \qquad y = \sum_{k=1}^{m} \lambda_k \theta_k x_k$$

where $x_k \in F(x_0^*)$, $\theta_k \in \mathbb{T}$, $0 \leqslant \lambda_k \leqslant 1$ for $k = 1, \ldots, m$, $\sum \lambda_k = 1$, and $m \in \mathbb{N}$ (in the real case it is possible to take $m = 2$ and the argument simplifies). From the above two equations, we get that $\operatorname{Re} x_0^*(y) > 1 - \delta^2$ and so

$$\sum_{k=1}^{m} \lambda_k \operatorname{Re} \theta_k > 1 - \delta^2.$$

We write

$$A = \big\{k \in \{1, \ldots, m\} \,:\, \operatorname{Re} \theta_k > 1 - \delta\big\}, \quad B = \{1, \ldots, m\} \setminus A, \quad \text{and} \quad \mu_A = \sum_{k \in A} \lambda_k\,,$$

and prove that

$$\mu_A > 1 - \delta \quad \text{and so} \quad \sum_{k \in B} \lambda_k < \delta$$

(which, in particular, shows that A is nonempty). Indeed, we have

$$1 - \delta^2 < \sum_{k=1}^{m} \lambda_k \operatorname{Re} \theta_k \leqslant \mu_A + (1 - \delta)(1 - \mu_A)$$

and an obvious simplification gives the result. On the one hand, for $k \in A$ we have $\operatorname{Re} \theta_k > 1 - \delta$ and an straightforward computation gives that

$$|\theta_k - 1| < \sqrt{2\delta} \text{ for every } k \in A.$$

Finally, we consider $x = \sum_{k\in A} \frac{\lambda_k}{\mu_A} x_k \in F(x_0^*) \subset B$ and estimate $\|x_0 - x\|$ as

$$\begin{aligned}\|x_0 - x\| &\leqslant \|y - x\| + \delta^2/2 = \left\| \sum_{k=1}^{m} \lambda_k \theta_k x_k - \sum_{k\in A} \frac{\lambda_k}{\mu_A} x_k \right\| + \delta^2/2 \\ &\leqslant \left\| \sum_{k\in A} \lambda_k (\theta_k - 1) x_k \right\| + \left\| \sum_{k\in A} \lambda_k \left(\frac{1}{\mu_A} - 1 \right) x_k \right\| + \left\| \sum_{k\in B} \lambda_k \theta_k x_k \right\| + \delta^2/2 \\ &< \sum_{k\in A} \lambda_k \sqrt{2\delta} + \sum_{k\in A} \lambda_k \left(\frac{1}{\mu_A} - 1 \right) + (1 - \mu_A) + \delta^2/2 \\ &< \mu_A \sqrt{2\delta} + (1 - \mu_A) + (1 - \mu_A) + \delta^2/2 < \sqrt{2\delta} + 2\delta + \delta^2/2 < \varepsilon. \qquad \square\end{aligned}$$

In the case of a space with 1-unconditional basis, it is posible to get more information about the set C in Proposition 2.1. On the other hand, in this case lushness is equivalent to the alternative Daugavet property. All of this are collected in the next statement.

Corollary 2.4. *Let X be a separable Banach sequence space with absolute norm which has the alternative Daugavet property. Then there exist a rounded set $C \subset S_{X'}$ (norming for X) satisfying $|x'| \in \{0,1\}^{\mathbb{N}}$ for every $x' \in C$, and a dense subset $B \subset S_X$ such that, for every $x \in B$, there is an $x' \in C$ such that $x'(x) = 1$.*

We will use the following result which is a particular case of [17, Lemma 3.2] and we state it for completeness.

Lemma 2.5. [17, Lemma 3.2] *Let X be a separable Banach sequence space with absolute norm and let $x' \in S_{X'}$ such that $|x^{**}(x')| = 1$ for every $x^{**} \in \operatorname{ext}(B_{X^{**}})$. Then $|x'(n)| \in \{0,1\}$ for every $n \in \mathbb{N}$.*

Proof of Corollary 2.4. As we commented in the introduction, separable Banach sequence spaces with absolute norm have 1-unconditional basis and then, it follows from [13, Corollary 3.2] that the alternative Daugavet property and lushness are equivalent for them. Therefore, X is lush and Proposition 2.1 applies, providing two sets $C \subset S_{X'}$ and $B \subset S_X$ which do the job. Actually, the only assertion which is not directly given by Proposition 2.1 is that $|x'| \in \{0,1\}^{\mathbb{N}}$ for every $x' \in C$, but this follows from Lemma 2.5. $\square$

3. The main results

We are now able to prove the main result of the paper, i.e., that among Banach spaces with 1-unconditional basis, the only ones which have polynomial numerical index of order 2 equal to 1 are c_0 and ℓ_∞^m in the complex case, and $\mathbb{R}$ in the real case.

Theorem 3.1. *Let X be a Banach space with 1-unconditional basis and $n^{(2)}(X) = 1$. If X is a real space, then $X = \mathbb{R}$. If X is a complex space, then either $X = c_0$ or there exists $m \in \mathbb{N}$ such that $X = \ell_\infty^m$.*

Since the proof of this result is mainly deduced from [17, Proposition 4.1], for the sake of completeness, we state here this result in the setting of sequence spaces.

Lemma 3.2. [17, Proposition 4.1] *Let X be a Banach sequence space with absolute norm. Suppose that there exists a dense subset B in S_X and a set $C \subset S_{X'}$ such that $|x'| \in \{0,1\}^{\mathbb{N}}$ for every $x' \in C$ and satisfying that for every $x \in B$ there is $x' \in C$ with $x'(x) = 1$. If $n^{(2)}(X) = 1$, then given $x^* \in B_{X^*}$ and $k, j \in \mathbb{N}$ such that $|x^*(e_j)| = |x^*(e_k)| = 1$, we get $j = k$.*

Proof of Theorem 3.1. Since the finite-dimensional case is covered by [17, Theorems 4.2 and 4.3], we suppose that the basis of X is infinite for the sake of simplicity, but the proof for finite bases follows the same lines.

As it was commented in the introduction, if X has 1-unconditional basis, then it is (isometrically isomorphic to) a separable Banach sequence space with absolute norm. Since $n^{(2)}(X) = 1$, it follows that $n(X) = 1$ [4, Proposition 2.5] and so that X has the alternative Daugavet property [20, Lemma 2.3]. Therefore, we may apply Corollary 2.4 to get a set $C \subset S_{X'}$ (norming for X) satisfying that $|x'| \in \{0,1\}^{\mathbb{N}}$ for every $x' \in C$, and a dense subset $B \subset S_X$ such that, for every $x \in B$, there exists $x' \in C$ such that $x'(x) = 1$. The existence of such sets is exactly the hypothesis of Lemma 3.2. So, we conclude that there is no $x' \in S_{X^*}$ and $k, j \in \mathbb{N}$, $k \neq j$, such that $|x'(k)| = |x'(j)| = 1$. Applying this to the elements of the set C, whose coordinates are of modulus 0 or 1, we get that

$$C \subseteq \left\{\omega\, e_n' \,:\, \omega \in \mathbb{T},\, n \in \mathbb{N}\right\} \subset B_{X^*}.$$

Being C norming for X, this gives that

$$\|x\| = \sup\{|x(n)| \,:\, n \in \mathbb{N}\} \qquad (x \in X).$$

Since X is the closed linear span of $\{e_n \,:\, n \in \mathbb{N}\}$ we deduce that X is isometric to c_0.

Finally, in the complex case, $n^{(2)}(c_0) = 1$ and this space is possible. In the real case, $n^{(2)}(c_0) = 1/2$ and, therefore, the infinite-dimensional case gives no space. □

The rest of the paper is devoted to extend the result in Theorem 3.1 to nonseparable Banach sequence spaces with absolute norm. First of all, we have to check that the hypotesis of $n^{(2)}(X) = 1$ in this theorem can be weakened to the k-order alternative Daugavet property.

Analogously to the linear case, we say that a Banach space X has the *k-order alternative Daugavet property* (*k-ADP* for short) [5] if for every scalar k-homogeneous polynomial $p \in \mathcal{P}\left({}^k X; \mathbb{K}\right)$ and every $x \in X$, the norm equality

$$\max_{\theta \in \mathbb{T}} \|\operatorname{Id} + \theta\, p \otimes x\| = 1 + \|p \otimes x\|$$

holds (the norm is taken in the space $\mathcal{P}\left(X; X\right)$). In this case, $\|\operatorname{Id} + P\| = 1 + \|P\|$ or, equivalently, $v(P) = \|P\|$, for every compact polynomial $P \in \mathcal{P}\left({}^k X; X\right)$ (see [5, Corollary 1.2 and Proposition 1.3]). Let us observe that in the proof of Theorem 3.1 above, if we just assume that X has the 2-ADP, we still have that X has the alternative Daugavet property [5, Proposition 3.7] and the set C and B there are also given by Corollary 2.4. Moreover, going into the proof of [17, Proposition 4.1], one realizes that only compact polynomials (actually, rank-two polynomials) are used and so the hypothesis of $n^{(2)}(X) = 1$ can be weakened to the 2-ADP. Therefore, we actually have the following formally stronger result.

Remark 3.3. *Let X be a separable Banach sequence space with absolute norm which has the 2-order alternative Daugavet property. If X is real, then $X = \mathbb{R}$. If X is complex, then either $X = c_0$ or $X = \ell_\infty^m$ for some $m \in \mathbb{N}$.*

It is possible to extend the results of Theorem 3.1 to the nonseparable case, but only to those sequence spaces for which the Köthe dual is norming. We only state the infinite-dimensional case since the finite-dimensional case is covered by the theorem.

Corollary 3.4. *Let X be an infinite-dimensional Banach sequence space with absolute norm such that X' is norming for X. In the complex case, if $n^{(2)}(X) = 1$, then $c_0 \subset X \subset \ell_\infty$ isometrically. In the real case we always have that $n^{(2)}(X) < 1$.*

We need a technical lemma whose proof takes ideas from [13, Lemma 3.8].

Lemma 3.5. *Let X be a Banach sequence space with absolute norm such that X' is norming for X and let k be a positive integer. Denote by E the closed linear span of the set of canonical basis vectors e_n, $n \in \mathbb{N}$. If X has the k-ADP, then E also has the k-ADP.*

Proof. It is enough to show that for $x_0 \in S_E$, $p \in \mathcal{P}\left({}^k E; \mathbb{K}\right)$ with $\|p\| = 1$ and $\varepsilon > 0$ fixed, we may find $y \in B_E$ and $\theta \in \mathbb{T}$ such that

$$|p(y)| > 1 - \varepsilon \quad \text{and} \quad \|\theta x_0 + y\| > 2 - \varepsilon$$

(see [5, Corollary 1.2 and Proposition 1.3]). As X' is norming for X, $X \subseteq E''$ with equality of norms (this can be easily deduce from the results of [19, pp. 29-30]) and therefore, as $E^* = E'$,

$$X \subset E'' = (E')' = (E^*)' \subseteq E^{**}$$

with equality of norms.

Now consider $\bar{p} \in \mathcal{P}\left({}^k E^{**}; \mathbb{K}\right)$ the Aron-Berner extension of p, which satisfies $\|\bar{p}\| = 1$ [7], consider $q = \bar{p}|_X$ and observe that $\|q\| = 1$ in X. As X has the k-ADP, we use again [5, Corollary 1.2 and Proposition 1.3] to find $x \in S_X$ and $\theta \in \mathbb{T}$ such that

$$|\bar{p}(x)| = |q(x)| > 1 - \varepsilon \quad \text{and} \quad \|\theta x + x_0\| > 2 - \varepsilon.$$

Next we use Davie-Gamelin result [7] to find a net $\{x_\alpha\}$ in B_E which is polynomial star convergent to $x \in S_{E^{**}}$ (i.e. for every polynomial h on X, $h(x_\alpha) \longrightarrow \bar{h}(x)$). In particular

$$|p(x_\alpha)| \longrightarrow |\bar{p}(x)| = |q(x)| > 1 - \varepsilon \quad \text{and} \quad \limsup_\alpha \|\theta x_\alpha + x_0\| \geqslant \|\theta x + x_0\| > 2 - \varepsilon.$$

Therefore, there is α such that for $y = x_\alpha \in B_E$ one has

$$|p(y)| > 1 - \varepsilon \qquad \text{and} \qquad \|\theta y + x_0\| > 2 - \varepsilon$$

as desired. □

Proof of Corollary 3.4. Let E be the closed linear span of the set of canonical basis vectors e_n, $n \in \mathbb{N}$. If $n^{(2)}(X) = 1$, then X has the 2-ADP and so does E by the above lemma. But E is separable and so Remark 3.3 applies. In the complex case, we get that $E = c_0$. So the fact that $E \subset X \subset E''$ shows that $c_0 \subset X \subset \ell_\infty$ isometrically. In the real case, we get that $E = \mathbb{R}$, which is impossible since we have supposed that X is infinite-dimensional. Therefore, $n^{(2)}(X) < 1$. □

It is not possible to get more in Corollary 3.4, as the following remarks show.

Remarks 3.6.

(a) *Let X be any subspace of ℓ_∞ containing (the canonical copy of) c_0. Then $n^{(k)}(X) = 1$ for every $k \in \mathbb{N}$.* Indeed, we just have to use [6, Theorem 3.2.f] with $K = \beta\mathbb{N}$, $U = \mathbb{N}$ which is open and dense, and observe that $C(K) = \ell_\infty$ and $Y = \{f \in C(K) : f(\beta\mathbb{N} \setminus \mathbb{N}) = 0\} = c_0$.

(b) *Let us consider the following closed subspace of the complex space ℓ_∞:*

$$X = \left\{x \in \ell_\infty, : \|x\|_\infty < \infty, \lim_{n\to\infty} x(2n) = 0\right\}$$

endowed with the ℓ_∞-norm. Then X is a Banach sequence space with absolute norm and X' is norming for X. As $c_0 \subset X \subset \ell_\infty$, item (a) above gives that $n^{(k)}(X) = 1$ for every $k \in \mathbb{N}$.

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