ISOMETRIES ON EXTREMELY NON-COMPLEX BANACH SPACES

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Abstract. Given a separable Banach space E, we construct an extremely non-complex Banach space (i.e. a space satisfying that $\|\operatorname{Id}+T^2\| = 1 + \|T^2\|$ for every bounded linear operator T on it) whose dual contains E^* as an L-summand. We also study surjective isometries on extremely non-complex Banach spaces and construct an example of a real Banach space whose group of surjective isometries reduces to $\pm\operatorname{Id}$, but the group of surjective isometries of its dual contains the group of isometries of a separable infinite-dimensional Hilbert space as a subgroup.

1. Introduction

All the Banach spaces in this paper will be real. Given a Banach space X, we write X^* for the topological dual, $L(X)$ for the space of all bounded linear operators, $W(X)$ for the space of weakly compact operators and $\operatorname{Iso}(X)$ for the group of surjective isometries.

A Banach space X is said to be *extremely non-complex* if the norm equality

$$\|\operatorname{Id}+T^2\| = 1 + \|T^2\|$$

holds for every $T \in L(X)$. This concept was introduced very recently by the authors in [20], where several different examples of $C(K)$ spaces are shown to be extremely non-complex. For instance, this is the case for some perfect compact spaces K constructed by the first author [19] such that $C(K)$ has few operators (in the sense that every operator is a weak multiplier). There are other examples of extremely non-complex $C(K)$ spaces which contain complemented copies of ℓ_∞ or $C(2^\omega)$ (and so, they do not have few operators). It is trivial that $X = \mathbb{R}$ is extremely non-complex. The existence of infinite-dimensional extremely non-complex Banach spaces had been asked in [16, Question 4.11], where possible generalizations of the Daugavet equation were investigated. We recall that an operator S defined on a Banach space X satisfies the *Daugavet equation* [4] if

$$\|\operatorname{Id}+S\| = 1 + \|S\|$$

and that the space X has the *Daugavet property* [17] if the Daugavet equation holds for every rank-one operator on X. We refer the reader to [2, 3, 17, 27] for background on the Daugavet property. Let us observe that X is extremely non-complex if the Daugavet equation holds for the square of every (bounded linear) operator on X. Spaces X in which the square of every rank-one operator on X satisfies the Daugavet equation are studied in [23], where it is shown that their unit balls do not have strongly exposed points. In particular, the unit ball of an extremely non-complex Banach space (of dimension greater than one) does not have strongly exposed points and, therefore, the space does not have the Radon-Nikodým property, even the more it is not reflexive.

The name of extremely non-complex comes from the fact that a real Banach space X is said to have a *complex structure* if there exists $T \in L(X)$ such that $T^2 = -\operatorname{Id}$, so extremely non-complex spaces lack of complex structures in a very strong way. Let us also comment that no

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hyperplane (actually no finite-codimensional subspace) of an extremely non-complex Banach space admits a complex structure. The existence of infinite-dimensional real Banach spaces admitting no complex structure is known since the 1950's, when J. Dieudonné [5] showed that this is the case of the James' space $\mathcal{J}$. We refer the reader to the very recent papers by V. Ferenczi and E. Medina Galego [10, 11] and references therein for a discussion about complex structures on spaces and on their hyperplanes.

Our first goal in this paper is to present examples of extremely non-complex Banach spaces which are not isomorphic to $C(K)$ spaces. Namely, it is proved in section 3 that if K is a compact space such that all operators on $C(K)$ are weak multipliers, L is a closed nowhere dense subset of K and E is a subspace of $C(L)$, then the space

$$C_E(K\|L) = \{f \in C(K) \, : \, f|_L \in E\}$$

is extremely non-complex (see section 2 for the definitions and basic facts about this kind of spaces). It is also shown that there are extremely non-complex $C_E(K\|L)$ spaces which are not isomorphic to $C(K')$ spaces. On the other hand, some spaces $C_E(K\|L)$ are isometric to spaces $C(K')$ for some compact K'. This can be used to note that the extremely non-complex spaces of the form $C_E(K\|L)$ may have many operators besides weak multipliers (see Remark 3.14).

The next aim is to show that $\operatorname{Iso}(X)$ is a "discrete" Boolean group when X is extremely non-complex. Namely, we show that $T^2 = \mathrm{Id}$ for every $T \in \operatorname{Iso}(X)$ (i.e. $\operatorname{Iso}(X)$ is a Bolean group and so, it is commutative) and that $\|T_1 - T_2\| \in \{0, 2\}$ for every $T_1, T_2 \in \operatorname{Iso}(X)$. Next, we discuss the relationship with the set of all unconditional projections on X and the possibility of this set to be a Boolean algebra. This is the content of section 4. In section 5 we particularize these results to spaces $C_E(K\|L)$ which are extremely non-complex, getting necessary conditions on the elements of $\operatorname{Iso}\big(C_E(K\|L)\big)$. In particular, if $C(K)$ is extremely non-complex, we show that the only homeomorphism from K to K is the identity, obtaining that $\operatorname{Iso}\big(C(K)\big)$ is isomorphic to the Boolean algebra of clopen sets of K.

Section 6 is devoted to apply all the results above to get an example showing that the behavior of the group of isometries with respect to duality can be extremely bad. Namely, we show that for every separable Banach space E, there is a Banach space $\widetilde{X}(E)$ such that $\operatorname{Iso}\big(\widetilde{X}(E)\big) = \{\mathrm{Id}, -\mathrm{Id}\}$ and $\widetilde{X}(E)^* = E^* \oplus_1 Z$, so $\operatorname{Iso}\big(\widetilde{X}(E)^*\big)$ contains $\operatorname{Iso}(E^*)$ as a subgroup. To do so, we have to modify a construction of a connected compact space K with few operators given by the first named author in [19, §5] and use our construction of $C_E(K\|L)$ for a nowhere dense $L \subset K$. For the special case $E = \ell_2$, we have $\operatorname{Iso}\big(\widetilde{X}(\ell_2)\big) = \{\mathrm{Id}, -\mathrm{Id}\}$, while $\operatorname{Iso}\big(\widetilde{X}(\ell_2)^*\big)$ contains infinitely many uniformly continuous one-parameter semigroups of surjective isometries. Let us comment that, in sharp contrast with the examples above, when a Banach space X is *strongly unique predual*, the group $\operatorname{Iso}(X^*)$ consists exactly of the conjugate operators to the elements of $\operatorname{Iso}(X)$. Quite a lot of spaces are actually strong unique preduals. We refer the reader to [14] for more information.

We finish this introduction with some needed notation. If X is a Banach space, we write B_X to denote the closed unit ball of X and given a convex subset $A \subseteq X$, $\operatorname{ext}(A)$ denotes the set of extreme points of A. A closed subspace Z of X is an *L-summand* if $X = Z \oplus_1 W$ for some closed subspace W of X, where $\oplus_1$ denotes the ℓ_1-sum. A closed subspace Y of a Banach space X is said to be an *M-ideal* of X if the annihilator $Y^\perp$ of Y is an L-summand of X^*. We refer the reader to [15] for background on L-summands and M-ideals.

2. Notation and preliminary results on the spaces $C_E(K\|L)$

All along the paper, K will be a (Hausdorff) compact (topological) space and $L \subseteq K$ will stand for a nowhere dense closed subset. Given a closed subspace E of $C(L)$, we will consider the subspace of $C(K)$ given by

$$C_E(K\|L) = \{f \in C(K) \, : \, f|_L \in E\}.$$

This notation is compatible with the Semadeni's book [25, II. 4] notation of

$$C_0(K\|L) = \{f \in C(K) : f|_L = 0\}.$$

This space can be identified with the space $C_0(K \setminus L)$ of those continuous functions $f : K \setminus L \longrightarrow \mathbb{R}$ vanishing at infinity.

By the Riesz representation theorem, the dual space of $C(K)$ is isometric to the space $M(K)$ of Radon measures on K, i.e. signed, Borel, scalar-valued, countably additive and regular measures. More precisely, given $\mu \in M(K)$ and $f \in C(K)$, the duality is given by

$$\mu(f) = \int f d\mu.$$

We recall that $C_0(K\|L)$ is an M-ideal of $C(K)$ [15, Example I.1.4(a)], meaning that $C_0(K\|L)^\perp$ is an L-summand in $C(K)^*$. This fact allows to show the following well-known result.

Lemma 2.1. $C_0(K\|L)^* \equiv \{\mu \in M(K) : |\mu|(L) = 0\}$.

Proof. Since $C_0(K\|L)$ is an M-ideal in $C(K)$, Proposition 1.12 and Remark 1.13 of [15] allow us to identify $C_0(K\|L)^*$ with the subspace of $C(K)^* = M(K)$ given by

$$C_0(K\|L)^\# = \{\mu \in M(K) : |\mu|(K) = |\mu|(K \setminus L)\} = \{\mu \in M(K) : |\mu|(L) = 0\}. \qquad \square$$

When we consider the space $C_E(K\|L)$, it still makes sense to talk about functionals corresponding to the measures on K, namely one understands them as the restriction of the functional from $C(K)$ to $C_E(K\|L)$. However, given a functional belonging to $C_E(K\|L)^*$ one may have several measures on K associated with it. The next result describes the dual of a $C_E(K\|L)$ space for an arbitrary $E \subseteq C(L)$. It is worth mentioning that its proof is an extension of that appearing in [21, Theorem 3.3].

Lemma 2.2. *Let K be a compact space, L a closed subset of K and $E \subseteq C(L)$. Then,*

$$C_E(K\|L)^* \equiv C_0(K\|L)^* \oplus_1 C_0(K\|L)^\perp \equiv C_0(K\|L)^* \oplus_1 E^*.$$

Proof. We write $P : C(K) \longrightarrow C(L)$ for the restriction operator, i.e.

$$[P(f)](t) = f(t) \qquad (t \in L,\ f \in C(K)).$$

Then, $C_0(K\|L) = \ker P$ and $C_E(K\|L) = \{f \in C(K) : P(f) \in E\}$. Since $C_0(K\|L)$ is an M-ideal in $C(K)$, it is a fortiori an M-ideal in $C_E(K\|L)$ by [15, Proposition I.1.17], meaning that

$$C_E(K\|L)^* \equiv C_0(K\|L)^* \oplus_1 C_0(K\|L)^\perp \equiv C_0(K\|L)^* \oplus_1 \left[C_E(K\|L)/C_0(K\|L)\right]^*.$$

Now, it suffices to prove that the quotient $C_E(K\|L)/C_0(K\|L)$ is isometrically isomorphic to E. To do so, we define the operator $\Phi : C_E(K\|L) \longrightarrow E$ given by $\Phi(f) = P(f)$ for every $f \in C_E(K\|L)$. Then Φ is well defined, $\|\Phi\| \leqslant 1$, and $\ker \Phi = C_0(K\|L)$. To see that the canonical quotient operator $\widetilde{\Phi} : C_E(K\|L)/C_0(K\|L) \longrightarrow E$ is a surjective isometry, it suffices to show that

$$\Phi\big(\{f \in C_E(K\|L) : \|f\| < 1\}\big) = \{g \in E : \|g\| < 1\}.$$

Indeed, the left-hand side is contained in the right-hand side since $\|\Phi\| \leqslant 1$. Conversely, for every $g \in E \subseteq C(L)$ with $\|g\| < 1$, we just use Tietze's extension theorem to find $f \in C(K)$ such that $\Phi(f) = f|_L = g$ and $\|f\| = \|g\|$. $\square$

If $\phi \in C_E(K\|L)^*$, by the above lemma we have $\phi = \phi_1 + \phi_E$ with $\phi_1 \in C_0(K\|L)^*$ and $\phi_E \in C_0(K\|L)^\perp \equiv E^*$. Observe that ϕ_1 can be isometrically associated with a measure on $K \setminus L$ by Lemma 2.1. which will be denoted $\phi|_{K\setminus L}$. Given a subset $A \subseteq K$ satisfying $A \cap L = \emptyset$, $\phi|_A$ will stand for the measure $\phi|_{K\setminus L}$ restricted to A.

The next result is a straightforward consequence of the above two lemmas.

Lemma 2.3. *Let $\phi \in C_E(K\|L)^*$ and $x \in K \setminus L$. Then*
$$\|\phi\| = \big\|\phi|_{\{x\}}\big\| + \big\|\phi|_{K\setminus(L\cup\{x\})}\big\| + \|\phi_E\|.$$

The next easy lemma describes the set of extreme points in the unit ball of the dual of $C_E(K\|L)$ and gives a norming set for $C_E(K\|L)$. We recall that a subset A of the unit ball of the dual of a Banach space X is said to be *norming* if
$$\|x\| = \sup\{|\phi(x)| \,:\, \phi \in A\} \qquad (x \in X).$$

Lemma 2.4. *Let K be a compact space, L a nowhere dense closed subset and $E \subseteq C(L)$. We consider the set*
$$\mathcal{A} = \big\{\theta\,\delta_y \,:\, y \in K \setminus L,\, \theta \in \{-1,1\}\big\} \subset C_E(K\|L)^*.$$
Then:

(a) $\operatorname{ext}(B_{C_E(K\|L)^*}) = \mathcal{A} \cup \operatorname{ext}(B_{E^*})$.
(b) *$\mathcal{A}$ is norming for $C_E(K\|L)$.*
(c) *Therefore, $\mathcal{A}$ is weak*-dense in* $\operatorname{ext}(B_{C_E(K\|L)^*})$.

Proof. By Lemma 2.2 and the description of the extreme points of the unit ball of an ℓ_1-sum of Banach spaces [15, Lemma I.1.5], we have
$$\operatorname{ext}(B_{C_E(K\|L)^*}) = \operatorname{ext}(B_{C_0(K\setminus L)^*}) \cup \operatorname{ext}(B_{E^*}).$$
It suffices to recall that $\operatorname{ext}(B_{C_0(K\setminus L)^*}) = \mathcal{A}$ (see [12, Theorem 2.3.5] for instance) to get (a). The fact that $\mathcal{A}$ is norming for $C_E(K\|L)$ is a direct consequence of the fact that $K \setminus L$ is dense in K. Finally, every norming set is weak*-dense in $\operatorname{ext}(B_{C_E(K\|L)^*})$ by the Hahn-Banach theorem and the reversed Krein-Milman theorem. □

We introduce one more ingredient which will play a crucial role in our arguments. Given an operator $U \in L\big(C_E(K\|L)^*\big)$, we consider the function
$$g_U : K \setminus L \longrightarrow [-\|U\|, \|U\|], \qquad g_U(x) = U(\delta_x)(\{x\}) \qquad \big(x \in K \setminus L\big).$$
This obviously extends to operators on $C_E(K\|L)$ by passing to the adjoint, that is, for $T \in L\big(C_E(K\|L)\big)$ one can consider $g_{T^*} : K \setminus L \longrightarrow [-\|T\|, \|T\|]$. This is a generalization of a tool used in [26] under the name "stochastic kernel". One of the results in that paper can be generalized to the following.

Lemma 2.5. *Let K be a compact space, L a nowhere dense closed subset of K, $E \subseteq C(L)$, and $T \in L\big(C_E(K\|L)\big)$. If the set $\{x \in K \setminus L \,:\, g_{T^*}(x) \geqslant 0\}$ is dense in $K \setminus L$, then T satisfies the Daugavet equation.*

Proof. We use Lemmas 2.2, 2.3 and 2.4 to get

$$\begin{aligned}
\text{(1)} \qquad \|\operatorname{Id} + T^*\| &\geqslant \sup_{x\in K\setminus L} \|\delta_x + T^*(\delta_x)\| \\
&= \sup_{x\in K\setminus L} |1 + T^*(\delta_x)(\{x\})| + \|T^*(\delta_x)|_{(K\setminus(L\cup\{x\}))}\| + \|T^*(\delta_x)|_E\| \\
&= \sup_{x\in K\setminus L} |1 + T^*(\delta_x)(\{x\})| - |T^*(\delta_x)(\{x\})| + \|T^*(\delta_x)\| \\
&\geqslant \sup_{x\in K\setminus L} 1 + T^*(\delta_x)(\{x\}) - |T^*(\delta_x)(\{x\})| + \|T^*(\delta_x)\|.
\end{aligned}$$

Now, we claim that the set $\{x \in K \setminus L \,:\, \|T^*(\delta_x)\| > \|T\| - \varepsilon\}$ is nonempty and open in $K \setminus L$ for every $\varepsilon > 0$. Indeed, take a norm one function $f \in C_E(K\|L)$ such that $\|T(f)\| > \|T\| - \varepsilon$ and use the fact that $K \setminus L$ is dense in K to find $x \in K \setminus L$ satisfying
$$\|T^*(\delta_x)\| \geqslant |T^*(\delta_x)(f)| = |T(f)(x)| > \|T\| - \varepsilon.$$

To show that $\{x \in K \setminus L : \|T^*(\delta_x)\| > \|T\| - \varepsilon\}$ is open we prove that the mapping

$$x \longmapsto \|T^*(\delta_x)\| \qquad (x \in K \setminus L)$$

is lower semicontinuous. To do so, since T^* is weak* continuous and $\|\cdot\|$ is weak* lower semicontinuous, it suffices to observe that the mapping which sends x to δ_x is continuous with respect to the weak* topology on $C_E(K\|L)^*$. But this is so since, for $a \in \mathbb{R}$ and $f \in C_E(K\|L)$, the preimage of the subbasic set $\{\phi \in C_E(K\|L)^* : \phi(f) < a\}$ in this topology is $\{x \in K \setminus L : f(x) < a\}$ which is open in $K \setminus L$.

To finish the proof, we use the hypothesis to find $x_0 \in K \setminus L$ satisfying

$$\|T^*(\delta_{x_0})\| > \|T\| - \varepsilon \qquad \text{and} \qquad g_{T^*}(x_0) = T^*(\delta_{x_0})(\{x_0\}) \geqslant 0$$

and we use it in (1) to obtain

$$\|\operatorname{Id} + T^*\| \geqslant 1 + \|T^*(\delta_{x_0})\| > 1 + \|T\| - \varepsilon$$

which implies that $\|\operatorname{Id} + T\| = \|\operatorname{Id} + T^*\| \geqslant 1 + \|T\|$ since ε was arbitrary. □

3. Spaces $C_E(K\|L)$ when $C(K)$ has few operators

Let us start fixing some notation and terminology that will be used throughout the section. If $g : K \longrightarrow \mathbb{R}$ is a bounded Borel function, we will consider the operator $g \operatorname{Id} : C(K)^* \longrightarrow C(K)^*$ which sends the functional which is the integration of a function $f \in C(K)$ with respect to a measure μ to the functional which is the integration of the product fg with respect to μ.

Definition 3.1. Let K be a compact space and $T \in L(C(K))$. We say that T is a *weak multiplier* if $T^* = g \operatorname{Id} + S$ where $g : K \longrightarrow \mathbb{R}$ is a bounded Borel function on K and $S \in W\big(C(K)^*\big)$.

This definition was given in [19] in an equivalent form (see [19, Definition 2.1] and [19, Theorem 2.2]).

Definition 3.2. We say that an open set $V \subseteq K$ is *compatible* with L if and only if $L \subseteq V$ or $L \cap \overline{V} = \emptyset$. In the first case, the notation $C_E(\overline{V}\|L)$ has the same meaning as in the previous section. If $L \cap \overline{V} = \emptyset$, we will write $C_E(\overline{V}\|L)$ just to denote $C(\overline{V})$. Let us also observe that if $L \subseteq V$, then

$$C_E(\overline{V}\|L)^* \equiv C_0(\overline{V}\|L)^* \oplus_1 C_0(\overline{V}\|L)^\perp \equiv C_0(\overline{V}\|L)^* \oplus_1 E^*$$

since Lemma 2.2 applies to $\overline{V}$.

Given an open set $V \subseteq K$ compatible with L, we consider the restriction operator $P^{\overline{V}} : C_E(K\|L)^* \longrightarrow C_E(\overline{V}\|L)^*$ given by

$$P^{\overline{V}}(\phi) = \phi|_{\overline{V}\setminus L} + \phi_E$$

for $\phi = \phi|_{K\setminus L} + \phi_E$ where $\phi|_{K\setminus L} \in C_0(K\|L)^*$ and $\phi_E \in C_0(K\|L)^\perp \equiv E^*$. Observe that ϕ_E can also be viewed as an element of $C_0(\overline{V}\|L)^\perp$ since the spaces $C_0(\overline{V}\|L)^\perp$ and $C_0(K\|L)^\perp$ are isometrically isomorphic (both coincide with E^*).

Given an open set $V \subseteq K$ compatible with L and $h : K \longrightarrow [0,1]$ a continuous function constant on L with support included in V, we denote by $P_{\overline{V}} : C_E(K\|L) \longrightarrow C_E(\overline{V}\|L)$ and $I_{h,\overline{V}} : C_E(\overline{V}\|L) \longrightarrow C_E(K\|L)$ the operators defined by

$$P_{\overline{V}}(f) = f|_{\overline{V}} \qquad \text{and} \qquad I_{h,\overline{V}}(\widetilde{f}) = h\widetilde{f}$$

for $f \in C_E(K\|L)$ and $\widetilde{f} \in C_E(\overline{V}\|L)$ respectively. We observe that $I_{h,\overline{V}}$ is well defined, that is, $h\widetilde{f}$ is a function in $C(K)$ with $h\widetilde{f}|_L \in E$ (indeed, $h\widetilde{f}$ is continuous in V as a product of two continuous functions and it is continuous in $K \setminus \operatorname{Supp}(h)$ as a constant function, since these two sets form an open cover of K we have that $h\widetilde{f}$ is continuous in K; being h constant on L, it is clear that $h\widetilde{f}|_L \in E$). Finally, If $V_1 \subseteq K$ is an open set compatible with L satisfying $\overline{V} \subseteq V_1$ we will also use the notation $P_{\overline{V}}$ for the restriction operator from $C_E(\overline{V}_1\|L)$ to $C_E(\overline{V}\|L)$. In the next result we gather some easy facts concerning these operators.

Lemma 3.3. *Let V and h be as above. Then, the following hold:*

(a) $P^*_{\overline{V}}(\phi)(f)=\phi(f|_{\overline{V}})$ *for* $\phi\in C_E(\overline{V}\|L)^*$ *and* $f\in C_E(K\|L)$.
(b) $I^*_{h,\overline{V}}(\phi)(\widetilde{f})=\phi(\widetilde{f}h)$ *for* $\phi\in C_E(K\|L)^*$ *and* $\widetilde{f}\in C_E(\overline{V}\|L)$.
(c) *If V_0 is an open set such that $\overline{V}_0\subseteq V$ and $h|_{\overline{V}_0}\equiv 1$, then $(I^*_{h,\overline{V}}P^*_{\overline{V}})(\mu)=\mu$ for every $\mu\in C(K)^*$ with* $\operatorname{Supp}(\mu)\subseteq V_0$.
(d) *If $E=C(L)$, then $C_E(K\|L)=C(K)$, $C_E(\overline{V}\|L)=C(\overline{V})$, and $P^{\overline{V}}P^*_{\overline{V}}(\mu)=\mu$ for every $\mu\in C(\overline{V})^*$.*

Proof. (a) and (b) are obvious from the definitions of the operators. To prove (c) we fix $\widetilde{f}\in C_E(\overline{V}\|L)$, $\mu\in C(K)^*$ with $\operatorname{Supp}(\mu)\subseteq V_0$ and we observe that

$$\begin{aligned}(I^*_{h,\overline{V}}P^*_{\overline{V}})(\mu)(\widetilde{f})&=P^*_{\overline{V}}(\mu)\big(I_{h,\overline{V}}(\widetilde{f})\big)=P^*_{\overline{V}}(\mu)(\widetilde{f}h)\\&=\mu\left((\widetilde{f}h)|_{\overline{V}}\right)=\int(\widetilde{f}h)|_{\overline{V}}\,d\mu=\int_{V_0}\widetilde{f}\,d\mu=\mu(\widetilde{f}).\end{aligned}$$

The first two assertions of (d) are obvious. For the third one, given $\mu\in C(\overline{V})^*$ and $\widetilde{f}\in C(\overline{V})$, use the regularity of the measure $P^*_{\overline{V}}(\mu)$ to find an open set $V_n\subseteq K$ satisfying

$$\overline{V}\subseteq V_n\qquad\text{and}\qquad\left|P^*_{\overline{V}}(\mu)\right|(V_n\setminus\overline{V})<\frac{1}{n}\tag{2}$$

for every $n\in\mathbb{N}$. Next, take $f_n\in C(K)$ satisfying

$$f_n|_{\overline{V}}\equiv\widetilde{f},\qquad f_n|_{K\setminus V_n}\equiv 0,\qquad\text{and}\qquad\|f_n\|=\|\widetilde{f}\|$$

for every $n\in\mathbb{N}$, and observe that

$$\begin{aligned}P^{\overline{V}}P^*_{\overline{V}}(\mu)(\widetilde{f})&=\big(P^*_{\overline{V}}(\mu)\big)|_{\overline{V}}(\widetilde{f})=\int_{\overline{V}}\widetilde{f}\,d\big(P^*_{\overline{V}}(\mu)\big)|_{\overline{V}}\\&=\int_K f_n\,dP^*_{\overline{V}}(\mu)-\int_{V_n\setminus\overline{V}}f_n\,dP^*_{\overline{V}}(\mu)\\&=P^*_{\overline{V}}(\mu)(f_n)-\int_{V_n\setminus\overline{V}}f_n\,dP^*_{\overline{V}}(\mu)=\mu(\widetilde{f})-\int_{V_n\setminus\overline{V}}f_n\,dP^*_{\overline{V}}(\mu).\end{aligned}$$

Therefore, using (2) and letting $n\to\infty$, it follows that $P^{\overline{V}}P^*_{\overline{V}}(\mu)(\widetilde{f})=\mu(\widetilde{f})$. □

Our first application uses the above operators in the simple case in which $E=C(L)$.

Proposition 3.4. *Let K be a compact space, let V_0, V_1, and V_2 be open nonempty subsets of K such that $\overline{V}_0\subseteq V_1$, and let $R:C(\overline{V}_2)\longrightarrow C(\overline{V}_1)$ be a linear operator. Suppose that all operators on $C(K)$ are weak multipliers. Then, there are a Borel function $g:\overline{V}_1\longrightarrow\mathbb{R}$ with support included in $\overline{V}_1\cap\overline{V}_2$ and a weakly compact operator $S:C(\overline{V}_1)^*\longrightarrow C(\overline{V}_2)^*$ such that*

$$R^*(\mu)=g\mu+S(\mu)$$

for every $\mu\in C(K)^*$ with $\operatorname{Supp}(\mu)\subseteq V_0$.

Proof. We fix a continuous function $h:K\longrightarrow[0,1]$ satisfying $h|_{\overline{V}_0}\equiv 1$ and $h|_{(K\setminus V_1)}\equiv 0$ and we define $R_0\in L(C(K))$ by

$$R_0(f)=I_{h,\overline{V_1}}RP_{\overline{V}_2}(f)\tag{3}$$

for $f \in C(K)$. Hence, there are a bounded Borel function $\widehat{g} : K \longrightarrow \mathbb{R}$ and a weakly compact operator $\widehat{S} \in L(C(K)^*)$ such that $R_0^*(\mu) = \widehat{g}\mu + \widehat{S}(\mu)$ for $\mu \in C(K)^*$, which allows us to write

$$P^{\overline{V}_2} R_0^* P^*_{\overline{V}_1} = P^{\overline{V}_2} \widehat{g}\, \mathrm{Id}_{C(K)^*}\, P^*_{\overline{V}_1} + P^{\overline{V}_2} \widehat{S} P^*_{\overline{V}_1}. \tag{4}$$

We claim that, considering the weakly compact operator given by $S = P^{\overline{V}_2}\widehat{S}P^*_{\overline{V}_1}$ and defining the functions $\breve{g} : K \longrightarrow \mathbb{R}$ and $g : \overline{V}_1 \longrightarrow \mathbb{R}$ by

$$\breve{g}(x) = \begin{cases} \widehat{g}(x) & \text{if } x \in \overline{V}_1 \cap \overline{V}_2 \\ 0 & \text{if } x \notin \overline{V}_1 \cap \overline{V}_2 \end{cases} \qquad \text{and} \qquad g = \breve{g}|_{\overline{V}_1}$$

the following holds for $\mu \in C(\overline{V}_1)^*$:

$$\big(P^{\overline{V}_2} R_0^* P^*_{\overline{V}_1}\big)(\mu) = g\mu + S(\mu). \tag{5}$$

Indeed, for $\mu \in C(\overline{V}_1)^*$ and $f \in C(\overline{V}_2)$ we observe that

$$\begin{aligned}\big(P^{\overline{V}_2}\widehat{g}\, \mathrm{Id}_{C(K)^*}\, P^*_{\overline{V}_1}\big)(\mu)(f) &= (\widehat{g}P^*_{\overline{V}_1}(\mu))|_{\overline{V}_2}(f) = \int_{\overline{V}_2} \widehat{g}f\, dP^*_{\overline{V}_1}(\mu) \\ &= \int_{\overline{V}_1\cap\overline{V}_2} \widehat{g}f\, dP^*_{\overline{V}_1}(\mu) = \int_{\overline{V}_1\cap\overline{V}_2} \breve{g}f\, dP^*_{\overline{V}_1}(\mu) = \int_K \breve{g}f\, dP^*_{\overline{V}_1}(\mu)\end{aligned}$$

and, for $n \in \mathbb{N}$, we use Lusin's Theorem (see [22, Theorem 21.4], for instance) to find a compact set $K_n \subseteq K$ such that

$$\breve{g}|_{K_n} \text{ is continuous on } K_n, \quad |P^*_{\overline{V}_1}(\mu)|(K\setminus K_n) < \frac{1}{n}, \quad \text{and} \quad |\mu|\big(\overline{V}_1 \setminus (\overline{V}_1 \cap K_n)\big) < \frac{1}{n}. \tag{6}$$

Using Tietze's extension Theorem we may find a continuous function $g_n : K \longrightarrow \mathbb{R}$ satisfying

$$g_n|_{K_n} = \breve{g}|_{K_n} \qquad \text{and} \qquad \|g_n\| = \|\breve{g}|_{K_n}\| \leqslant \|\breve{g}\|$$

for every $n \in \mathbb{N}$. Now it is easy to check that

$$\begin{aligned}\int_K \breve{g}f\, dP^*_{\overline{V}_1}(\mu) &= \int_K g_n f\, dP^*_{\overline{V}_1}(\mu) + \int_{K\setminus K_n} (\breve{g}-g_n)f\, dP^*_{\overline{V}_1}(\mu) \\ &= P^*_{\overline{V}_1}(\mu)(g_n f) + \int_{K\setminus K_n} (\breve{g}-g_n)f\, dP^*_{\overline{V}_1}(\mu) \\ &= \mu\big((g_n f)|_{\overline{V}_1}\big) + \int_{K\setminus K_n} (\breve{g}-g_n)f\, dP^*_{\overline{V}_1}(\mu) \\ &= \int_{\overline{V}_1} g_n f d\mu + \int_{K\setminus K_n} (\breve{g}-g_n)f\, dP^*_{\overline{V}_1}(\mu) \\ &= \int_{\overline{V}_1} g f d\mu + \int_{\overline{V}_1\setminus(\overline{V}_1\cap K_n)} (g_n-\breve{g}) f d\mu + \int_{K\setminus K_n} (\breve{g}-g_n)f\, dP^*_{\overline{V}_1}(\mu) \\ &= \mu(gf) + \int_{\overline{V}_1\setminus(\overline{V}_1\cap K_n)} (g_n-\breve{g}) f d\mu + \int_{K\setminus K_n} (\breve{g}-g_n)f\, dP^*_{\overline{V}_1}(\mu)\end{aligned}$$

which, letting $n \to \infty$ and using (6), implies that

$$\int_K \breve{g}f\, dP^*_{\overline{V}_1}(\mu) = \mu(gf)$$

and, therefore,

$$\big(P^{\overline{V}_2}\widehat{g}\, \mathrm{Id}_{C(K)^*}\, P^*_{\overline{V}_1}\big)(\mu)(f) = \mu(gf).$$

This, together with (4) and the definition of S, finishes the proof of the claim. On the other hand by (3), we can write

$$P^{\overline{V}_2} R_0^* P^*_{\overline{V}_1} = P^{\overline{V}_2} P^*_{\overline{V}_2} R^* I^*_{h,\overline{V}_1} P^*_{\overline{V}_1}.$$

So, if the support of μ is included in V_0, by Lemma 3.3, parts (c) and (d) we obtain

$$P^{\overline{V}_2} R_0^* P^*_{\overline{V}_1}(\mu) = R^*(\mu)$$

and, consequently, $R^*(\mu) = g\mu + S(\mu)$ follows from (5). □

Remark 3.5. The result above shows that if every operator on $C(K)$ is a weak multiplier then, in the above sense, there are also few operators on $C(\overline{V})$ for V open (since for such a closed set it is possible to define an appropriate function h as in the proof). In general, one cannot replace closures of open sets by general closed sets: it is shown in [9] that under CH, there are compact K's as above which contain $\beta\mathbb{N}$ (and, of course, there are many operators on $C(\beta\mathbb{N}) \equiv \ell_\infty$). On the other hand, using the set-theoretic principle $\diamondsuit$, it is also shown in [9] that there are K's such that for every infinite closed $K' \subseteq K$, all operators on the space $C(K')$ are weak multipliers.

Corollary 3.6. *Let K be a compact space, let V_0, V_1, and V_2 be open nonempty subsets of K such that $\overline{V}_0 \subseteq V_1$ and $\overline{V}_1 \cap \overline{V}_2 = \emptyset$, and let $R : C(\overline{V}_2) \longrightarrow C(\overline{V}_1)$ be a linear operator. Suppose that all operators on $C(K)$ are weak multipliers. Then, $P_{\overline{V}_0} R$ is weakly compact.*

Proof. By Proposition 3.4 there is a weakly compact operator $S : C^*(\overline{V}_1) \longrightarrow C^*(\overline{V}_2)$ such that $R^*(\mu) = S(\mu)$ for every measure μ with support included in $\overline{V}_0$. In other words, $(P_{\overline{V}_0} R)^* = R^* P^*_{\overline{V}_0}$ is weakly compact and so, by Gantmacher theorem, $P_{\overline{V}_0} R$ is weakly compact. □

The following result is an easy consequence of the Dieudonné-Grothendieck theorem which we state for the sake of clearness.

Lemma 3.7. *Let K be a compact space, X a Banach space and $S : X^* \longrightarrow C(K)^*$ a weakly compact operator. Then, for every bounded subset $B \subseteq X^*$, the set*

$$\big\{x \in K \,:\, \exists \phi \in B \text{ so that } S(\phi)(\{x\}) \neq 0\big\}$$

is countable.

Proof. Suppose that the set

$$\big\{x \in K \,:\, \exists \phi \in B \text{ so that } S(\phi)(\{x\}) \neq 0\big\}$$

is uncountable for some bounded set $B \subseteq X^*$. Then, there is $\varepsilon > 0$ so that the set

$$\big\{x \in K \,:\, \exists \phi \in B \text{ so that } |S(\phi)(\{x\})| \geqslant \varepsilon\big\}$$

is infinite, which contradicts the fact of being $S(B)$ relatively weakly compact by the Dieudonné-Grothendieck theorem (see [6, Theorem VII.14], for instance). □

Lemma 3.8. *Let K be a compact space, let V_0, V_1, and V_2 be open nonempty subsets of K compatible with L such that $\overline{V}_0 \subseteq V_1$ and $\overline{V}_1 \cap L = \emptyset$, and let $T : C_E(K\|L) \longrightarrow C_E(K\|L)$ be a linear operator. Then, there exists an operator $R : C(\overline{V}_1) \longrightarrow C_E(\overline{V}_2\|L)$ such that*

$$R^*(\phi)|_{\overline{V}_0} = (T^* P^*_{\overline{V}_2})(\phi)|_{\overline{V}_0}$$

for all $\phi \in C_E(\overline{V}_2\|L)^*$.

Proof. Take a continuous function $h : K \longrightarrow [0,1]$ satisfying $h|_{\overline{V}_0} \equiv 1$ and $h|_{(K\setminus V_1)} \equiv 0$, and define the operator $R = P_{\overline{V}_2} T I_{h,\overline{V}_1}$. Given $\phi \in C_E(\overline{V}_2\|L)^*$ and $f \in C(\overline{V}_1)$ with $\operatorname{Supp}(f) \subseteq \overline{V}_0$, by parts (b) and (a) of Lemma 3.3 and using the facts $h|_{V_0} \equiv 1$ and $\operatorname{Supp}(f) \subseteq \overline{V}_0$, we can write

$$R^*(\phi)(f) = I^*_{h,\overline{V}_1} T^* P^*_{\overline{V}_2}(\phi)(f) = T^* P^*_{\overline{V}_2}(\phi)(fh) = T^* P^*_{\overline{V}_2}(\phi)(f)$$

which finishes the proof. □

We are ready to state and prove the main result of the section.

Theorem 3.9. *Let K be a perfect compact space such that all operators on $C(K)$ are weak multipliers, let $L \subseteq K$ be closed and nowhere dense, and E a closed subspace of $C(L)$. Then, $C_E(K\|L)$ is extremely non-complex.*

Proof. Fixed $T \in L\big(C_E(K\|L)\big)$, we have to show that its square satisfies the Daugavet equation. By Lemma 2.5, it is enough to prove that the set $\{x \in K \setminus L \,:\, g_{(T^2)^*}(x) \geqslant 0\}$ is dense in $K \setminus L$. To do so, we proceed ad absurdum: suppose that there is an open set $U_1 \subseteq K$ such that $\overline{U}_1 \cap L = \emptyset$ and $g_{(T^2)^*}(x) < 0$ for each $x \in U_1$. By going to a subset, we may w.l.o.g. assume that $\overline{U}_1$ is a G_δ set. Therefore, we can find open sets $W_n \subseteq K$ such that $\bigcap_{n\in\mathbb{N}} W_n = \overline{U}_1$, $\overline{W}_{n+1} \subseteq W_n$, and $K \setminus W_n$ is the closure of an open set containing L for every $n \in \mathbb{N}$. Next, we fix a nonempty open set $U_0 \subseteq K$ with $\overline{U}_0 \subseteq U_1$, and we observe that it is uncountable (since K is perfect) so, there is $\varepsilon > 0$ such that the set

$$A = \{x \in U_0 \,:\, g_{(T^2)^*}(x) < -\varepsilon\}$$

is uncountable. Moreover, we claim that there is $n_0 \in \mathbb{N}$ such that the set

$$B = \left\{x \in A \,:\, |T^*(\delta_x)|(W_{n_0} \setminus \overline{U}_1)) < \frac{\varepsilon}{2\|T\|}\right\}$$

is uncountable. Indeed, fixed $x \in A$, the regularity of the measure $T^*(\delta_x)$ implies that there is $n \in \mathbb{N}$ so that $|T^*(\delta_x)|(W_n \setminus \overline{U}_1)) < \frac{\varepsilon}{2\|T\|}$ which gives us

$$A = \bigcup_{n\in\mathbb{N}} \left\{x \in A \,:\, |T^*(\delta_x)|(W_n \setminus \overline{U}_1)) < \frac{\varepsilon}{2\|T\|}\right\}$$

and the uncountability of A finishes the argument.

For $x \in B$, we write

$$\phi_x = T^*(\delta_x)|_{K\setminus(L\cup W_{n_0})} + T^*(\delta_x)_E \in C_E\big((K \setminus W_{n_0})\|L\big)^*$$

and we can decompose $T^*(\delta_x)$ as follows:

$$T^*(\delta_x) = T^*(\delta_x)|_{\overline{U}_1} + T^*(\delta_x)|_{W_{n_0}\setminus\overline{U}_1} + \phi_x\,.$$

Hence, for every $x \in B$, we get

$$-\varepsilon > \big[(T^*)^2(\delta_x)\big](\{x\}) = T^*\big[T^*(\delta_x)|_{\overline{U}_1} + T^*(\delta_x)|_{W_{n_0}\setminus\overline{U}_1} + \phi_x\big]\{x\}. \tag{7}$$

However, the following claims show that this is impossible.

Claim 1. $\big\|T^*\big[T^*(\delta_x)|_{W_{n_0}\setminus\overline{U}_1}\big]\big\| < \varepsilon/2$.

Proof of claim 1. It follows obviously from the choice of n_0.

Claim 2. The function $x \longmapsto T^*(T^*(\delta_x)|_{\overline{U}_1})(\{x\})$ is non-negative for all but countably many $x \in B$.

Proof of claim 2. By Lemma 3.8 applied to $V_0 = U_1$, $V_1 = W_{n_0}$ and $V_2 = U_1$, we obtain an operator $R : C(\overline{W}_{n_0}) \longrightarrow C(\overline{U}_1)$ such that

$$R^*(\psi)|_{\overline{U}_1} = T^* P^*_{\overline{U}_1}(\psi)|_{\overline{U}_1}$$

for $\psi \in C(\overline{U}_1)^*$. On the other hand, by Proposition 3.4 applied to R and $V_0 = U_0$, $V_1 = U_1$, $V_2 = W_{n_0}$, we get a bounded Borel function $g : \overline{U}_1 \longrightarrow \mathbb{R}$ and a weakly compact operator $S : C(\overline{U}_1)^* \longrightarrow C(\overline{W}_{n_0})^*$ such that

$$R^*(\mu) = g\mu + S(\mu)$$

for every μ with support in U_0. In particular, for $x \in B$ we have

$$T^*(\delta_x)|_{\overline{U}_1} = T^* P^*_{\overline{U}_1}(\delta_x)|_{\overline{U}_1} = R^*(\delta_x)|_{\overline{U}_1} = g\delta_x + S(\delta_x)|_{\overline{U}_1} = g(x)\delta_x + S(\delta_x)|_{\overline{U}_1}$$

and using this twice, we get

$$\begin{aligned}
T^*(T^*(\delta_x)|_{\overline{U}_1})(\{x\}) &= T^*\Big[g(x)\delta_x + S(\delta_x)|_{\overline{U}_1}\Big](\{x\}) = g(x)T^*(\delta_x)(\{x\}) + T^*\big[S(\delta_x)|_{\overline{U}_1}\big](\{x\})\\
&= g(x)T^*(\delta_x)|_{\overline{U}_1}(\{x\}) + R^*\big[S(\delta_x)|_{\overline{U}_1}\big](\{x\})\\
&= g(x)^2 + g(x)S(\delta_x)|_{\overline{U}_1}(\{x\}) + \big(gS(\delta_x)|_{\overline{U}_1}\big)(\{x\}) + S\big[S(\delta_x)|_{\overline{U}_1}\big](\{x\})\\
&= g(x)^2 + (2gP^{\overline{U}_1}S)(\delta_x)(\{x\}) + (SP^{\overline{U}_1}S)(\delta_x)(\{x\})
\end{aligned}$$

for every $x \in B$. Finally, since $gP^{\overline{U}_1}S$ and $SP^{\overline{U}_1}S$ are weakly compact operators, we conclude by Lemma 3.7 that $(gP^{\overline{U}_1}S)(\delta_x)(\{x\})$ and $(SP^{\overline{U}_1}S)(\delta_x)(\{x\})$ are zero for all but countably many x, completing the proof of the claim.

Claim 3. $T^*(\phi_x)(\{x\}) = 0$ for all but countably many $x \in U_0$.

Proof of claim 3. By Lemma 3.8 applied to $V_0 = U_0$, $V_1 = U_1$, and $V_2 = K \setminus \overline{W}_{n_0+1}$ we obtain an operator $R : C(\overline{U}_1) \longrightarrow C_E\big((K \setminus W_{n_0+1})\|L\big)$ such that $R^*(\phi)(\{x\}) = T^*P^*_{K\setminus W_{n_0}}(\phi)(\{x\})$ for $x \in U_0$ and $\phi \in C_E\big((K \setminus W_{n_0+1})\|L\big)^*$. We denote $J : C_E\big((K \setminus W_{n_0+1})\|L\big) \longrightarrow C(K \setminus W_{n_0+1})$ the inclusion operator and we apply Corollary 3.6 for the operator JR and the open sets $V_0 = K\setminus\overline{W}_{n_0}$, $V_1 = K \setminus \overline{W}_{n_0+1}$, and $V_2 = U_1$ to obtain that the operator $P_{K\setminus W_{n_0}}JR$ is weakly compact.

Besides, we recall that $\phi_x \in C_E\big((K \setminus W_{n_0}\|L)\big)^*$ and that it can be viewed as an element of $C_E\big((K \setminus W_{n_0+1}\|L)\big)^*$ by just extending it by zero outside $K \setminus W_{n_0}$. For $x \in U_0$ we take $\widetilde{\phi}_x$ a Hahn-Banach extension of ϕ_x to $C(K \setminus W_{n_0})$ and we observe that $J^*P^*_{K\setminus W_{n_0}}(\widetilde{\phi}_x) = \phi_x$. Indeed, for $f \in C_E\big((K \setminus W_{n_0+1})\|L\big)$ we have that

$$\begin{aligned}
J^*P^*_{K\setminus W_{n_0}}(\widetilde{\phi}_x)(f) &= \widetilde{\phi}_x\big(P_{K\setminus W_{n_0}}(Jf)\big) = \widetilde{\phi}_x\big(P_{K\setminus W_{n_0}}(f)\big)\\
&= \widetilde{\phi}_x\big(f|_{K\setminus W_{n_0}}\big) = \phi_x\big(f|_{K\setminus W_{n_0}}\big) = \phi_x(f).
\end{aligned}$$

Therefore, for $x \in U_0$, we can write

$$\begin{aligned}
T^*(\phi_x)(\{x\}) &= T^*P^*_{K\setminus W_{n_0}}(\phi_x)(\{x\}) = R^*(\phi_x)(\{x\})\\
&= R^*J^*P^*_{K\setminus W_{n_0}}(\widetilde{\phi}_x)(\{x\}) = (P_{K\setminus W_{n_0}}JR)^*(\widetilde{\phi}_x)(\{x\})
\end{aligned}$$

where we are identifying ϕ_x with its extension by zero to K. Now the proof of the claim is finished by just applying Lemma 3.7 to the operator $P_{K\setminus W_{n_0}}JR$.

Finally, the claims obviously contradict (7) completing the proof of the theorem. □

When $E = \{0\}$, we get a sufficient condition to get that a space of the form $C_0(K \setminus L)$ is extremely non-complex.

Corollary 3.10. *Let K be a compact space such that all operators on $C(K)$ are weak multipliers. Suppose $L \subseteq K$ is closed and nowhere dense. Then, $C_0(K \setminus L)$ is extremely non-complex.*

To show that there are extremely non-complex spaces of the form $C_E(K\|L)$ which are not isomorphic to the $C(K')$ spaces, we need the following (well-known) result which allows us to construct spaces $C_E(K\|L)$ for every perfect separable compact space K and every separable Banach space E.

Lemma 3.11. *Let K be a perfect compact space. Then:*

(a) *There is a nowhere dense closed subset $L \subset K$ such that L can be continuously mapped onto the Cantor set.*
(b) *Therefore, every separable Banach space E is (isometrically isomorphic to) a subspace of $C(L)$.*

Proof. (a). As K is perfect, given an nonempty open subset U in K and $x \in U$, there are two nonempty open subsets V_1, V_2 of U such that

$$\overline{V}_1 \cap \overline{V}_2 = \emptyset \qquad \text{and} \qquad x \notin \overline{V}_i \ (i = 1, 2).$$

This allows us to construct a family of open sets U_s for $s \in \{0,1\}^{<\omega}$ such that

$$U_\emptyset = K, \quad \overline{U}_{s\frown 0} \cap \overline{U}_{s\frown 1} = \emptyset, \quad \overline{U}_{s\frown 0}, \overline{U}_{s\frown 1} \subseteq U_s, \quad \text{and} \quad U_s \setminus [\overline{U}_{s\frown 0} \cup \overline{U}_{s\frown 1}] \neq \emptyset.$$

Take any point y_s in the above difference. Define L to be the set of all the accumulation points of the set $\{y_s : s \in \{0,1\}^{<\omega}\}$.

For $n \in \mathbb{N}$, let $f_n : K \longrightarrow [0,1]$ be such continuous functions that for all $s \in \{0,1\}^n$ we have $f_n[\overline{U}_{s\frown 0}] = \{0\}$ and $f_n[\overline{U}_{s\frown 1}] = \{1\}$ which can be easily obtained since $\overline{U}_s \cap \overline{U}_{s'} = \emptyset$ if $s, s' \in \{0,1\}^n$ are distinct. Let $f : K \longrightarrow [0,1]^{\mathbb{N}}$ be defined by $f(x)(n) = f_n(x)$. We claim that $f|_L$ satisfies the lemma. One can easily check that f is continuous since f_ns are continuous.

Because each $\overline{U}_s$ contains infinitely many points y_t, we have that $L \cap \overline{U}_s \neq \emptyset$ for each $s \in \{0,1\}^{<\omega}$. Note that if $x \in \overline{U}_s$ and $s \in \{0,1\}^n$, then $f(x)|\{0, ..., n-1\} = s$. So, as the image of L under f, is closed, it contains $\{0,1\}^{\mathbb{N}}$.

On the other hand if $x \in L$, then for each $n \in \mathbb{N}$ there is $s \in \{0,1\}^n$ such that $x \in \overline{U}_s$, this is because $K \setminus \bigcup\{\overline{U}_s : |s| = n\}$ contains only finitely many points y_t and hence no element of L. Thus $f_n(x) \in \{0,1\}$ if $x \in L$ which gives that $f[L] \subseteq \{0,1\}^{\mathbb{N}}$, which together with the previous observation gives that $f[L] = \{0,1\}^{\mathbb{N}}$.

Finally let us prove that L has empty interior, and so, as a closed set, it is nowhere dense. It is enough to see that L has empty interior in the subspace $\{y_s : s \in \{0,1\}^{<\omega}\} \cup L$. This is true since $\{y_s : s \in \{0,1\}^{<\omega}\}$ is dense and open in $\{y_s : s \in \{0,1\}^{<\omega}\} \cup L$, as each point y_s is isolated by $U_s \setminus [\overline{U}_{s\frown 0} \cup \overline{U}_{s\frown 1}]$ from the remaining points.

(b). Since the function $f|_L : L \longrightarrow 2^\omega$ of the above item is continuous and surjective, $C(2^\omega)$ embeds isometrically into $C(L)$ by just composing every element in $C(2^\omega)$ with $f|_L$. Since every separable Banach space E embeds isometrically into $C(2^\omega)$ (Banach-Mazur theorem), we get $E \subseteq C(L)$ isometrically. □

Remark 3.12. If K is a compact space such that all operators on $C(K)$ are weak multipliers, then it is easier to prove the existence of $L \subseteq K$ closed nowhere dense which maps onto $[0,1]$ giving also (b) above. Indeed, $C(K)$ is a Grothendieck space by [19, Theorem 2.4], so K has no convergent sequence (otherwise it would give rise to a complemented copy of c_0 contradicting the Grothendieck property). Now, take any discrete sequence $\{x_n : n \in \mathbb{N}\} \subseteq K$ and consider the set L of all its accumulation points. Then, L is perfect because an isolated point would produce a convergent subsequence of $\{x_n : n \in \mathbb{N}\}$, so L continuously maps onto $[0,1]$ [25, Theorem 8.5.4]. To see that L is nowhere dense we use the discreteness of $\{x_n : n \in \mathbb{N}\}$. If $U \subseteq K$ is open and intersects L, then there is $n_0 \in \mathbb{N}$ such that $x_{n_0} \in U$; but by the discreteness of $\{x_n : n \in N\}$, there is an open neighborhood V of x_{n_0} not containing the remaining x_n's and hence, disjoint from L. Therefore, $V \cap U$ is an open subset of U disjoint with L, proving that L is nowhere dense.

Now, we take a perfect compact space K such that every operator on $C(K)$ is a weak multiplier [19], and we use Lemma 3.11 to find a nowhere dense closed subset L such that $C(L)$ contains isometric copies of every separable Banach space. Then, for every $E \subset C(L)$, $C_E(K\|L)$ is extremely non-complex by Theorem 3.9 and $C_E(K\|L)^* = C_0(K\|L)^* \oplus_1 E^*$ by Lemma 2.2. If E is infinite-dimensional and reflexive, $C_E(K\|L)$ is not isomorphic to a $C(K')$ space, since $C(K')^*$ never contains complemented infinite-dimensional reflexive subspaces (see [1, Proposition 5.6.1], for instance). Let us state all what we have proved.

Example 3.13.

(a) *For every separable Banach space E, there is an extremely non-complex Banach space $C_E(K\|L)$ such that E^* is an L-summand in $C_E(K\|L)^*$.*

(b) If E is infinite-dimensional and reflexive, then such $C_E(K\|L)$ is not isomorphic to any $C(K')$ space.
(c) Therefore, *there are extremely non-complex Banach spaces which are not isomorphic to $C(K)$ spaces.*

We finish the section commenting that some $C(K')$ spaces with many operators which can be viewed as $C_E(K\|L)$ spaces where $C(K)$ has few operators and for which our previous results apply.

Remark 3.14. Let $L \subseteq K$ be a nowhere dense subset of a compact K as before. Consider the topological quotient map $q : K \longrightarrow K_L$, where K_L is obtained from K by identifying all points of L to one point. The canonical isometric embedding I_q of $C(K_L)$ into $C(K)$ defined by $I_q(f) = f \circ q$ has the image equal to the subspace of $C(K)$ consisting of all functions constant on L. Thus $C(K_L)$ is isometric to $C_E(K\|L)$, where E is the subspace of $C(L)$ of all constant functions. Hence, by the results above, if all operators on $C(K)$ are weak multipliers, then all spaces of the form $C(K_L)$ are extremely non-complex. It turns out that the spaces of [20] can be realized as spaces of this form. In particular, there are extremely non-complex spaces of the form $C_E(K\|L)$ which have many operators besides weak multipliers. For example, take K such that all operators on $C(K)$ are weak multipliers. Choose a discrete sequence (x_n) in K and let L be the set of its accumulation points. Then the sequence (x_n) has a unique accumulation point in K_L, that is, it is a convergent sequence. By a well known fact, this means that $C(K_L) \equiv C_E(K\|L)$ has a complemented copy of c_0 and so is not Grothendieck, hence it has more operators than weak multipliers by results of [19] (actually, many operators which are not weak multipliers can be directly obtained from automorphisms of the complemented copy of c_0 generated by permutations of the natural numbers).

4. Isometries on extremely non-complex spaces

The following result shows that the group of isometries of an extremely non complex Banach space is a discrete Boolean group.

Theorem 4.1. *Let X be an extremely non-complex Banach space. Then*

(a) *If $T \in \operatorname{Iso}(X)$, then $T^2 = \operatorname{Id}$.*
(b) *As a consequence, for every $T_1, T_2 \in \operatorname{Iso}(X)$, $T_1T_2 = T_2T_1$.*
(c) *For every $T_1, T_2 \in \operatorname{Iso}(X)$, $\|T_1 - T_2\| \in \{0, 2\}$.*

Proof. (a). Given $T \in \operatorname{Iso}(X)$, we define the operator $S = \frac{1}{\sqrt{2}}\big(T - T^{-1}\big)$ and we observe that $S^2 = \frac{1}{2}\,T^2 - \operatorname{Id} + \frac{1}{2}\,T^{-2}$. Since X is extremely non-complex, we get

$$1 + \|S^2\| = \|\operatorname{Id} + S^2\| = \left\|\frac{1}{2}\,T^2 + \frac{1}{2}\,T^{-2}\right\| \leqslant 1$$

and, therefore, $S^2 = 0$. This gives us that $\operatorname{Id} = \frac{1}{2}T^2 + \frac{1}{2}T^{-2}$. Finally, since Id is an extreme point of $L(X)$ (see [24, Proposition 1.6.6], for instance) and $\|T^2\| \leqslant 1$, $\|T^{-2}\| \leqslant 1$, we get $T^2 = \operatorname{Id}$.

(b). Commutativity comes routinely from the first part since $T_1T_2 \in \operatorname{Iso}(X)$, so

$$\operatorname{Id} = \big(T_1T_2\big)^2 = T_1T_2T_1T_2$$

which finishes the proof by just multiplying by T_1 from the left and by T_2 from the right.

(c). We start observing that $\|\operatorname{Id} - T\| \in \{0, 2\}$ for every $T \in \operatorname{Iso}(X)$. Indeed, from (a) we have

$$\big(\operatorname{Id} - T\big)^2 = \operatorname{Id} + \operatorname{Id} - 2T = 2(\operatorname{Id} - T),$$

which gives us that

$$2\|\operatorname{Id} - T\| = \big\|(\operatorname{Id} - T)^2\big\| \leqslant \|\operatorname{Id} - T\|^2.$$

Therefore, we get either $\|\operatorname{Id} - T\| = 0$ or $\|\operatorname{Id} - T\| \geqslant 2$. Now, if $T_1, T_2 \in \operatorname{Iso}(X)$ we observe that

$$\|T_1 - T_2\| = \|T_1(\operatorname{Id} - T_1T_2)\| = \|\operatorname{Id} - T_1T_2\| \in \{0, 2\}. \qquad \square$$

As an immediate consequence we obtain the following result. Let us observe that there is no topological consideration on the semigroup.

Corollary 4.2. *If X is an extremely non-complex Banach space and $\Phi : \mathbb{R}_0^+ \longrightarrow \operatorname{Iso}(X)$ is a one-parameter semigroup, then $\Phi(\mathbb{R}_0^+) = \{\mathrm{Id}\}$.*

Proof. Just observe that $\Phi(t) = \Phi(t/2 + t/2) = \Phi(t/2)^2 = \mathrm{Id}$ for every $t \in \mathbb{R}_0^+$. □

Let X be a Banach space. A projection $P \in L(X)$ is said to be *unconditional* if $2P - \mathrm{Id} \in \operatorname{Iso}(X)$ (equivalently, $\|2P - \mathrm{Id}\,\| = 1$). We write $\operatorname{Unc}(X)$ for the set of unconditional projections on X. It is straightforward to show that $P \in \operatorname{Unc}(X)$ if and only if $P = \frac{1}{2}(\mathrm{Id} + T)$ for some $T \in \operatorname{Iso}(X)$ with $T^2 = \mathrm{Id}$. It is then immediate that $\operatorname{Unc}(X)$ identifies with $\{T \in \operatorname{Iso}(X) : T^2 = \mathrm{Id}\}$ and both sets are Boolean groups: the group operation in $\{T \in \operatorname{Iso}(X) : T^2 = \mathrm{Id}\}$ is just the composition and so the group operation in $\operatorname{Unc}(X)$ is

$$(P_1, P_2) \longmapsto P_1 + P_2 - P_1P_2.$$

It also follows that all unconditional projections commute.

If X is extremely non-complex, the set $\{T \in \operatorname{Iso}(X) : T^2 = \mathrm{Id}\}$ is the whole $\operatorname{Iso}(X)$ (Theorem 4.1). We summarize all of this in the next result, where we will also discuss when these Boolean groups are actually Boolean algebras. The proof is completely straightforward. We refer the reader to the book [18, §1.8] for background on Boolean algebras of projections.

Proposition 4.3. *Let X be an extremely non-complex Banach space.*

(a) $\operatorname{Iso}(X)$ *is a Boolean group for the composition operation.*
(b) $\operatorname{Unc}(X)$ *is (equivalently,* $\operatorname{Iso}(X)$ *is isomorphic to) a Boolean algebra if, and only if, $P_1P_2 \in \operatorname{Unc}(X)$ for every $P_1, P_2 \in \operatorname{Unc}(X)$ if, and only if, $\|\,\mathrm{Id} + T_1 + T_2 - T_1T_2\| = 2$ for every $T_1, T_2 \in \operatorname{Iso}(X)$.*

We will show later that for many examples of extremely non-complex Banach spaces the set of unconditional projections is a Boolean algebra, but we do not know if this always happens.

5. Surjective isometries on extremely non-complex $C_E(K\|L)$ spaces

Our aim in this section is to describe the group of isometries of the spaces $C_E(K\|L)$ when they are extremely non-complex. We will deduce all the results from the following theorem.

Theorem 5.1. *Suppose that the space $C_E(K\|L)$ is extremely non-complex. Then, for every $T \in \operatorname{Iso}\big(C_E(K\|L)\big)$ there is a continuous function $\theta : K \setminus L \longrightarrow \{-1, 1\}$ such that*

$$[T(f)](x) = \theta(x)f(x)$$

for all $x \in K \setminus L$ and $f \in C_E(K\|L)$.

Proof. We divide the proof into several claims.

Claim 1. The set $D_0 = \{x \in K \setminus L \,:\, \exists\, y \in K \setminus L,\ \theta_0 \in \{-1, 1\} \text{ with } T^*(\delta_x) = \theta_0\delta_y\}$ is dense in K.

Proof of claim 1. Let W be a nonempty open subset of K. Since $K \setminus L$ is open and dense in K, there is V nonempty and open satisfying $V \subseteq W \cap (K \setminus L)$. Now, $\{\delta_x : x \in V\}$ is a subset of $\operatorname{ext}(B_{C_E(K\|L)})$ by Lemma 2.4, and it is easy to check that it is weak* open in $\operatorname{ext}(B_{C_E(K\|L)^*})$ (indeed, for $x_0 \in V$ take a non-negative $f \in C_0(K\|L)$ such that $f(x_0) = 1$ and $f(K \setminus V) = 0$, and observe that $\delta_{x_0} \in \{\delta_x : \delta_x(f) > 1/2\} \subset \{\delta_x : x \in V\}$). Now, being T^* a weak* continuous surjective isometry, the mapping

$$T^* : \big(\operatorname{ext}(B_{C_E(K\|L)^*}), w^*\big) \longrightarrow \big(\operatorname{ext}(B_{C_E(K\|L)^*}), w^*\big)$$

is a homeomorphism and so, the set $\{T^*(\delta_x) : x \in V\}$ is weak* open in $\operatorname{ext}(B_{C_E(K\|L)^*})$. Since, by Lemma 2.4, the set $\big\{\theta\,\delta_y : y \in K \setminus L,\ \theta \in \{-1, 1\}\big\}$ is weak* dense in $\operatorname{ext}(B_{C_E(K\|L)^*})$, there are

$x \in V$, $y \in K \setminus L$, and $\theta_0 \in \{-1, 1\}$ such that $T^*(\delta_x) = \theta_0 \delta_y$, which implies $x \in V \cap D_0 \subseteq W \cap D_0$, finishing the proof of claim 1.

Now, we can consider functions $\phi : D_0 \longrightarrow D_0$ and $\theta : D_0 \longrightarrow \{-1, 1\}$ such that

$$T^*(\delta_x) = \theta(x)\, \delta_{\phi(x)} \tag{8}$$

for all $x \in D_0$. Since $T^2 = \mathrm{Id}$ by Theorem 4.1, if $x \in D_0$ and $T^*(\delta_x) = \pm\delta_y$, then $T^*(\delta_y) = \pm\delta_x$ and so $y \in D_0$. Therefore, ϕ is a well defined function from D_0 into itself. Moreover, it also follows that

$$\phi^2(x) = x \quad \text{and} \quad \theta(x)\, \theta(\phi(x)) = 1 \qquad (x \in D_0). \tag{9}$$

Indeed, given $x \in D_0$, we use the fact that $(T^*)^2 = \mathrm{Id}$ to get

$$\delta_x = T^*(T^*(\delta_x)) = T^*(\theta(x)\delta_{\phi(x)}) = \theta(x)\theta(\phi(x))\, \delta_{\phi^2(x)}.$$

Claim 2. ϕ is a homeomorphism of D_0.

Proof of claim 2. As ϕ^2 is the identity on D_0, it is enough to prove that ϕ is continuous. To do so, fixed $x_0 \in D_0$ and an open subset W of $K \setminus L$ with $\phi(x_0) \in W$, we have to show that $\phi^{-1}(W \cap D_0)$ is a neighborhood of x_0 in D_0. Indeed, we consider a continuous function $f_0 \in C_0(K\|L) \subseteq C_E(K\|L)$ such that

$$f_0(\phi(x_0)) = 1 = \|f_0\| \qquad \text{and} \qquad f_0 \equiv 0 \text{ in } K \setminus W.$$

Since the mapping

$$x \longmapsto [T^*(\delta_x)](f_0) = \theta(x) f_0(\phi(x)) \qquad \bigl(x \in D_0\bigr)$$

is continuous at x_0, there is an open neighborhood U_0 of x_0 such that

$$\bigl||f_0(\phi(x))| - 1\bigr| \leqslant \bigl|\theta(x) f_0(\phi(x)) - \theta(x_0) f_0(\phi(x_0))\bigr| < \frac{1}{2} \quad \bigl(x \in U_0 \cap D_0\bigr).$$

Since $f_0 \equiv 0$ outside W, we get that $U_0 \cap D_0 \subseteq \phi^{-1}(W \cap D_0)$.

Claim 3. $\phi(x) = x$ for all $x \in D_0$.

Proof of claim 3. Suppose otherwise that there are $x_0, y_0 \in D_0$ such that $\phi(x_0) = y_0 \neq x_0$. Let $V_i \subseteq \overline{V}_i \subseteq K \setminus L$ with $i = 1, 2$ be open subsets of K satisfying

$$x_0 \in V_1, \quad y_0 \in V_2, \quad \overline{V}_1 \cap \overline{V}_2 = \emptyset, \quad \text{and} \quad V_1 \cap D_0 \subseteq \phi^{-1}(V_2 \cap D_0).$$

The last condition obviously implies $\phi(V_1 \cap D_0) \subseteq V_2$ and, since ϕ is a homeomorphism of D_0, it follows that $\phi(V_1 \cap D_0)$ is open in D_0. Therefore, we can find $V_0 \subseteq V_2$ an open subset of K such that $V_0 \cap D_0 = \phi(V_1 \cap D_0)$. Then, we may find $g \in C_0(K\|L) \subseteq C_E(K\|L)$ satisfying $g(x_0) = 1$, $g(y_0) = -1$, $g(x) \in [-1, 0]$ for $x \in V_0$, $g(x) \in [0, 1]$ for $x \in V_1$, and $g(x) = 0$ for $x \notin V_1 \cup V_0$. In particular, for $x \in D_0$, we have that

$$g(x)\, g(\phi(x)) \in [-1, 0].$$

Next, we define the operator $T_g : C_E(K\|L) \longrightarrow C_E(K\|L)$ by $T_g(f) = gf$, which is well defined (since $g|_L \equiv 0$) and satisfies $T_g^*(\delta_x) = g(x)\delta_x$ for each $x \in K \setminus L$. Finally, we consider the composition $S = T_g T$ and, for $x \in D_0$, we use (8) and (9) to write

$$(S^*)^2(\delta_x) = T^*(T_g^*(T^*(T_g^*(\delta_x)))) = \theta(x)\theta(\phi(x))g(x)g(\phi(x))\delta_{\phi^2(x)} = g(x)g(\phi(x))\delta_x.$$

This, together with our choice of g, tells us that

$$\bigl\|[\mathrm{Id} + (S^*)^2](\delta_x)\bigr\| \leqslant 1 \qquad \bigl(x \in D_0\bigr).$$

As D_0 is dense in K by claim 1 and $\mathrm{Id} + (S^*)^2$ is weak*-continuous, we deduce that $\|\mathrm{Id} + (S^*)^2\| \leqslant 1$. Now, the fact that $C_E(K\|L)$ is extremely non-complex implies that $S^2 = 0$ which is a contradiction since $(S^*)^2(\delta_{x_0}) = -\delta_{x_0} \neq 0$.

Claim 4. $D_0 = K \setminus L$.

Proof of claim 4. Let us fix $x_0 \in K \setminus L$. Since D_0 is dense in $K \setminus L$, we may find a net $(x_\lambda)_{\lambda\in\Lambda}$ in D_0 such that $(x_\lambda)_{\lambda\in\Lambda} \longrightarrow x_0$, so $\bigl(T^*(\delta_{x_\lambda})\bigr)_{\lambda\in\Lambda} \longrightarrow T^*(\delta_{x_0})$. But $T^*(\delta_{x_\lambda}) = \theta(x_\lambda)\, \delta_{x_\lambda}$, so the only possible accumulation points of the net $\big(T^*(\delta_{x_\lambda})\big)_{\lambda\in\Lambda}$ are $+\delta_{x_0}$ and $-\delta_{x_0}$. Therefore, $x_0 \in D_0$ as claimed.

Claim 5. θ is continuous on $K \setminus L$.

Proof of claim 5. We fix $x_0 \in K \setminus L$ and an open subset W of K such that $x_0 \in W \subseteq \overline{W} \subseteq K \setminus L$ and we take a function $f \in C_0(K\|L) \subseteq C_E(K\|L)$ satisfying $f|_{\overline{W}} \equiv 1$. Since the mapping

$$x \longmapsto \psi(x) = [T^*(\delta_x)](f) = \theta(x)\, f(x) \qquad \big(x \in K \setminus L\big)$$

is continuous and $\psi|_W \equiv \theta|_W$, we get the continuity of θ at x_0. □

We are now able to completely describe the set of surjective isometries in some special cases. The first one covers the case when K and $K \setminus L$ are connected.

Corollary 5.2. *Let K be a connected compact space such that $K \setminus L$ is also connected. Suppose that $C_E(K\|L)$ is extremely non-complex. Then,* $\operatorname{Iso}\big(C_E(K\|L)\big) = \{\mathrm{Id}, -\mathrm{Id}\}$.

Proof. Given $T \in \operatorname{Iso}\big(C_E(K\|L)\big)$, Theorem 5.1 gives a continuous function $\theta : K \setminus L \longrightarrow \{-1, 1\}$ such that $[T(f)](x) = \theta(x)\, f(x)$ for every $x \in K \setminus L$ and every $f \in C_E(K\|L)$. If $K \setminus L$ is connected, there are only two possible functions θ. Being L nowhere dense, the values of $[T(f)](x)$ for $x \in K \setminus L$ determine completely the function $T(f)$ for every $f \in C_E(K\|L)$. This gives only two possible surjective isometries, Id and $-\mathrm{Id}$. □

Corollary 5.3. *Suppose E is a subspace of $C(L)$ such that $C_E(K\|L)$ is extremely non-complex and for every $x \in L$, there is $f \in E$ such that $f(x) \neq 0$. If $T \in \operatorname{Iso}\big(C_E(K\|L)\big)$, then there is a continuous function $\theta : K \longrightarrow \{-1, 1\}$ such that $T(f) = \theta\, f$ for all $f \in C_E(K\|L)$.*

Proof. By Theorem 5.1, we may find $\theta' : K \setminus L \longrightarrow \{-1, 1\}$ continuous such that

$$[T(f)](x) = \theta'(x)\, f(x) \qquad \big(x \in K \setminus L,\ f \in C_E(K\|L)\big).$$

First, we note that θ' can be extended to a continuous function θ on K (indeed, if $x \in L$, there is an open neighborhood U of x on K and an $f \in C_E(K\|L)$ such that $f(y) \neq 0$ for every $y \in U$ and $\frac{T(f)|_U}{f|_U}$ is a continuous function on U which extends $\theta'|_{U\setminus L}$). Now, for each $f \in C_E(K\|L)$ we have

$$[T(f)](x) = \theta(x) f(x) \qquad \big(x \in K \setminus L\big),$$

so $T(f) = \theta f$ since they are two continuous functions which agree on a dense set. □

By just taking $E = C(L)$ in the above result, we get a description of all surjective isometries on an extremely non-complex $C(K)$ space. One direction is the above corollary, the converse result is just a consequence of the classical Banach-Stone theorem (see [12, Theorem 2.1.1], for instance).

Corollary 5.4. *Let K be a perfect Hausdorff space such that $C(K)$ is extremely non-complex. If $T \in \operatorname{Iso}\big(C(K)\big)$, then there is a continuous function $\theta : K \longrightarrow \{-1, 1\}$ such that $T(f) = \theta\, f$ for every $f \in C(K)$. Conversely, for every continuous function $\theta' : K \longrightarrow \{-1, 1\}$, the operator given by $T(f) = \theta'\, f$ for every $f \in C(K)$ is a surjective isometry. In other words, $\operatorname{Iso}\big(C(K)\big)$ is isomorphic to the Boolean algebra of clopen subsets of K.*

It follows from the above result and the Banach-Stone theorem on the representation of surjective isometries on $C(K)$ (see [12, Theorem 1.2.2] for instance) that the only homeomorphism of K is the identity.

Corollary 5.5. *Let K be a perfect Hausdorff space such that $C(K)$ is extremely non-complex. Then, the unique homeomorphism from K onto K is the identity.*

We finish the section with the study of the opposite extreme case, i.e. when $E = \{0\}$. Then, the hypothesis of Corollary 5.3 are not satisfied, but we obtain a description of the surjective isometries of the spaces $C_0(K\|L) \equiv C_0(K \setminus L)$ directly from Theorem 5.1. Again, the converse result comes from the Banach-Stone theorem (see [12, Corollary 2.3.12] for instance).

Corollary 5.6. *Let K be a compact Hausdorff space, $L \subset K$ closed nowhere dense, and suppose that $C_0(K \setminus L)$ is extremely non-complex. If $T \in \operatorname{Iso}\big(C_0(K \setminus L)\big)$, then there is a continuous function $\theta : K \setminus L \longrightarrow \{-1,1\}$ such that $T(f) = \theta\, f$ for every $f \in C_0(K \setminus L)$. Conversely, for every continuous function $\theta' : K \setminus L \longrightarrow \{-1,1\}$, the operator*

$$\big[T(f)\big](x) = \theta'(x)\, f(x) \qquad \big(x \in K \setminus L,\ f \in C_0(K \setminus L)\big)$$

is a surjective isometry. In other words, $\operatorname{Iso}\big(C_0(K \setminus L)\big)$ is isomorphic to the Boolean algebra of clopen subsets of $K \setminus L$.

Proof. The first part is a direct consequence of Theorem 5.1 for $E = \{0\}$. For the converse result, just observe that given any extension of θ' to L, the product $\theta' f : K \longrightarrow \mathbb{R}$ does not depend on the extension, belongs to $C_0(K\|L)$ and $\|\theta' f\|_\infty = \|f\|_\infty$. □

6. The construction of the main example

Our goal here is to construct a compact space K and a nowhere dense subset $L \subseteq K$ with very special properties which will allow us to provide the main example on surjective isometries and duality.

Theorem 6.1. *There exist a compact space K and a closed nowhere dense subset $L \subseteq K$ with the following properties:*

(a) *K and $K \setminus L$ are connected.*
(b) *There is a continuous mapping ϕ from L onto the Cantor set.*
(c) *Every operator on $C(K)$ is a weak multiplier.*

Proof. K is the compact space constructed in [19, §5]. The fact that all operators on $C(K)$ are weak multipliers is given in [19, Lemma 5.2].

We just need to find the appropriate L. We will assume the familiarity of the reader with the above construction of K in [19, § 5]. In particular, that $K \subseteq [0,1]^{2^\omega}$ is the inverse limit of $K_\alpha \subseteq [0,1]^\alpha$ for $\alpha \leqslant 2^\omega$ where $K_1 = [0,1]^2$. For $\beta \leqslant \alpha \leqslant 2^\omega$ the projection from $[0,1]^\alpha$ onto $[0,1]^\beta$ is denoted $\pi_{\beta,\alpha}$.

Choose any $N \subseteq [0,1]^2$ which is a copy of a Cantor set included in some subinterval of $[0,1]^2$. In particular, it is compact nowhere dense perfect and such that $[0,1]^2 \setminus N$ is connected. Let $N_\alpha = \pi_{1,\alpha}[N]$. We claim that $L = N_{2^\omega}$ works i.e., is nowhere dense in K and $K \setminus L$ is connected and there is a continuous mapping of L onto the Cantor set. The last part is clear as $\pi_{1,2^\omega}$ sends L onto N which is a homeomorphic copy of the Cantor set.

One proves by induction on $\alpha \leqslant 2^\omega$ that N_α is nowhere dense in K_α and $K_\alpha \setminus N_\alpha$ is connected. This is essentially a generalization of [19, Lemma 4.6] from a finite set to a nowhere dense set with a connected complement in $[0,1]^2$.

[19, Lemma 4.3.a] says that being nowhere dense is preserved when we pass by preimages from K_α to $K_{\alpha+1}$ so, as the limit stage is trivial, it follows that every N_α is nowhere dense in K_α. Therefore, L is nowhere dense in K.

So we are left with showing that $K_\alpha \setminus N_\alpha$ are connected. As in [19, Lemma 4.6], we prove by induction on α that there are $M^n_\alpha \subseteq K_\alpha$ such that

1) $\pi_{\alpha',\alpha}[M^n_\alpha] = M^n_{\alpha'}$ for $\alpha' \leqslant \alpha \leqslant \beta$,
2) M^n_α's are compact and connected,
3) $M^n_\alpha \cap N_\alpha = \emptyset$, $M^n_\alpha \subseteq M^{n+1}_\alpha$,
4) $\bigcup_{n\in N} M^n_\alpha$ is dense in $K_\alpha \setminus N_\alpha$.

We start by choosing M^n_1 to satisfy 2) - 4) and such that $[0,1]^2 \setminus \bigcup_{n\in N} M^n_\alpha$ is $N = N_1 \subseteq [0,1]^2$. The rest of the argument is exactly as in the last part of the proof of [19, Lemma 4.6]. □

We are now ready to present the main application of the results of the paper.

Theorem 6.2. *For every separable Banach space E, there is a Banach space $\widetilde{X}(E)$ such that* $\operatorname{Iso}\bigl(\widetilde{X}(E)\bigr) = \{\mathrm{Id}, -\mathrm{Id}\}$ *and* $\widetilde{X}(E)^* = E^* \oplus_1 Z$ *for a suitable space Z. In particular,* $\operatorname{Iso}\bigl(\widetilde{X}(E)^*\bigr)$ *contains* $\operatorname{Iso}(E^*)$ *as a subgroup.*

Proof. Consider the compact space K and the nowhere dense closed subset $L \subset K$ given in Theorem 6.1. As there is a surjective continuous function from L to the Cantor set, every separable Banach space E is a subset of $C(L)$. Let $\widetilde{X}(E)$ be $C_E(K\|L)$. Then, $\widetilde{X}(E)$ is extremely non complex since every operator on $C(K)$ is a weak multiplier and we may use Theorem 3.9. Now, since $K \setminus L$ is connected, we may apply Corollary 5.2 to get that $\operatorname{Iso}\bigl(\widetilde{X}(E)\bigr) = \{\mathrm{Id}, -\mathrm{Id}\}$. Finally, Lemma 2.2 gives us that $\widetilde{X}(E)^* = E^* \oplus_1 C_0(K\|L)^*$ and so $\operatorname{Iso}\bigl(\widetilde{X}(E)^*\bigr)$ contains $\operatorname{Iso}(E^*)$ as a subgroup (see [21, Proposition 2.4] for instance). $\square$

Let us comment that all the spaces $\widetilde{X}(E)$ constructed above are non-separable. We do not know whether separable spaces with the same properties can be constructed.

The case $E = \ell_2$ in Theorem 6.2 gives the following specially interesting example.

Example 6.3. *There is a Banach space $\widetilde{X}(\ell_2)$ such that* $\operatorname{Iso}\bigl(\widetilde{X}(\ell_2)\bigr) = \{\mathrm{Id}, -\mathrm{Id}\}$ *but* $\operatorname{Iso}\bigl(\widetilde{X}(\ell_2)^*\bigr)$ *contains* $\operatorname{Iso}(\ell_2)$ *as a subgroup. Therefore,* $\operatorname{Iso}\bigl(\widetilde{X}(\ell_2)\bigr)$ *is trivial, while* $\operatorname{Iso}\bigl(\widetilde{X}(\ell_2)^*\bigr)$ *contains infinitely many uniformly continuous one-parameter semigroups of surjective isometries.*

Recently, the second author of this paper constructed a Banach space $X(\ell_2)$ such that $\operatorname{Iso}\bigl(X(\ell_2)\bigr)$ does not contain any *uniformly continuous* one-parameter semigroup of surjective isometries, while $\operatorname{Iso}\bigl(X(\ell_2)^*\bigr)$ contains infinitely many of them [21, Example 4.1]. But it is not difficult to show that $\operatorname{Iso}\bigl(X(\ell_2)\bigr)$ does not reduce to $\{\mathrm{Id}, -\mathrm{Id}\}$ and, actually, it contains infinitely many *strongly continuous* one-parameter semigroups of surjective isometries. We refer the reader to the books [7, 8] for background on one-parameter semigroups of operators and to the monographs [12, 13] for more information on isometries on Banach spaces.

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