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EXTENSION OF ISOMETRIES BETWEEN UNIT SPHERES OF FINITE-DIMENSIONAL POLYHEDRAL BANACH SPACES

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Abstract. We prove that an onto isometry between unit spheres of finite-dimensional polyhedral Banach spaces extends to a linear isometry of the corresponding spaces.

1. Introduction

In 1987, D. Tingley proposed the following question [7]: let f be a bijective isometry between the unit spheres S_X and S_E of real Banach spaces X, E respectively. Is it true that f extends to a linear (bijective) isometry $F : X \longrightarrow E$ of the corresponding spaces? Let us mention that this is equivalent to the fact that the natural (positive) homogeneous extension of f (see (1)) is linear. He proved a useful partial result:

Theorem 1.1 (Tingley's theorem [7]). *If X and E are finite-dimensional Banach spaces and $f : S_X \longrightarrow S_E$ is a bijective isometry, then $f(-x) = -f(x)$ for all $x \in S_X$.*

We recall that the classical Mazur-Ulam theorem states that every surjective isometry between X and E is affine and that there is a result by P. Mankiewicz [5] which states that every bijective isometry between convex bodies of X and E can be uniquely extended to an affine isometry from X and E.

There is a number of publications devoted to Tingley's problem (see [2] for a survey of corresponding results) and, in particular, the problem is solved in positive for many concrete classical Banach spaces. Surprisingly, the question for general spaces remains open, even in dimension two.

Recently, L. Cheng and Y. Dong [1] attacked the problem for the class of polyhedral spaces (i.e. for those spaces whose unit sphere is a polyhedron). Unfortunately their interesting attempt failed by a mistake at the very end of the proof. The authors told to us in a private communication that they don't see how their proof can be repaired.

In this paper we present a new approach to Tingley's problem that enables us to save partially the Cheng-Dong result. Namely, we answer the problem in positive for **finite-dimensional polyhedral spaces**. The idea of the proof is to study the

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differentiability properties of f and of its homogeneous extension F. Although our main result is about polyhedral spaces, for the sake of possible applications, the technical differentiability lemmas are proved for general finite-dimensional normed spaces.

2. Notation

Throughout the paper X, E are m-dimensional Banach spaces over the field of reals, X^*, E^* are their dual spaces, S_X, B_X, stand for the unit sphere and unit ball of the corresponding space, $f : S_X \longrightarrow S_E$ is a bijective isometry and, finally, $F : X \longrightarrow E$ is the natural (positively) homogeneous extension of f, that is,

$$F(0) = 0, \qquad F(x) = \|x\|\, f\,(x/\|x\|) \quad \big(x \in X \setminus \{0\}\big). \tag{1}$$

Recall that, thanks to Tingley's theorem 1.1, $F(-x) = -F(x)$ for every $x \in X$, so F is homogeneous for the negative scalars as well.

We will use the notation $\rho(x, y) = \|x-y\|$ for the metric in both S_X and S_E. We will use the notations $x^*(x)$ and $\langle x^*, x\rangle$ to denote the action of $x^* \in X^*$ on $x \in X$, and we also use the same notations for the action of elements of E^* on elements of E.

For every $A \subset X$, we denote by $\operatorname{cone}(A) = \{tx : x \in A, t \geqslant 0\}$ the cone generated by A. For every $x \in S_X$, we denote by $\daleth(x) \subset X^*$ the nonempty set of support functionals of x, i.e. those $x^* \in X^*$ such that $\|x^*\| = x^*(x) = 1$. If $\daleth(x)$ consists of only one element, we say that x is a smooth point and the set of smooth points of S_X is denoted by $\Sigma(X)$. If $x \in \Sigma(X)$, we denote the unique element of $\daleth(x)$ as $\gamma(x)$. Recall that in finite-dimensional spaces, every smooth point of the unit sphere is actually a Fréchet differentiability point for the map $x \longmapsto \|x\|$. This means that for $x \in \Sigma(X)$, there is a function $\varepsilon_x(r)$ such that

$$\frac{\varepsilon_x(r)}{r} \underset{r\to 0}{\longrightarrow} 0 \quad \text{and} \quad \langle\gamma(x), z\rangle \leqslant \|z\| \leqslant (1 + \varepsilon_x(r))\langle\gamma(x), z\rangle \tag{2}$$

for every $z \in \operatorname{cone}(x + rB_X)$. For this and other standard facts from convex geometry we refer to Rockafellar's book [6]. Remark, that in the most valuable for us case of polyhedral spaces, $x \in \Sigma(X)$ if and only if x is an interior point of an $(m-1)$-dimensional face and $\varepsilon_x(r) = 0$ for sufficiently small r.

3. The differentiability lemmas

Lemma 3.1. *Let $x, y, y_n \in S_X$, $x \neq y$, such that $\frac{x-y}{\|x-y\|} \in \Sigma(X)$ and suppose that*

$$y_n \longrightarrow y, \ \ \frac{y - y_n}{\|y - y_n\|} \longrightarrow u \qquad \text{as } n \to \infty.$$

Then

$$\frac{\rho(x, y_n) - \rho(x, y)}{\rho(y, y_n)} \longrightarrow \left\langle \gamma\left(\frac{x-y}{\|x-y\|}\right), u\right\rangle \text{ as } n \to \infty. \tag{3}$$

Proof. If we denote $r_n = \|y - y_n\|/\|x - y\|$ then

$$\|(x - y_n) - (x - y)\| = r_n\|x - y\|,$$

i.e.

$$x - y_n \in \operatorname{cone}\left(\frac{x-y}{\|x-y\|} + r_n B_X\right),$$

and we can use (2) to get

$$\begin{aligned}\frac{\left\langle \gamma\left(\frac{x-y}{\|x-y\|}\right), x - y_n\right\rangle - \|x-y\|}{\|y-y_n\|} &\leqslant \frac{\rho(x,y_n) - \rho(x,y)}{\rho(y,y_n)} \\ &\leqslant \frac{(1+\varepsilon_{\frac{x-y}{\|x-y\|}}(r_n))\left\langle \gamma\left(\frac{x-y}{\|x-y\|}\right), x-y_n\right\rangle - \|x-y\|}{\|y-y_n\|}.\end{aligned}$$

Since $\|x-y\| = \left\langle \gamma\left(\frac{x-y}{\|x-y\|}\right), x-y\right\rangle$, we can continue as follows:

$$\begin{aligned}\left\langle \gamma\left(\frac{x-y}{\|x-y\|}\right), \frac{y-y_n}{\|y-y_n\|}\right\rangle &\leqslant \frac{\rho(x,y_n)-\rho(x,y)}{\rho(y,y_n)} \\ &\leqslant \left\langle \gamma\left(\frac{x-y}{\|x-y\|}\right), \frac{y-y_n}{\|y-y_n\|}\right\rangle + \frac{\varepsilon_{\frac{x-y}{\|x-y\|}}(r_n)}{r_n\|x-y\|}\left\langle \gamma\left(\frac{x-y}{\|x-y\|}\right), x-y_n\right\rangle.\end{aligned}$$

Passing to limit when $n \to \infty$, we get the desired result. □

For $y \in S_X$, we write $D_y = \{x \in S_X : \|x+y\| < 2\}$, which is a relatively open subset of S_X, and observe that D_y consists of those points of the sphere for which the line interval $]x,y[= \{\lambda x + (1-\lambda)y : 0 < \lambda < 1\}$ lies in the open unit ball. Also observe that $D_y = \{x \in S_X : \rho(-y,x) < 2\}$ so, thanks to Tingley's Theorem 1.1, f maps bijectively D_y onto $D_{f(y)}$. We denote by W_y the set of those $x \in D_y$ for which

$$\frac{x-y}{\|x-y\|} \in \Sigma(X) \quad \text{and} \quad \frac{f(x)-f(y)}{\|f(x)-f(y)\|} \in \Sigma(E).$$

Lemma 3.2. *$D_y \setminus W_y$ is negligible in D_y so, in particular, W_y is dense in D_y.*

Proof. Consider the function $g : D_y \longrightarrow S_X$, $g(x) = \frac{x-y}{\|x-y\|}$ for every $x \in D_y$. Then, g is injective, $g(D_y)$ is relatively open, and g, as well as g^{-1} are locally Lipschitz. Since $S_X \setminus \Sigma(X)$ is negligible in S_X, $g^{-1}(S_X \setminus \Sigma(X))$ is negligible in D_y, i.e. the set $\left\{x \in D_y : \frac{x-y}{\|x-y\|} \notin \Sigma(X)\right\}$ is negligible in D_y. Analogously, from the fact that $S_E \setminus \Sigma(E)$ is negligible in S_E, we deduce that the set $\left\{x \in D_y : \frac{f(x)-f(y)}{\|f(x)-f(y)\|} \notin \Sigma(E)\right\}$ is negligible in D_y. Finally, $D_y \setminus W_y$ is the union of two negligible sets. □

We say that a subset A of the unit sphere of the dual of a Banach space Z is *total* if for every $z \in Z$, there is $z^* \in A$ such that $z^*(z) \neq 0$. The set A is said to be *1-norming* if $\sup\{|z^*(z)| : z^* \in A\} = \|z\|$ for every $z \in Z$.

Lemma 3.3. *For every $y \in S_X$ the set*

$$\left\{\gamma\left(\frac{x-y}{\|x-y\|}\right) : x \in W_y\right\}$$

is total over X, and

$$\left\{\gamma\left(\frac{f(x)-f(y)}{\|f(x)-f(y)\|}\right) : x \in W_y\right\}$$

is total over E. Moreover, if $y \in \Sigma(X)$ (resp. $f(y) \in \Sigma(E)$), then the corresponding set is 1-norming.

Proof. Let us start with the "moreover" part. If y is a smooth point of S_X, then

$$\left\{\frac{x-y}{\|x-y\|}\ :\ x \in D_y\right\} \supset \{z \in S_X\ :\ \langle \gamma(y), z\rangle < 0\},$$

i.e. it contains the intersection of the sphere with an open half-space. This together with the density of W_y in D_y makes the "moreover" part evident.

For the main part of the statement, denote by A the relative interior in S_X of the set $\left\{\frac{x-y}{\|x-y\|}\ :\ x \in D_y\right\}$. Since $\left\{\frac{x-y}{\|x-y\|} : x \in W_y\right\}$ is dense in A,

$$\overline{\operatorname{conv}}\left\{\gamma\left(\frac{x-y}{\|x-y\|}\right) : x \in W_y\right\} \supset \bigcup_{a\in A} \gimel(a).$$

So it is sufficient to show that for every $z \in X$ there is $a \in A$ and $x^* \in \gimel(a)$ such that $x^*(z) \neq 0$. Consider the two-dimensional subspace $Z \subset X$ spanned by y and z. If y is a smooth point of S_Z, then the job is done by the same reason as in the "moreover" part. If y is not a smooth point of S_Z, then $a = -y \in A$ is not a smooth point of S_Z neither, so at least one of support functionals in this point a must take a non-zero value at z.

The same argument works for the set $\left\{\gamma\left(\frac{f(x)-f(y)}{\|f(x)-f(y)\|}\right)\ :\ x \in W_y\right\}$. □

Lemma 3.4. *For every $y \in S_X$ and for every sequence (y_n) on S_X converging to y, if the sequence $\left(\frac{y-y_n}{\|y-y_n\|}\right)$ is convergent, then so is the sequence $\left(\frac{f(y)-f(y_n)}{\|f(y)-f(y_n)\|}\right)$. Moreover, for every $x \in W_y$*

$$\left\langle \gamma\left(\tfrac{x-y}{\|x-y\|}\right), \lim_{n\to\infty} \tfrac{y-y_n}{\|y-y_n\|}\right\rangle = \left\langle \gamma\left(\tfrac{f(x)-f(y)}{\|f(x)-f(y)\|}\right), \lim_{n\to\infty} \tfrac{f(y)-f(y_n)}{\|f(y)-f(y_n)\|}\right\rangle \tag{4}$$

Proof. Denote $u = \lim\limits_{n\to\infty} \frac{y-y_n}{\|y-y_n\|}$. Assume at first that $\lim\limits_{n\to\infty} \frac{f(y)-f(y_n)}{\|f(y)-f(y_n)\|}$ exists, and denote it v. Then, according to Lemma 3.1, we have

$$\begin{aligned}\left\langle \gamma\left(\frac{x-y}{\|x-y\|}\right), u\right\rangle &= \lim_{n\to\infty} \frac{\rho(x,y_n)-\rho(x,y)}{\rho(y,y_n)}\\ &= \lim_{n\to\infty} \frac{\rho(f(x),f(y_n))-\rho(f(x),f(y))}{\rho(f(y),f(y_n))}\\ &= \left\langle \gamma\left(\frac{f(x)-f(y)}{\|f(x)-f(y)\|}\right), v\right\rangle.\end{aligned}$$

This proves (4). Now, assume that v_1, v_2 are limits of some subsequences of the sequence $\left(\frac{f(y)-f(y_n)}{\|f(y)-f(y_n)\|}\right)$. Applying for these subsequences the already proved condition (4) we get that

$$\left\langle \gamma\left(\frac{f(x)-f(y)}{\|f(x)-f(y)\|}\right), v_1\right\rangle = \left\langle \gamma\left(\frac{f(x)-f(y)}{\|f(x)-f(y)\|}\right), v_2\right\rangle$$

for all $x \in W_y$. By Lemma 3.3 this means that $v_1 = v_2$. □

For every $y \in S_X$, we write Λ_y to denote the set of all limiting points of the expression

$$\frac{y-z}{\|y-z\|}$$

when $z \to y$, $z \in S_X$ (Λ_y is the set of tangent directions) and we observe that

(a) if $y \in \Sigma(X)$, then $\Lambda_y = S_{\ker \gamma(y)}$, i.e. it is the unit sphere of a hyperplane,
(b) otherwise, Λ_y is the intersection of the unit sphere with the boundary of the supporting cone $\{u \in X : x^*(u) \geqslant 0 \ \forall x^* \in \daleth(y)\}$ and, in particular, $\operatorname{lin} \Lambda_y = X$.

Let us also observe that Lemma 3.4 means that the correspondence

$$\lim_{n\to\infty} \frac{y-y_n}{\|y-y_n\|} \longrightarrow \lim_{n\to\infty} \frac{f(y)-f(y_n)}{\|f(y)-f(y_n)\|}$$

defines a bijective map between Λ_y and $\Lambda_{f(y)}$. We write $F_y : \Lambda_y \longrightarrow \Lambda_{f(y)}$ for this map. With this notation we can rewrite (4) as follows: for every $x \in W_y$, $u \in \Lambda_y$

$$\left\langle \gamma\left(\frac{x-y}{\|x-y\|}\right), u\right\rangle = \left\langle \gamma\left(\frac{f(x)-f(y)}{\|f(x)-f(y)\|}\right), F_y(u)\right\rangle \tag{5}$$

Lemma 3.5. *The map F_y extends to a linear isomorphism between $\operatorname{lin} \Lambda_y$ and $\operatorname{lin} \Lambda_{f(y)}$ (we will denote this extension again by F_y). Moreover, if $y \in \Sigma(X)$ and $f(y) \in \Sigma(E)$, then this linear isomorphism is an isometry.*

Proof. Let $v_1, \ldots, v_N \in \Lambda_y$ and $a_1, \ldots, a_N \in \mathbb{R}$. By Lemma 3.3, the set

$$\left\{\gamma\left(\frac{f(x)-f(y)}{\|f(x)-f(y)\|}\right) : x \in W_y\right\}$$

is total over E. Since $\dim E < \infty$, this set of functionals is norming with some constant $C > 0$. Therefore,

$$\begin{aligned}
\left\|\sum_{j=1}^N a_j F_y(v_j)\right\| &\leqslant C \sup\left\{\left|\left\langle \gamma\left(\frac{f(x)-f(y)}{\|f(x)-f(y)\|}\right), \sum_{j=1}^N a_j F_y(v_j)\right\rangle\right| : x \in W_y\right\} \\
&= C \sup\left\{\left|\sum_{j=1}^N a_j \left\langle \gamma\left(\frac{f(x)-f(y)}{\|f(x)-f(y)\|}\right), F_y(v_j)\right\rangle\right| : x \in W_y\right\} \\
&= C \sup\left\{\left|\sum_{j=1}^N a_j \left\langle \gamma\left(\frac{x-y}{\|x-y\|}\right), v_j\right\rangle\right| : x \in W_y\right\} \\
&= C \sup\left\{\left|\left\langle \gamma\left(\frac{x-y}{\|x-y\|}\right), \sum_{j=1}^N a_j v_j\right\rangle\right| : x \in W_y\right\} \\
&\leqslant C \left\|\sum_{j=1}^N a_j v_j\right\|.
\end{aligned}$$

This demonstrates the possibility of a linear extension and we may interchange the rolles of X and E to get the reversed inequality and so an isomorphism. The "moreover" part follows from the "moreover" part of Lemma 3.3 since, in such a case, $C = 1$. $\square$

The next goal is to study what happens with the supporting functionals in a non-smooth point $y \in S_X$.

Lemma 3.6. *Let (x_n) be a sequence in W_y such that $(x_n) \longrightarrow -y$. Assume that $\gamma(\frac{x_n - y}{\|x_n - y\|}) \longrightarrow y^* \in \daleth(-y)$. Then, there exists $e^* := \lim\limits_{n\to\infty} \gamma\left(\frac{f(x_n)-f(y)}{\|f(x_n)-f(y)\|}\right)$ and*

$$\langle y^*, u\rangle = \langle e^*, F_y(u)\rangle \tag{6}$$

for every $u \in \Lambda_y$.

Proof. Denote $z_n = \frac{f(x_n)-f(y)}{\|f(x_n)-f(y)\|}$. At first assume that $\lim\limits_{n\to\infty} \gamma(z_n) =: e^*$ exists, then (6) is just a limiting case of (5). Now suppose that e_1^* and e_2^* are limits of some subsequences of $\big(\gamma(z_n)\big)$. Then (6) is valid for both e_1^*, e_2^*, so for every $u \in \Lambda_y$

$$\langle e_1^* - e_2^*, F_y(u)\rangle = 0.$$

Also, evidently, $e_1^*(f(y)) = e_2^*(f(y)) = -1$, so $e_1^* - e_2^* \in \big[\Lambda_{f(y)} \cup \{f(y)\}\big]^{\perp} = \{0\}$. $\square$

Denote by M_y^* the set of elements in S_{X^*} of the form $\lim\limits_{n\to\infty} \gamma(\frac{x_n-y}{\|x_n-y\|})$, where (x_n) is a sequence in W_y converging to $-y$ and observe that $M_y^* \subset \daleth(-y)$. We write $M_{f(y)}^* \subset \daleth(-f(y))$ for the set of elements in S_{E^*} of the form $\lim\limits_{n\to\infty} \gamma(\frac{f(x_n)-f(y)}{\|f(x_n)-f(y)\|})$, where (x_n) is a sequence in W_y converging to $-y$. Equivalently, $M_{f(y)}^*$ is the set of elements in S_{E^*} of the form $\lim\limits_{n\to\infty} \gamma(\frac{z_n-f(y)}{\|z_n-f(y)\|})$, where (z_n) is a sequence in S_E converging to $-f(y)$ such that $\|z_n - f(y)\| < 2$, $z_n \in \Sigma(E)$ and $f^{-1}(z_n) \in \Sigma(X)$.

In the same way as in the definition of F_y, we can now define a bijective map $G_y : M_y^* \longrightarrow M_{f(y)}^*$ by

$$G_y\left(\lim_{n\to\infty} \gamma\left(\frac{x_n - y}{\|x_n - y\|}\right)\right) := \lim_{n\to\infty} \gamma\left(\frac{f(x_n)-f(y)}{\|f(x_n)-f(y)\|}\right).$$

Then (6) can be re-written as

$$\langle y^*, u\rangle = \langle G_y(y^*), F_y(u)\rangle \tag{7}$$

for every $y^* \in M_y^*$ and for every $u \in \operatorname{lin}\Lambda_y$. Now, as in Lemma 3.5 and taking into account that the closed convex hull of M_y^* equals to $\daleth(-y) = -\daleth(y)$, we can deduce the following.

Lemma 3.7. *G_y extends to a linear isomorphism between $\operatorname{lin}\daleth(y)$ and $\operatorname{lin}\daleth(f(y))$ (we will denote this extension again as G_y) satisfying that $G_y(\daleth(y)) = \daleth(f(y))$ and*

$$\langle y^*, u\rangle = \langle G_y y^*, F_y u\rangle$$

for all $y^* \in \operatorname{lin}\daleth(y)$, $u \in \operatorname{lin}\Lambda_y$. Therefore, $\dim\operatorname{lin}\daleth(y) = \dim\operatorname{lin}\daleth(f(y))$ and, in particular, f maps smooth points into smooth points.

Proof. Recall first that outside of (7) we know that $\langle y^*, y\rangle = \langle G_y(y^*), f(y)\rangle = -1$ for every $y^* \in M_y^*$. Let $v_1^*, \dots, v_N^* \in M_y^*$, $a_1, \dots, a_N \in \mathbb{R}$. The set $\Lambda_{f(y)} \cup \{f(y)\}$ spans all the E, which means, thanks to the finite-dimensionality of E, that this set is norming for E^* with some constant $C > 0$. So, writing $\vee$ to denote the maximum of two numbers, we have

$$\begin{aligned}
&\left\| \sum_{j=1}^{N} a_j G_y(v_j^*) \right\| \\
&\quad \leqslant C \left(\sup \left\{ \left| \left\langle \sum_{j=1}^{N} a_j G_y(v_j^*), F_y(u) \right\rangle \right| : u \in \Lambda_y \right\} \bigvee \left| \left\langle \sum_{j=1}^{N} a_j G_y(v_j^*), f(y) \right\rangle \right| \right) \\
&\quad = C \left(\sup \left\{ \left| \sum_{j=1}^{N} a_j \left\langle G_y(v_j^*), F_y(u) \right\rangle \right| : u \in \Lambda_y \right\} \bigvee \left| \sum_{j=1}^{N} a_j \left\langle G_y(v_j^*), f(y) y \right\rangle \right| \right) \\
&\quad = C \left(\sup \left\{ \left| \sum_{j=1}^{N} a_j \left\langle v_j^*, u \right\rangle \right| : u \in \Lambda_y \right\} \bigvee \left| \sum_{j=1}^{N} a_j \left\langle v_j^*, y \right\rangle \right| \right) \\
&\quad = C \left(\sup \left\{ \left| \left\langle \sum_{j=1}^{N} a_j v_j^*, u \right\rangle \right| : u \in \Lambda_y \right\} \bigvee \left| \left\langle \sum_{j=1}^{N} a_j v_j^*, y \right\rangle \right| \right) \\
&\quad \leqslant C \left\| \sum_{j=1}^{N} a_j v_j^* \right\| .
\end{aligned}$$

This demonstrates the possibility of linear extension. $\square$

4. The main results

Recall that F stands for the homogeneous extension of f, see (1). We denote

$$[F'(y)](z) = \lim_{a \to 0^+} \frac{1}{a} \left(F(y + az) - F(y) \right)$$

the derivative of F at point y in direction z. This is just the first step in the definition of the Gateaux differential: F is Gateaux differentiable if $[F'(y)](z)$ depends on z linearly and continuously. In the finite-dimensional case, continuity follows from linearity. We also denote $H(y, z) \subset \daleth(y)$ the set of all $y^* \in \daleth(y)$ such that

$$\lim_{a \to 0^+} \frac{1}{a} \left(\|y + az\| - 1 \right) = y^*(z) \tag{8}$$

and observe that $H(y, z) \neq \emptyset$ by the convexity of the norm.

Lemma 4.1. *For every $y \in S_X$, $z \in X$, $y^* \in H(y, z)$, we have $z - \langle y^*, z \rangle y \in \operatorname{lin} \Lambda_y$ and*

$$[F'(y)](z) = \langle y^*, z \rangle f(y) + F_y \left(z - \langle y^*, z \rangle y \right).$$

Proof. Observe that

$$\lim_{a \to 0^+} \frac{1}{a} \left(\frac{y + az}{\|y + az\|} - y \right) = \lim_{a \to 0^+} \frac{1}{a} \left(y + az - \|y + az\| y \right) = z - y^*(z) y,$$

and denote

$$u := \lim_{a \to 0^+} \frac{\frac{y+az}{\|y+az\|} - y}{\left\| \frac{y+az}{\|y+az\|} - y \right\|} = \frac{z - y^*(z) y}{\|z - y^*(z) y\|}$$

which, evidently, belongs to Λ_y. Now we can calculate the limit that we need as follows:

$$
\begin{aligned}
[F'(y)](z) &= \lim_{a\to 0^+} \frac{1}{a}\left(\|y+az\| f\left(\frac{y+az}{\|y+az\|}\right) - f(y)\right)\\
&= \lim_{a\to 0^+} \frac{1}{a}\left(\|y+az\|-1\right) f\left(\frac{y+az}{\|y+az\|}\right)\\
&\qquad\qquad + \lim_{a\to 0^+} \frac{1}{a}\left(f\left(\frac{y+az}{\|y+az\|}\right) - f(y)\right)\\
&= y^*(z)f(y) + \lim_{a\to 0^+} \frac{1}{a}\left\|\frac{y+az}{\|y+az\|} - y\right\| \cdot \lim_{a\to 0^+} \frac{f\left(\frac{y+az}{\|y+az\|}\right) - f(y)}{\left\|f\left(\frac{y+az}{\|y+az\|}\right) - f(y)\right\|}\\
&= y^*(z)f(y) + \|z - y^*(z)y\| F_y(u) = y^*(z)f(y) + F_y(z - y^*(z)y). \qquad \square
\end{aligned}
$$

We are now ready to present the most important results of the paper. The first one contains two sufficient conditions assuring the differentiability of F.

Theorem 4.2. *In the following cases we can guaranty the Gateaux differentiability of F in the point $y \in S_X$:*

1. *if $y \in \Sigma(X)$,*
2. *if $\operatorname{lin} \daleth(y) = X^*$.*

Proof. (1). If $y \in \Sigma(X)$, then $H(y,z) = \{\gamma(y)\}$,

$$[F'(y)](z) = \langle \gamma(y), z\rangle f(y) + F_y\left(z - \langle \gamma(y), z\rangle y\right),$$

so it linearly depends on z.

(2). In this case y is not a smooth point, so $\operatorname{lin} \Lambda_y = X$, and $F_y(y)$ is correctly defined. Let us prove that $f(y) - F_y(y) = 0$. In fact, according to Lemma 3.7, $\dim \operatorname{lin} \daleth(y) = \dim \operatorname{lin} \daleth(f(y))$, consequently $\operatorname{lin} \daleth(f(y)) = E^*$. This implies that it is sufficient to show that $\langle G_y(y^*), f(y) - F_y(y)\rangle = 0$ for all $y^* \in \daleth(y)$. In fact, according to the same Lemma 3.7

$$
\begin{gathered}
\langle G_y(y^*), f(y) - F_y(y)\rangle = \langle G_y(y^*), f(y)\rangle - \langle G_y(y^*), F_y(y)\rangle\\
1 - \langle y^*, y\rangle = 0.
\end{gathered}
$$

Now, fix $x^* \in \daleth(y)$ and let us show that for every $z \in X$

$$[F'(y)](z) = \langle x^*, z\rangle f(y) + F_y\left(z - \langle x^*, z\rangle y\right). \tag{9}$$

This will give us the linearity of $[F'(y)](z)$ in the variable z. Let us check (9). According to Lemma 4.1 for $y^* \in H(y,z)$ we have the representation

$$[F'(y)](z) = \langle y^*, z\rangle f(y) + F_y\left(z - \langle y^*, z\rangle y\right).$$

Let us compare this with (9):

$$
\begin{aligned}
&\left(\langle y^*, z\rangle f(y) + F_y\left(z - \langle y^*, z\rangle y\right)\right) - \left(\langle x^*, z\rangle f(y) + F_y\left(z - \langle x^*, z\rangle y\right)\right)\\
&= \langle y^* - x^*, z\rangle \left(f(y) - F_y(y)\right) = 0. \qquad \square
\end{aligned}
$$

Two easy consequences can be stated.

Corollary 4.3. *If* $\dim X = 2$*, then* F *is Gateaux differentiable in all non-zero points.*

Corollary 4.4. *If* X *is smooth (i.e. if every point of* S_X *is smooth), then* F *is Gateaux differentiable in all non-zero points.*

Finally, we state the main result of the paper.

Theorem 4.5. *Let* X *be an* m*-dimensional polyhedral space,* E *a finite-dimensional Banach space and* $f : S_X \longrightarrow S_E$ *a bijective isometry. Then, the homogeneous extension* F *of* f *is a linear operator and, therefore, a linear isometry.*

Proof. It is shown in [7, p. 377] (using Mankiewicz result [5]), that for every cone C_j generated by an $(m-1)$-dimensional face of S_X there is a linear operator A_j, such that $F(y) = A_j y$ for $y \in C_j$. In every vertex, according to (2) of Theorem 4.2, F is Gateaux differentiable, so all the A_j that correspond to faces that meet in this vertex are the same. This means that all A_j are the same linear operator A and so $F = A$. □

5. Concluding remarks

From the Tingley's problem about bijective isometries of spheres one can extract two weaker questions:

(1) If such an isometry exists, is it true that the corresponding spaces are isomorphic?
(2) If such an isometry exists, is it true that the corresponding spaces are isometric?

Of course, the first question is meaningful only in the infinite-dimensional case. Remark that, since the homogeneous extension F of the the bijective isometry $f : S_X \longrightarrow S_E$ is a Lipschitz homeomorphism [1, Proposition 4.1], the question (1) is closely related to a still open problem of whether Lipschitz homeomorphism of separable Banach spaces implies linear isomorphism. This problem have been studied by a number of extraordinary mathematicians, and there are many deep and interesting partial results ([3], [4]).

The second question is quite interesting even for finite-dimensional spaces. Our Lemma 3.5 means, in particular, that for a smooth space X the existence of a bijective isometry $f : S_X \longrightarrow S_E$ implies that every 1-codimensional subspace of X is isometric to a 1-codimensional subspace of E, and this correspondence between 1-codimensional subspaces is bijective. If $\dim X \geqslant 3$, then this condition is quite restrictive and we wonder whether it implies that X and E are isometric.

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