NUMERICAL RADIUS OF RANK-ONE OPERATORS ON BANACH SPACES

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Abstract. We study the rank-one numerical index of a Banach space, namely the infimum of the numerical radii of those rank-one operators on the space which have norm-one. We show that the rank-one numerical index is always greater or equal than $1/e$. We also present properties of this index and some examples.

1. Introduction

The aim of this paper is to study the *rank-one numerical index* of Banach spaces. This concept has been recently introduced in [12] to relate the numerical range and the usual norm of rank-one operators on L_p-spaces as an analog to the deeply studied numerical index of Banach spaces.

Let us recall the relevant notation and definitions. Given a Banach space X over the field $\mathbb{K}$ ($=\mathbb{R}$ or $\mathbb{C}$), X^* will stand for its topological dual, S_X is the unit sphere of X, and $L(X)$ is the Banach algebra of all (bounded linear) operators on X. The *numerical radius* is the semi-norm defined on $L(X)$ by

$$v(T) = \sup\{|x^*(Tx)| \ : \ x^* \in S_{X^*},\, x \in S_X,\, x^*(x) = 1\} \quad \big(T \in L(X)\big).$$

Very often, v is actually a norm and it is equivalent to the operator norm $\|\cdot\|$. Thus, it is natural to consider the so-called *numerical index* of the space X, namely the constant $n(X)$ defined by

$$n(X) = \inf\{v(T) \ : \ T \in L(X),\, \|T\| = 1\}.$$

Equivalently, $n(X)$ is the greatest constant $k \geqslant 0$ such that $k\|T\| \leqslant v(T)$ for every $T \in L(X)$. Note that $0 \leqslant n(X) \leqslant 1$, and $n(X) > 0$ if and only if v and $\|\cdot\|$ are equivalent norms on $L(X)$.

Let us present some results about numerical index of Banach spaces. We refer the reader to the expository paper [8] and references therein and the more recent papers [5, 12, 13]. First, some examples of Banach spaces whose numerical index is known are the following: for a Hilbert space H of dimension greater than one, one has $n(H) = 1/2$ in the complex case and $n(H) = 0$ in the real case. $L_1(\mu)$ spaces and their isometric preduals have numerical index 1 so, in particular, $n(C(K)) = 1$ for every compact topological space K. The disk algebra $A(\mathbb{D})$ and H^∞ are other examples of Banach spaces with numerical index one. The numerical index of $L_p(\mu)$ spaces is still unknown, but it has been recently shown that $n(L_p[0,1]) = n(\ell_p)$ and that, in the real case, this number is strictly positive when $p \neq 2$. Some results about the numerical index that can be interesting for our discussion are the following: it is known that $v(T) = v(T^*)$ for every $T \in L(X^*)$, so it follows that $n(X^*) \leqslant n(X)$ for every Banach space X; this inequality may be strict. The numerical index is continuous with respect

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to the Banach-Mazur distance between equivalent norms and this gives that the set of values of the numerical index of a Banach space up to renorming is an interval (actually, a non-trivial interval). The numerical index of the c_0-, ℓ_1-, or ℓ_∞-sum of a family of spaces is equal to the infimum of the numerical index of the spaces and the numerical indices of the vector-valued function spaces $C(K,X)$, $L_1(\mu,X)$, and $L_\infty(\mu,X)$ are equal to the numerical index of the range space.

Real and complex spaces behave differently with respect to the numerical index. Indeed, the set of values of the numerical index of real Banach spaces fills the whole interval $[0,1]$, while for complex Banach spaces it fills the interval $[1/\mathrm{e},1]$. The fact that $n(X)\geqslant 1/\mathrm{e}$ in the complex case, known as the Bonehnblust-Karlin theorem, has important consequences specially in the theory of Banach algebras.

As mentioned above, the *rank-one numerical index* of a Banach space X was introduced in [12] as the constant

$$\begin{aligned} n_1(X) &= \max\{k\geqslant 0 \ : \ k\|T\|\geqslant v(T)\ \forall T\in L(X) \text{ with } \dim(T(X))\leqslant 1\} \\ &= \inf\{v(T) \ : \ T\in L(X),\, \|T\|=1,\, \dim(T(X))\leqslant 1\}. \end{aligned}$$

The main motivation to study rank-one operators and the rank-one numerical index is that in an arbitrary Banach space, only for rank-one operators it is possible to give a formula for the operator norm.

It is proved in [12] that for every $1<p<\infty$ and every atomless measure μ, one has

$$n_1(L_p(\mu))\geqslant p^{-\frac{2}{p}}q^{-\frac{2}{q}}$$

where $q=p/(p-1)$ is the conjugate exponent to p. It is also shown that $n_1(H)=1/2$ for every real or complex Hilbert space H of dimension greater than one.

While the definition of rank-one numerical index was first given in [12], the study of numerical radius of rank-one operators was initiated much earlier. For instance, in the 1999 paper [9], the authors proved a number of results for Banach spaces with numerical index one, but they claimed that all of them are also true for Banach spaces with rank-one numerical index equal to one, since in all the proofs only rank-one operators are used. Actually, in [16] it is introduced the so-called alternative Daugavet property. A Banach space X has the *alternative Daugavet property* if the norm equality

$$\max_{|\theta|=1}\|\operatorname{Id}+\theta T\| = 1+\|T\|$$

holds for all rank-one operators on the space and, in such a case, all compact operators also satisfy that equation (actually, this is true for all operators not fixing a copy of ℓ_1, as has been recently proved in [1, Corollary 5.6]). The relation of this property with the rank-one numerical index comes from the fact known from the 1970's [4] that for $T\in L(X)$,

$$v(T)=\|T\| \quad\Longleftrightarrow\quad \max_{|\theta|=1}\|\operatorname{Id}+\theta T\| = 1+\|T\|.$$

Therefore, a Banach space X has the alternative Daugavet property if and only if $n_1(X)=1$ and, in such a case, we actually have $v(T)=\|T\|$ for every operator $T\in L(X)$ which does not fix a copy of ℓ_1 (in particular, for compact operators). It also follows that for a finite-dimensional space X, if $n_1(X)=1$, then $n(X)=1$. This result is false in the infinite-dimensional setting, an example being $C([0,1],\ell_2)$. Some interesting examples are the following: $C(K,X)$ has the alternative Daugavet property if and only if X does or K is perfect; the spaces $L_1(\mu,X)$ and $L_\infty(\mu,X)$ have the alternative Daugavet property if and only if X does or μ has no atoms.

In this paper, we fist prove that $n_1(X)\geqslant 1/\mathrm{e}$ for every Banach space X (a result which is new in the real case) and provide an example of a two-dimensional space for which the equality is true. Next, we study stability properties of the rank-one numerical index. Among others, we show that the rank-one numerical index of a c_0-, ℓ_1-, and ℓ_∞-sum of spaces is the infimum of the rank-one numerical index of the summands, we calculate the rank-one numerical index of $C(K,X)$, $L_1(\mu,X)$ and $L_\infty(\mu,X)$, and we show that $n_1(X^*)=n_1(X)$ when X is an L-embedded space. We also prove that the rank-one numerical index is continuous with respect to equivalent norms. Finally, we present an example of a Banach space X such that $n(X)<n_{\mathrm{comp}}(X)<n_1(X)$ (where $n_{\mathrm{comp}}(X)$ is the compact numerical index, see section 4 for the definition) and a number of other interesting examples.

2. A sharp lower bound for the rank-one numerical index

Our first goal is to give a lower bound for the rank-one numerical index valid in the real case and to show that it is best possible.

Theorem 2.1. *Let X be a real Banach space. Then,*

$$n_1(X)\geqslant \frac{1}{\mathrm{e}}\,.$$

We recall that given a real or complex Banach space X, one can define the exponential function on $L(X)$ by

$$\exp(T)=\mathrm{Id}+\sum_{k=1}^{\infty}\frac{T^k}{k!}\qquad \big(T\in L(X)\big)$$

and that it follows from [2, Theorem 3.4] that

$$\|\exp(\alpha T)\|\leqslant \mathrm{e}^{|\alpha|v(T)}\qquad \big(T\in L(X),\ \alpha\in\mathbb{K}\big). \tag{1}$$

Proof of Theorem 2.1. Let us fix a rank-one operator $T\in L(X)$. We find $x_0^*\in X^*$, $x_0\in X$ such that

$$Tx=x_0^*(x)\,x_0\quad (x\in X)$$

and write $\lambda=x_0^*(x_0)$. It is immediate to check that for each $\alpha\in\mathbb{R}$ one has

$$\exp(\alpha T)=\begin{cases}\mathrm{Id}+\alpha T & \text{if } \lambda=0\\ \mathrm{Id}+\frac{\mathrm{e}^{\lambda\alpha}-1}{\lambda}\,T & \text{if } \lambda\neq 0.\end{cases} \tag{2}$$

Now, if $v(T)=0$ then $\lambda=0$ (indeed, if $x_0=0$ the result is clear; otherwise, just pick $y^*\in S_{X^*}$ such that $y^*(x_0)=\|x_0\|$, write $y=x_0/\|x_0\|\in S_X$ and observe that $y^*(y)=1$ and $\lambda=y^*(Ty)$). Therefore, equations (1) and (2) give in this case that

$$\|\mathrm{Id}+\alpha T\|=\|\exp(\alpha T)\|\leqslant 1\qquad(\alpha\in\mathbb{R}).$$

This obviously implies that $T=0$ and thus $\|T\|\leqslant \mathrm{e}\,v(T)$.

If otherwise $v(T)\neq 0$, we can assume without loss of generality that $v(T)=1$ and so we have to show that $\|T\|\leqslant\mathrm{e}$. We distinguish two cases depending on wether $\lambda=0$ or not. Suppose first that $\lambda=0$. Then, using equations (1) and (2) for $\alpha=1$ and $\alpha=-1$, we obtain

$$\|\mathrm{Id}+T\|\leqslant\mathrm{e}\qquad\text{and}\qquad\|\mathrm{Id}-T\|\leqslant\mathrm{e}$$

which gives

$$\|T\|=\left\|\tfrac{1}{2}(\mathrm{Id}+T)-\tfrac{1}{2}(\mathrm{Id}-T)\right\|\leqslant\frac{\mathrm{e}}{2}+\frac{\mathrm{e}}{2}=\mathrm{e},$$

as desired. Finally, when $\lambda\neq 0$ one can use (1) and (2) to obtain

$$\left\|\mathrm{Id}+\frac{\mathrm{e}^{\lambda\alpha}-1}{\lambda}\,T\right\|=\|\exp(\alpha T)\|\leqslant\mathrm{e}^{|\alpha|}\qquad(\alpha\in\mathbb{R}).$$

Using this for $\alpha = 1$ and $\alpha = -1$ one gets

$$\left\| \mathrm{Id} + \frac{\mathrm{e}^{\lambda} - 1}{\lambda} T \right\| \leqslant \mathrm{e} \qquad \text{and} \qquad \left\| \mathrm{Id} + \frac{\mathrm{e}^{-\lambda} - 1}{\lambda} T \right\| \leqslant \mathrm{e}$$

and, therefore, one has

$$\left| \frac{\mathrm{e}^{\lambda} - \mathrm{e}^{-\lambda}}{2\lambda} \right| \|T\| \leqslant \frac{1}{2} \left\| \mathrm{Id} + \frac{\mathrm{e}^{\lambda} - 1}{\lambda} T \right\| + \frac{1}{2} \left\| \mathrm{Id} + \frac{\mathrm{e}^{-\lambda} - 1}{\lambda} T \right\| \leqslant \mathrm{e}\,.$$

The desired inequality follows now from the fact that $\inf\limits_{\lambda \neq 0} \left| \dfrac{\mathrm{e}^{\lambda} - \mathrm{e}^{-\lambda}}{2\lambda} \right| = 1.$ □

The following example shows that the inequality above is sharp. Let us comment that it is the real version of the space given in [7] of a complex two-dimensional space with numerical index equal to $1/\mathrm{e}$.

Example 2.2. *There is a real two-dimensional Banach space X with $n_1(X) = 1/\mathrm{e}$.* Indeed, consider the function $\Phi : [0, +\infty[\longrightarrow \mathbb{R}$ given by

$$\Phi(t) = \begin{cases} e^{t/e} & \text{if } t \in [0, e] \\ t & \text{if } t \geqslant e. \end{cases} \qquad (t \in [0, +\infty[)$$

Then, by [4, Proposition 3.1] the mapping $\|\cdot\| : \mathbb{R}^2 \longrightarrow [0, +\infty[$ given by

$$\|(\alpha, \beta)\| = \begin{cases} |\alpha| \Phi(\frac{|\beta|}{|\alpha|}) & \text{if } \alpha \neq 0 \\ |\beta| & \text{if } \alpha = 0 \end{cases} \qquad \big((\alpha, \beta) \in \mathbb{R}^2\big)$$

defines a norm on $\mathbb{R}^2$. Now denote $X = (\mathbb{R}^2, \|\cdot\|)$ and consider the shift operator $S \in L(X)$ given by $S(\alpha, \beta) = (0, \alpha)$. Using Lemma 3.3 in [4] one obtains that

$$\|S\| = 1 \quad \text{and} \quad v(S) = \sup \frac{\phi'}{\phi} = 1/\mathrm{e}\,,$$

which give $n_1(X) \leqslant 1/\mathrm{e}$, as desired.

3. Some properties of the rank-one numerical index

We present in this section properties of the rank-one numerical index. In many cases, they are analogous to the ones of the classical numerical index, but also there are some properties which present a different behavior.

Our first goal is to show that the rank-one numerical index behaves in the expected way when suitable sums of Banach spaces are considered. Given an arbitrary family $\{X_\lambda : \lambda \in \Lambda\}$ of Banach spaces, we denote by $\big[\bigoplus_{\lambda \in \Lambda} X_\lambda\big]_{c_0}$ the c_0-sum of the family and $\big[\bigoplus_{\lambda \in \Lambda} X_\lambda\big]_{\ell_p}$ denotes the ℓ_p-sum of the family for a given p with $1 \leqslant p \leqslant \infty$.

For the case of c_0-, ℓ_1-, and ℓ_∞-sums we have the following expected result.

Proposition 3.1. *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of Banach spaces. Then*

$$n_1\Big(\Big[\bigoplus_{\lambda \in \Lambda} X_\lambda\Big]_{c_0}\Big) = n_1\Big(\Big[\bigoplus_{\lambda \in \Lambda} X_\lambda\Big]_{\ell_1}\Big) = n_1\Big(\Big[\bigoplus_{\lambda \in \Lambda} X_\lambda\Big]_{\ell_\infty}\Big) = \inf\{n_1(X_\lambda) \ : \ \lambda \in \Lambda\}.$$

The proof of the proposition above is just an adaptation of the one given in [17, Proposition 1] for the numerical index. Indeed, it is enough to check that when in this proof one starts with rank-one operators, all operators involved are also rank-one.

In the already cited paper [17] it is also commented that the numerical index of an ℓ_p-sum is less or equal than the numerical index of the summands. This result has been generalized to absolute sums of Banach spaces in [14, §2]. Again, all the proofs can be adapted to the rank-one numerical index since when one starts with a rank-one operator, then all the operators appearing are rank-one. But, actually, we are now presenting a more general result which is new even for the classical numerical index. We will use this result later on.

Proposition 3.2. *Let X be a Banach space and Y,Z closed subspaces of X such that $X = Y \oplus Z$ and $\|y_1 + z\| = \|y_2 + z\|$ for $z \in Z$ and $y_1, y_2 \in Y$ with $\|y_1\| = \|y_2\|$. Then,*

$$n(X) \leqslant n(Y) \qquad \textit{and} \qquad n_1(X) \leqslant n_1(Y).$$

We need a lemma which gives, in the hypotheses of the above result, the possibility of extending an operator from Y to X with the same norm and numerical radius.

Lemma 3.3. *Let X be a Banach space and Y,Z nontrivial closed subspaces of X such that $X = Y \oplus Z$ and $\|y_1 + z\| = \|y_2 + z\|$ for every $z \in Z$ and every $y_1, y_2 \in Y$ with $\|y_1\| = \|y_2\|$. Then, given an operator $T \in L(Y)$, the operator $\widetilde{T} \in L(X)$ defined by*

$$\widetilde{T}(y + z) = Ty \qquad (y \in Y, z \in Z),$$

satisfies $\|\widetilde{T}\| = \|T\|$ and $v(\widetilde{T}) = v(T)$.

Proof. We start with two easy observations. First, the hypothesis gives us that the projections to Y and Z given by the decomposition $X = Y \oplus Z$ have norm one. Indeed, given $y \in Y$ and $z \in Z$, one has $y = \frac{1}{2}(y + z) + \frac{1}{2}(y - z)$ which, using the fact that $\|y - z\| = \|-y + z\| = \|y + z\|$, gives

$$\|y\| \leqslant \frac{1}{2}\|y + z\| + \frac{1}{2}\|y - z\| = \|y + z\|$$

and, analogously, we get $\|z\| \leqslant \|y + z\|$.

Secondly, we show that X^* is isometrically isomorphic to $Y^* \oplus Z^*$. To do so, recall that $X^* = Z^\perp \oplus Y^\perp$ and observe that $Z^\perp \equiv Y^*$ and $Y^\perp \equiv Z^*$. Indeed, consider the mapping $J : Z^\perp \longrightarrow Y^*$ given by $Jz^\perp = z^\perp|_Y$ for $z^\perp \in Z^\perp$. Taking into account that $z^\perp(y + z) = z^\perp(y)$ and $\|y + z\| \geqslant \|y\|$ for $z^\perp \in Z^\perp, y \in Y, z \in Z$, it is clear that $\|Jz^\perp\| = \|z^\perp\|$. To see that J is onto, fix $y^* \in Y^*$, take $x^* \in X^*$ a Hahn-Banach extension of y^*, and write $x^* = z^\perp + y^\perp$ for some $z^\perp \in Z^\perp$ and $y^\perp \in Y^\perp$. Then one has $Jz^\perp = z^\perp|_Y = x^*|_Y = y^*$. Analogous arguments show that $Y^\perp \equiv Z^*$. Summarizing, we have proved that $X^* \equiv Y^* \oplus Z^*$ and that the action on X is given by

$$[y^* + z^*](y + z) = y^*(y) + z^*(z) \qquad \big(y + z \in X,\ y^* + z^* \in X^*\big).$$

Now, since Y is 1-complemented in X, it is clear that $\|\widetilde{T}\| \leqslant \|T\|$ and the reversed inequality is always true. To show that $v\big(\widetilde{T}\big) \geqslant v(T)$, fixed $y \in S_Y$ and $y^* \in S_{Y^*}$ satisfying $y^*(y) = 1$, take a Hahn-Banach extension $x^* \in S_{X^*}$ of y^* and observe that $x^*(y) = 1$ and $x^*(\widetilde{T}(y)) = y^*(Ty)$. Therefore,

$$|y^*(Ty)| = \big|x^*(\widetilde{T}(y))\big| \leqslant v(\widetilde{T})$$

and we get the inequality taking supremum.

Finally, to prove the inequality $v(\widetilde{T}) \leqslant v(T)$, fixed $x \in S_X$ and $x^* \in S_{X^*}$ satisfying $x^*(x) = 1$, there are $y \in Y, z \in Z, y^* \in Y^*, z^* \in Z^*$ such that

$$x = y + z, \quad x^* = y^* + z^*, \quad \text{and} \quad \operatorname{Re} x^*(x) = \operatorname{Re} \big(y^*(y) + z^*(z)\big) = 1.$$

Moreover, it holds that $\|y\| \leqslant \|x\|$ and $\|y^*\| \leqslant \|x^*\|$. Hence, if we show that $\operatorname{Re} y^*(y) = \|y^*\|\|y\|$, then

$$|x^*(\widetilde{T}x)| = |y^*(Ty)| \leqslant v(T)\|y^*\|\|y\| \leqslant v(T)\|x^*\|\|x\| = v(T)$$

and the proof will be finished. To do so, given $\varepsilon > 0$, take $y_\varepsilon \in B_Y$ with $\|y_\varepsilon\| = \|y\|$ and such that

$$\operatorname{Re} y^*(y_\varepsilon) > \|y^*\|\|y_\varepsilon\| - \varepsilon.$$

By hypothesis, we have that $\|y_\varepsilon + z\| = \|y + z\| = 1$ and, therefore,

$$\begin{aligned} \operatorname{Re} y^*(y) + \operatorname{Re} z^*(z) &= \operatorname{Re} [y^* + z^*](y + z) = 1 \\ &\geqslant \operatorname{Re} [y^* + z^*](y_\varepsilon + z) = \operatorname{Re} y^*(y_\varepsilon) + \operatorname{Re} z^*(z) \\ &> \|y^*\|\|y_\varepsilon\| - \varepsilon + \operatorname{Re} z^*(z) \end{aligned}$$

which gives $\operatorname{Re} y^*(y) > \|y^*\|\|y\| - \varepsilon$. Finally, the arbitrariness of ε tells us that $\operatorname{Re} y^*(y) \geqslant \|y^*\|\|y\|$. □

Proof of Proposition 3.2. For the numerical index the result is an obvious consequence of the above lemma, since for every $T \in L(Y)$ with $T \neq 0$, one has

$$n(X) \leqslant \frac{v(\widetilde{T})}{\|\widetilde{T}\|} = \frac{v(T)}{\|T\|}.$$

Taking infimum on $T \in L(Y)$ with $T \neq 0$, we get $n(X) \leqslant n(Y)$ as desired. The result for the rank-one numerical index is exactly the same taking into account that when T is a rank-one operator, then $\widetilde{T}$ is also a rank-one operator. □

As a particular case of Proposition 3.2 we have that the rank-one numerical index of an absolute sum of Banach spaces is less or equal than the rank-one numerical index of each of the summands (see [14, §2] for definitions and for a different proof for the case of the classical numerical index). We will only give here two particular cases.

Corollary 3.4. *Let Λ be a nonempty set, let $\{X_\lambda \ : \ \lambda \in \Lambda\}$ be a family of Banach spaces and $1 < p < \infty$. Then*

$$n_1\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{\ell_p}\Big) \leqslant \inf\{n_1(X_\lambda) \ : \ \lambda \in \Lambda\}.$$

Corollary 3.5.

(a) *Let E be $\mathbb{R}^m$ endowed with an absolute norm, let $X_1, \ldots, X_m$ be Banach spaces and write X to denote their E-sum. Then*

$$n_1(X) \leqslant \min\big\{n_1(X_1), \ldots, n_1(X_m)\big\}.$$

(b) *Let E be a Banach space with a one-unconditional (infinite) basis, let $\{X_j \ : \ j \in \mathbb{N}\}$ be a sequence of Banach spaces and let X denote their E-sum. Then*

$$n_1(X) \leqslant \inf\big\{n_1(X_j) \ : \ j \in \mathbb{N}\big\}.$$

The next result deals with the study of the rank-one numerical index of vector valued spaces. Its behavior differs from that of the classical numerical index.

Proposition 3.6. *Let K be a compact Hausdorff space, μ a positive measure, and X a Banach space. Then, the following hold:*

$$n_1\big(C(K,X)\big) = \begin{cases} 1 & \text{if } K \text{ is perfect,} \\ n_1(X) & \text{if } K \text{ is not perfect,} \end{cases}$$

$$n_1\big(L_1(\mu,X)\big) = \begin{cases} 1 & \text{if } \mu \text{ has no atoms,} \\ n_1(X) & \text{if } \mu \text{ has atomic part,} \end{cases}$$

$$n_1\big(L_\infty(\mu,X)\big) = \begin{cases} 1 & \text{if } \mu \text{ has no atoms,} \\ n_1(X) & \text{if } \mu \text{ has atomic part.} \end{cases}$$

Proof. The proof for $L_1(\mu,X)$ and $L_\infty(\mu,X)$ follows the same lines, so we only give the one for the L_1-case. Indeed, it is known that $L_1(\mu,X)$ is isometrically isomorphic to a space of the form

$$\ell_1(\Gamma,X)\oplus_1 L_1(\nu,X)$$

for suitable set Γ and atomless measure ν, being Γ empty when μ is actually atomless. Now, the result follows from Proposition 3.1 and the fact that $L_1(\nu,X)$ has the alternative Daugavet property [16] and so, $n_1\big(L_1(\nu,X)\big)=1$.

Let us now prove the result for $C(K,X)$. If K is perfect, $C(K,X)$ has the alternative Daugavet property [16] and so $n_1\big(C(K,X)\big)=1$. If K has an isolated point, then X is an ℓ_∞-summand of $C(K,X)$ and so Proposition 3.1 gives us that $n_1\big(C(K,X)\big)\leqslant n_1(X)$. For the reversed inequality, we just follow the first part of the proof of [17, Theorem 5] but considering rank-one operators: there, for a given operator $T\in L(C(K,X))$ with $\|T\|=1$, an operator $S\in L(X)$ such that $\|S\|\simeq\|T\|$ and $v(T)\simeq v(S)$ is constructed. If one observes that when T is rank-one, then S is also rank-one, the mentioned fact shows that $n_1(C(K,X))\geqslant n_1(X)$. □

We may use the above result to give an example of a Banach space X such that $n_1(X^*)<n_1(X)$. Indeed, the space $X=C([0,1],\ell_2)$ satisfies $n_1(X)=1$ but $X^*\equiv L_1(\mu,\ell_2)$ for some measure μ which clearly contains atoms, so $n_1(X^*)=n_1(\ell_2)=1/2$. Let us comment that this kind of examples have appeared previously in the literature (see [16, Example 4.4]) using a characterization of the alternative Daugavet property for C^*-algebras and von Neumann preduals. On the other hand, for a von Neumann algebra A, it is shown in [16, Theorem 4.2] that A has the alternative Daugavet property if and only if its predual A_* does (equivalently, $n_1(A)=1$ iff $n_1(A_*)=1$) and this result was generalized to L-embedded spaces in [10, Proposition 2.3]. Actually, we may give a more general result covering any value of the rank-one numerical index. We recall that a Banach space X is said to be *L-embedded* if $X^{**}=X\oplus_1 X_s$ for some closed subspace X_s of X^{**}.

Proposition 3.7. *Let X be an L-embedded space. Then, $n_1(X)=n_1(X^*)$.*

The proof is just an adaptation of the one given in [10, Theorem 2.1] for the case of the classical numerical index taking into account that in this proof, if one starts with rank-one operators then all operators envolved are rank-one.

We finish this section with a result on the continuity of the rank-one numerical index of Banach spaces, analogous to the one given in [6] for the classical numerical index. Actually, the proofs are just adaptations to the new index of the ones given there, so we will only comment the changes. We need some definitions and notation used in the cited paper [6] which were actually taken from [3, §18].

Given a Banach space X, we denote by $\mathcal{E}(X)$ the set of all equivalent norms on X. This is an arcwise connected metric space when provided with the distance

$$d(p,q) = \log\left(\min\{k \geqslant 1 \ : \ p \leqslant kq,\, q \leqslant kp\}\right) \qquad (p,q \in \mathcal{E}(X)).$$

If $p \in \mathcal{E}(X)$ and $T \in L(X)$, we write $v_p(T)$ for the numerical radius of T in the space (X,p), that is,

$$v_p(T) = \sup\{|x^*(Tx)| \,:\, x \in X,\, x^* \in X^*,\, p(x) = p(x^*) = x^*(x) = 1\}$$

and $n_1(X,p)$ will be the rank-one numerical index of the Banach space (X,p). Finally, we consider the set

$$\mathcal{N}_1(X) = \{n_1(X,p) \ : \ p \in \mathcal{E}(X)\}$$

which represents all the values of the rank-one numerical index that X may have up to equivalent renorming.

Proposition 3.8. *Let X be a Banach space.*

(a) *The mapping $(p,T) \longmapsto v_p(T)$ from $\mathcal{E}(X) \times L(X)$ to $\mathbb{R}$ is uniformly continuous on bounded sets.*
(b) *As a consequence, the mapping $p \longmapsto n_1(X,p)$ from $\mathcal{E}(X)$ to $\mathbb{R}$ is continuous.*
(c) *Hence, $\mathcal{N}_1(X)$ is an interval.*
(d) *If $\dim(X) > 1$, then $1/\mathrm{e} \in \mathcal{N}_1(X)$.*

Proof. Item (a) follows from an easy refinement of the proof of [3, Corollary 18.4], as it was commented in [6]. (b) follows from (a) in the same manner as the continuity of the classical numerical index is deduced in [6]. Indeed, fix $p_0 \in \mathcal{E}(X)$, let B be an open ball centered at p_0 and $S = \{T \in L(X) \ : \ p_0(T) = 1 \ \dim(T(X)) = 1\}$, where we use the same symbol for a norm on X and the associated operator norm. It follows from (a) that the mapping $\Psi : B \times S \longrightarrow \mathbb{R}$ given by

$$\Psi(p,T) = \frac{v_p(T)}{p(T)} \qquad (p \in B,\ T \in S)$$

is uniformly continuous, which implies that the mapping

$$p \longmapsto \inf\left\{\Psi(p,T) \ : \ T \in S\right\} = n_1(X,p)$$

is continuous on B. (c) is an obvious consequence of (b). Finally, to prove (d) we take a two-dimensional subspace Y of X, and write $X = Y \oplus Z$ for suitable Z. Now, let W be a two-dimensional space with $n_1(W) = 1/\mathrm{e}$ (Example 2.2 for the real case and [7] for the complex case). Then, we have $X \simeq W \oplus_1 Z$, and Proposition 3.1 tells us that $n_1(W \oplus_1 Z) = \min\{n_1(W), n_1(Z)\} = n_1(W)$. □

4. Some examples and remarks

Our goal in this section is to provide some interesting examples relating the rank-one numerical index with other indices. Let us start by defining some more indices. Given a Banach space X, for every $r \in \mathbb{N}$ we define the *rank-r numerical index* by

$$n_r(X) = \inf\{v(T) \ : \ T \in S_{L(X)},\, \dim(T(X)) \leqslant r\}$$

and the *compact numerical index* by

$$n_{\text{comp}}(X) = \inf\{v(T) \ : \ T \in S_{L(X)},\, T \text{ compact}\}.$$

It is immediate that $n_r(X) \geqslant n_{r+1}(X) \geqslant n_{\text{comp}}(X) \geqslant n(X)$ for every $r \in \mathbb{N}$.

We start providing a real Banach space whose compact numerical index is strictly between the classical numerical index and the rank-one numerical index. Let us recall that when $n_1(X) = 1$ for a Banach space X, then $n_{\text{comp}}(X) = 1$ [16, Theorem 2.2].

Example 4.1. *There exists a real Banach space X such that*

$$n(X) < n_{\mathrm{comp}}(X) < n_1(X).$$

Indeed, fix a sufficiently large even number k such that $\tan\left(\frac{\pi}{2k}\right) < 1/\mathrm{e}$ and take X_k to be the two-dimensional real Banach space whose unit ball is the $2k$-sided regular polygon centered at the origin, having one of its vertices on the point $(1, 0)$. Now, consider the space

$$X = C([0,1], \ell_2) \oplus_1 X_k.$$

Then, we have that $n(X) = n(\ell_2) = 0$ by [17, Proposition 1 and Theorem 5], that $n_1(X) \geqslant 1/\mathrm{e}$ by Theorem 2.1, and that $n_{\mathrm{comp}}(X) = n_{\mathrm{comp}}(X_k) = n(X_k) = \tan\left(\frac{\pi}{2k}\right)$ by [11, Theorem 5] and Proposition 3.1.

Let us comment that we do not know if the equality $n_{\mathrm{comp}}(X) = n_1(X)$ holds for every complex Banach space X.

The next result we present is that for finite-dimensional spaces, the values of the rank-one and the rank-two numerical indices are sometimes related. We start with a lemma which does not require the space to be finite-dimensional.

Lemma 4.2. *Let X be a Banach space. If there is $T \in L(X)$ with $\dim(T(X)) = 2$ and $v(T) = 0$, then $n_1(X) \leqslant \frac{1}{2}$.*

Proof. By [19, Thorem 2.1], $Y = T(X)$ is a two-dimensional well-embedded Hilbert subspace of X. That is (see [19, p. 430] and [19, Proposition 1.11]), there exists a subspace Z of X such that $X = Y \oplus Z$ and $\|y_1 + z\| = \|y_2 + z\|$ for every $z \in Z$ and every $y_1, y_2 \in Y$ with $\|y_1\| = \|y_2\|$. Now, Proposition 3.2 gives that $n_1(X) \leqslant n_1(Y)$. Finally, we have that $n_1(Y) = \frac{1}{2}$ by [12, Proposition 3.3] since Y is a Hilbert space with dimension greater than one. $\square$

As an immediate consequence we obtain that the numerical indices of rank-one and rank-two operators are linked for finite-dimensional Banach spaces.

Corollary 4.3. *Let X be a finite-dimensional (real) space with $n_2(X) = 0$. Then, $n_1(X) \leqslant \frac{1}{2}$.*

For two-dimensional spaces, the result actually deals with the classical numerical index.

Corollary 4.4. *Let X be a two-dimensional (real) space. If $n_1(X) > 1/2$, then $n(X) > 0$.*

We do not know whether the above result is true for arbitrary Banach spaces.

When X is a two-dimensional real Hilbert space, one has that $n_2(X) = 0$ and $n_1(X) = 1/2$. In the next example we show that something similar can happen for the numerical indices of higher rank.

Example 4.5. *For every even number r there is a Banach space X_r of dimension r with $n_r(X_r) = 0$ and $n_s(X_r) > 0$ for every $s < r$.* Indeed, fixed an even number r, [18, Theorem 3.10] provides us with an r-dimensional real Banach space X_r and an onto operator $T_0 \in L(X_r)$ such that

$$\{T \in L(X_r) \,:\, v(T) = 0\} = \{\lambda T_0 \,:\, \lambda \in \mathbb{R}\}.$$

It follows that $n_r(X_r) = 0$ since $T_0 \neq 0$ and that $n_s(X_r) > 0$ for every $s < r$ since the only non-null operators with numerical radius zero are the non-null multiples of T_0 which have rank r.

We may use the above example to produce an analogue to Example 4.1 for the numerical indices of higher rank.

Example 4.6. *For every $r \in \mathbb{N}$, there exists a real Banach space X such that*

$$n(X) < n_{\text{comp}}(X) < n_r(X).$$

Indeed, write $Y = X_{r+2}$ or $Y = X_{r+1}$ of the above example depending on whether r is even or odd. We have then $n_r(Y) > 0$ and $n_{\text{comp}}(Y) = 0$ since $n_{r+1}(Y) = 0$ or $n_{r+2}(Y) = 0$ depending on our choice of Y. From Proposition 3.8.a (just the same proof as (b) there), we deduce that both n_{comp} and n_r are continuous with respect to equivalent norm. Therefore, as we may find polyhedral norms as close to the norm of Y as we want, there exists a polyhedral norm such that, calling W to the space Y endowed with the this norm, we still have $n_{\text{comp}}(W) < n_r(W)$. Moreover, since W is polyhedral it cannot contain an isometric copy of $\mathbb{C}$, so Theorem 2.4 in [15] tells us that $n_{\text{comp}}(W) \neq 0$. Now, we may follow the lines of the proof of Example 4.1 and consider the space

$$X = C([0,1], \ell_2) \oplus_1 W$$

which satisfies $n(X) = n(\ell_2) = 0$ by [17, Proposition 1 and Theorem 5], $n_r(X) = n_r(W)$ and $n_{\text{comp}}(X) = n_{\text{comp}}(W)$ since $C([0,1], \ell_2)$ has the alternative Daugavet property and Proposition 3.1 is also true for the rank-r and the compact numerical indices.

We do not know if there is a Banach space X such that $n_{\text{comp}}(X) \neq \inf\limits_{r\in\mathbb{N}} n_r(X)$. If such an example exists, it cannot have the approximation property since in that case compact operators can be approximated in norm, and hence in numerical radius, by finite-rank operators.

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