ON REMOTALITY FOR CONVEX SETS IN BANACH SPACES

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Abstract. We show that every infinite dimensional Banach space has a closed and bounded convex set that is not remotal. This settles a problem raised by Sababheh and Khalil in [8].

1. Introduction

Let X be a real Banach space and let $E \subset X$ be a bounded set. We write $\mathrm{ext}(E)$ for the set of extreme points of E and $\overline{\mathrm{co}}(E)$ for the closed (in the norm topology) convex hull of E. If τ is a locally convex topology in X, we will write $\overline{\mathrm{co}}^{\tau}(E)$ to denote the τ-closed convex hull of E. We denote by B_X the closed unit ball of X.

The set E is said to be *remotal* from a point $x \in X$, if there exists a point $e_0 \in E$ such that $D(x,E) = \sup\{\|x-e\| : e \in E\} = \|x-e_0\|$. The point e_0 is called a *farthest point* of E from x. E is said to be *remotal* (*densely remotal*) if it is remotal from all (on a dense set) $x \in X$. Let $F(x,E) = \{e \in E : D(x,E) = \|x-e\|\}$. In general this set can be empty. A well known result of Lau ([5]) says that any weakly compact set is densely remotal. It seems to be open, the question of whether every infinite dimensional Banach space has a closed and bounded *convex* set that is not remotal. This question was actually raised in [8] and some partial positive answers were given in [8] and [7] in the case of reflexive Banach spaces and Banach spaces that fail the Schur property. The aim of this note is to give a positive answer to this question. We follow the notation and terminology of [8] and [7].

Let us outline the content of this paper. Let X be an infinite dimensional Banach space and let X^* be its topological dual. Using a classical integral representation theorem, we first show that X^* has a weak*-compact convex set K that is not remotal. This should be compared with [2, Proposition 1] where the authors exhibited a weak*-compact convex set $C \subset \ell^1$ that has no farthest points. To prove the general result, we use a stronger form of integral representation theorem for closed convex bounded sets with the Radon-Nikodým property (RNP for short) due to Edgar ([4], see [6, Theorem 16.12]). Let $E \subset X$ be a weakly closed and bounded set. An interesting problem that is open is to determine conditions on $\overline{\mathrm{co}}(E)$ so that $\overline{\mathrm{co}}(E)$ is remotal from x implies that E is remotal from x. We will give an example showing that E being norm closed in a reflexive space is not enough for the validity of Theorem A in [8].

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2. Main result

We first prove a weak*-version of [8, Theorem A]. In order to produce a weak*-compact convex non-remotal set, it is enough to show that if E is a weak*-compact set having no vector of maximum length, then the same is true of $\overline{\mathrm{co}}^{\mathrm{weak}^*}(E)$ (weak*-closed convex hull). For a compact convex set $K \subset X^*$ and for a probability measure μ on K, let $\gamma(\mu) \in K$ denote its resultant (or weak integral) with the property

$$\big[\gamma(\mu)\big](x) = \int_K k(x)\,d\mu(x) \qquad \big(x \in X\big).$$

We refer to [3, 6] for the results on integral representations we use here.

Theorem 1. *Let X be an infinite dimensional Banach space. Let $E \subset X^*$ be a weak*-closed and bounded set having no vector of maximum length. Then the weak*-closed convex hull K of E has no vector of maximum length. Equivalently, if E is not remotal from a point $x \in X$, then neither is K.*

Proof. Let $M = D(0,E) = \sup\{\|e\| : e \in E\} = \sup\{\|k\| : k \in K\}$. Suppose that there exists $x_0^* \in K$ such that $\|x_0^*\| = M$. Let μ be a probability measure on K with $\mu(E) = 1$ and such that $\gamma(\mu) = x_0^*$ (see [6, Proposition 1.1]). We fix $\varepsilon > 0$ and take $x \in X$ such that $\|x\| = 1$ and $x_0^*(x) > M - \varepsilon$. Now,

$$M - \varepsilon < x_0^*(x) = \int_K x^*(x)\,d\mu(x^*) = \int_E x^*(x)\,d\mu(x^*) \leqslant \int_E \|x^*\|\,d\mu \leqslant M.$$

Letting $\varepsilon \downarrow 0$, we get that $\displaystyle\int_E \|x^*\|\,d\mu(x^*) = M$ and so, $M = \|k\|$ μ-a.e. Hence $M = \|e\|$ for some $e \in E$. A contradiction. The last part of the statement is equivalent to the first one just by translation. ∎

Corollary 2. *Let X be an infinite dimensional Banach space. Then there exists a weak*-compact convex set $K \subset X^*$ that is not remotal.*

Proof. Since X is infinite dimensional, by the well-known Josefson-Nissenzweig theorem (see [3, p. 219]), there exists a sequence $\{x_n^*\}_{n\geqslant 1}$ of unit vectors such that $x_n^* \longrightarrow 0$ in the weak*-topology. Consider the set

$$E = \left\{ \frac{n}{n+1}\,x_n^* \ :\ n \in \mathbb{N} \right\} \cup \{0\},$$

which is clearly a weak*-compact set having no vector of maximum length. Thus, by the above theorem, the weak*-closed convex hull K of E does not have vectors of maximum length, so K is not remotal from 0. ∎

Remark 3. The arguments in Theorem 1 and Corollary 2 also work in the case of a weakly compact set E and its closed convex hull $K = \overline{\mathrm{co}}(E)$ (actually, the argument simplifies in this case and ε is not necessary). Thus, *in a Banach space X that fail the Schur property, by taking a sequence $\{x_n\}_{n\geqslant 1}$ of unit vectors which converges to 0 in the weak topology, we get that the set*

$$K = \overline{\mathrm{co}}\left(\left\{ \frac{n}{n+1}\,x_n \ :\ n \in \mathbb{N} \right\} \cup \{0\} \right)$$

is nonremotal from 0 *(alternatively, the set does not have any vector of maximal length).* This gives an alternative proof of the main result from [7].

Remark 4. From the above arguments it is easy to see that *for a weak***-compact set* $E \subset X^*$ *and for any* $x^* \in X^*$*, if the set* $F(x^*, K)$ *of farthest points in the weak***-closed convex hull* K *of* E *to* x^* *is non-empty, then it has a point of* E*.* However the method of proof in [7] has the advantage that it shows that there is an extreme point of K in $F(x^*, K)$. Then by Milman's theorem [3, p. 151], such an extreme point is also in E.

The following easy example shows that the hypothesis of weak*-closedness can not be omitted on the set E in Theorem 1 (weak-compactness in the case of Remark 3).

Example 5. *Let* $\{e_n\}_{n\geqslant 1}$ *denote the canonical vector basis in* ℓ^2*. Let* $X = \mathbb{K} \oplus_\infty \ell^2$*, where* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$ *is the base field and* $\oplus_\infty$ *means the* ℓ^∞*-direct sum. Consider the set*

$$E = \left\{ \left(\frac{n}{n+1}, \frac{n}{n+1} e_n \right) \ : \ n \in \mathbb{N} \right\}.$$

Then E *is a norm closed set which is not remotal from* 0*. Since* $\overline{\text{co}}(E) = \overline{\text{co}}^{\,\text{weak}}(E)$ *by Mazur's theorem and* $\{e_n\}_{n\geqslant 1} \longrightarrow 0$ *in the weak topology,* $(1, 0) \in \overline{\text{co}}(E)$ *and so,* $\overline{\text{co}}(E)$ *is remotal from* 0*.*

Remark 6. Let X be a Banach space and let $E \subset X$ be a weakly closed and bounded set. We do not know if remotality of $K = \overline{\text{co}}(E)$ from a point always implies that of E. Since any strongly exposed point of K clearly lies in E, the answer is affirmative if the farthest point in K is actually strongly exposed. We may also ask whether the above question has a positive answer for RNP sets (see [1, § 3] for these concepts).

We are now able to present the main result of our paper.

Theorem 7. *Let* X *be an infinite dimensional Banach space. Then, there exists a closed and bounded convex set* K *that is not remotal.*

Proof. As before, we will construct a closed and bounded set E which is not remotal from 0 and show that $K = \overline{\text{co}}(E)$ is also not remotal from 0.

In view of Remark 3 (or of [7]), we may assume without loss of generality that X has the Schur property. Since X is infinite dimensional, by Rosenthal's ℓ_1 Theorem (see [3, § XI]), X contains an isomorphic copy of ℓ^1. Let $|||\cdot|||$ denote the norm on X^*. Now we will be done if we can construct in every Banach space $Y = (\ell_1, |||\cdot|||)$ isomorphic to ℓ_1, a closed convex bounded set $K \subseteq Y$ which is not remotal from 0. Let us write τ for the weak*-topology of ℓ_1 as dual of c_0 inherited in Y. This is a locally convex topology on Y weaker than the norm topology and any τ-closed norm-bounded set is compact in this topology. Observe now that $|||\cdot|||$ is not necessarily weak*-lower semi-continuous (i.e. Y may not be a dual space) so, on the one hand, Corollary 2 does not apply and, on the other hand, B_Y may not be τ-closed.

Let $\{e_n\}_{n\geqslant 1}$ be the canonical basis of ℓ^1. Consider the set

$$E = \left\{ \frac{n}{n+1} \frac{e_n}{|||e_n|||} \ : \ n \in \mathbb{N} \right\} \cup \{0\} \subseteq B_Y$$

which is τ-compact since $\{e_n\}_{n\geqslant 1}$ τ-converges to 0 and $|||\cdot|||$ is equivalent to the usual norm of ℓ_1. We consider the set $K = \overline{\text{co}}^{\tau}(E) \subset Y$, which is τ-compact since it is τ-closed and norm-bounded (indeed, E is contained in the τ-closed set MB_{ℓ_1} for some $M > 0$, so $K \subset MB_{\ell_1}$).

Claim. $K \subseteq B_Y$. Indeed, since ℓ^1 (and so Y) has the RNP, K is a set with the RNP. Therefore, we have $K = \overline{\text{co}}\big(\text{ext}(K)\big)$ (closure in norm, see [1, § 3]). As K and E are τ-compact, Milman's theorem gives us that $\text{ext}(K) \subseteq E$ (see [3, p. 151]). Therefore, we have

$$K = \overline{\text{co}}\big(\text{ext}(K)\big) \subseteq \overline{\text{co}}(E) \subseteq \overline{\text{co}}^{\tau}(E) = K,$$

so $K = \overline{\text{co}}(E) \subseteq B_Y$ as claimed.

Suppose K is remotal from 0 in Y. As $D(0, E) = 1$ and $K \subseteq B_Y$, we also have $D(0, K) = 1$. Therefore, there is a vector $y_0 \in K$ with $|||y_0||| = 1$, and we may pick a functional $y_0^* \in Y^*$ with

$$|||y_0^*||| = 1 \qquad \text{and} \qquad y_0^*(y_0) = 1.$$

As K is a separable closed convex bounded set with the RNP, Edgar's integral representation theorem ([4], see [6, Theorem 16.12]), gives us that there exists a probability measure μ on K with $\mu\big(\text{ext}(K)\big) = 1$ (so $\mu(E) = 1$) such that

$$1 = y_0^*(y_0) = \int_K y_0^*(y)\, d\mu(y) = \int_E y_0^*(y)\, d\mu(y) \leqslant \int_E |||y|||\, d\mu(y) \leqslant 1.$$

Therefore, $|||y||| = 1$ μ-a.e. in E, which is clearly false. Thus we get a contradiction and K is nonremotal from 0. $\square$

Since remotality from 0 is equivalent to having a vector of maximal norm, we get the following corollary.

Corollary 8. *Let X be an infinite-dimensional Banach space. Then there is a closed convex set K contained in the* open *unit ball of X such that* $\sup\{\|x\| : x \in K\} = 1$.

Remark 9. Similar to Remark 4 (see also Remark 6), let us note that *for a separable weakly closed and bounded set E such that its closed convex hull K has the RNP, our arguments show that if $F(x, K) \neq \emptyset$ then it has an extreme point of K.*

Remark 10. *Going into the details of the proofs of Remark 3 and Theorem 7, one realizes that for every infinite-dimensional Banach space X, there is a locally convex Hausdorff topology τ, which is weaker than the norm topology and such that there is a τ-compact convex set K which is not remotal (from 0).* Indeed, if X does not have the Schur property, then the set K is actually weak compact. Otherwise, X contains a subspace Y isomorphic to ℓ_1, and the set $K \subset Y$ is compact for the topology τ' of Y which it inherits from the weak* topology of ℓ_1 as dual of c_0. Since we may extend the topology τ' of Y to a locally convex Hausdorff topology τ of X (still weaker than the norm topology of X), we get that K is τ-compact, as desired.

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