ON ORDER STRUCTURE AND OPERATORS IN $L_\infty(\mu)$

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Abstract. It is known that there is a continuous linear functional on L_∞ which is not narrow. On the other hand, every order-to-norm continuous AM-compact operator from $L_\infty(\mu)$ to a Banach space is narrow. We study order-to-norm continuous operators acting from $L_\infty(\mu)$ with a finite atomless measure μ to a Banach space. One of our main results asserts that every order-to-norm continuous operator from $L_\infty(\mu)$ to $c_0(\Gamma)$ is narrow while not every such an operator is AM-compact.

1. Introduction

1.1. General.

Our notation and terminology are standard (see [11], [12] for Banach spaces and [2] for vector lattices). We consider Banach spaces over the reals only. By $\mathcal{L}(X,Y)$ and $\mathcal{L}(X)$ we denote the spaces of all continuous linear operators acting from X to Y and from X to X respectively. We consider the space $L_p(\mu)$, $1 \leqslant p \leqslant \infty$, on a measure space (Ω, Σ, μ) with a finite atomless measure μ. For the Lebesgue measure space on $[0,1]$ we just write L_p instead of $L_p(\mu)$. By $\mathbf{1}_A$ we denote the characteristic function of a set $A \in \Sigma$, and $A = B \sqcup C$ for $A, B, C \in \Sigma$ means that both $A = B \cup C$ and $B \cap C = \emptyset$ hold, up to a measure null set. For any $x \in L_p(\mu)$ the support of x is defined by $\operatorname{supp} x = \{\omega \in \Omega : \ x(\omega) \neq 0\}$. Clearly, it is defined, up to a measure null set.

1.2. Order convergence and order-to-norm continuous operators.

Let E be a vector lattice and let (x_α) be a net in E. The notation $x_\alpha \downarrow 0$ is used to mean that the net (x_α) is decreasing (in the non-strict sense) and $\inf\limits_\alpha x_\alpha = 0$. A net (x_α) in E *order converges to an element* $x \in E$ (notation $x_\alpha \xrightarrow{\text{o}} x$) if there exists a net $u_\alpha \downarrow 0$ in E with $|x_\alpha - x| \leqslant u_\alpha$ for all α. A subset M of E is said to be *order bounded* when there are $x, y \in E$ such that $x \leqslant m \leqslant y$ for all $m \in M$.

Let E be a vector lattice and X be a Banach space. A map $f : E \to X$ is called *order-to-norm continuous* if order converging nets from E it sends to norm converging nets in X, and

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order-to-norm σ-continuous if order converging sequences from E it sends to norm converging sequences in X.

A map $f : E \to X$ is said to be *AM-compact* if $f(A)$ is a relatively compact set in X for any order bounded set $A \subset E$.

1.3. **Narrow operators.**

The notion of narrow operator (as a generalization of the notion of compact operator) was introduced and studied by Plichko and Popov in [14] for operators acting from a symmetric (in other terminology, rearrangement invariant) function space E with absolutely continuous norm to a Banach space (or, more generally, to an F-space) X. Although the assumption of absolute continuity of the norm in E is not used in the definition, it essentially has been used by mathematicians in investigations on narrow operators for a long time. The definition can be considered on a Köthe function space (as was considered in [4]).

Definition 1.1. Let E be a Köthe function space (Banach, or even F-space) on an atomless finite measure space (Ω, Σ, μ) and X be a Banach (or F-space). An operator $T \in \mathcal{L}(E, X)$ is called *narrow* if for every $A \in \Sigma$ and every $\varepsilon > 0$ there exists an $x \in E$ such that $|x| = \mathbf{1}_A$, $\displaystyle\int_\Omega x\, d\mu = 0$ and $\|Tx\| < \varepsilon$.

Here the conditions on x mean that $x = \mathbf{1}_B - \mathbf{1}_C$ for some $B, C \in \Sigma$ with $A = B \sqcup C$ and $\mu(B) = \mu(C)$. The condition $\displaystyle\int_\Omega x\, d\mu = 0$ can be equivalently removed if the norm of E is absolutely continuous [14]. We recall that the norm of a Köthe function space E is *absolutely continuous* if for every $x \in E$ and every decreasing sequence (A_n) in Σ with empty intersection, one has $\lim_n \|x \cdot \mathbf{1}_{A_n}\| = 0$. Of course, the norm of $L_\infty(\mu)$ is not absolutely continuous and we do not know whether the condition $\displaystyle\int_\Omega x\, d\mu = 0$ can be removed in this case. We will discuss some particular cases in Section 3.

The notion of narrow operator naturally generalizes that of compact operator when the domain space has absolutely continuous norm. So, the Daugavet property for compact operators on L_1 proved by Lozanovskii and the pseudo-Daugavet property for compact operators on L_p with $1 < p < \infty$, $p \neq 2$ established by Benyamini and Lin [3] were extended to narrow operators in [14].

Recently, some investigations have appeared on narrow operators defined on the space L_∞ the norm of which is not absolutely continuous. A striking example of a continuous linear functional on L_∞ which is not narrow was constructed in [13]. Let $\overline{\Sigma}$ be the σ-algebra of measurable subsets of $[0,1]$ which are identified up to measure null sets. Observe that $\overline{\Sigma}$ is also a Boolean algebra, so let $\mathcal{U}$ be any ultrafilter on $\overline{\Sigma}$ in the sense of [5, p. 72]. Then the linear functional $f_\mathcal{U} : E \to \mathbb{R}$ defined by

$$f_\mathcal{U}(x) = \lim_{A \in \mathcal{U}} \frac{1}{\lambda(A)} \int_A x d\lambda$$

is obviously bounded and it is not narrow. Indeed, for each $x \in L_\infty$ of the form $x = \mathbf{1}_A - \mathbf{1}_B$ where $[0,1] = A \sqcup B$ one has $f_\mathcal{U}(x) = \pm 1$ depending on whether $A \in \mathcal{U}$ or $B \in \mathcal{U}$. Let us comment that the reason why $f_\mathcal{U}$ is not narrow is that it is not order-to-norm continuous.

It should be mentioned that in [8], Kadets and Popov introduced the following notion of narrow operator defined on $C(K)$-spaces: an operator $T \in \mathcal{L}\big(C(K), X\big)$ is called *C-narrow* if for every open nonempty set $G \subseteq K$ and every $\varepsilon > 0$ there exists an $x \in C(K)$ such that $\|x\| = 1$, $\operatorname{supp} x \subseteq G$ and $\|Tx\| < \varepsilon$. In view of the possible consideration of $L_\infty(\mu)$ as a $C(K_\mu)$-space, we have two different definitions of a narrow operator. Using arguments of [6, Lemma 3.1], one can show that the definition of C-narrow operator for $L_\infty(\mu) = C(K_\mu)$ means exactly the following: $T \in \mathcal{L}\big(L_\infty(\mu), X\big)$ is narrow if and only if, for any $A \in \Sigma$, $\mu(A) > 0$ and every $\varepsilon > 0$ there exists an $x \in L_\infty(A)$ such that $\|x\| = 1$ and $\|Tx\| < \varepsilon$. In other words, if the restriction $T|_{L_\infty(A)}$ of T to $L_\infty(A)$ is not an isomorphic embedding. Observe that if μ is finite, then the inclusion operator $J : L_\infty(\mu) \to L_1(\mu)$ is C-narrow but not narrow.

1.4. **Motivation.**

In the recent paper [9], it has been shown that the following two results on narrow operators in L_p, $1 \leqslant p < \infty$ obtained in [14] can be extended to the case of $p = \infty$:

(A) there exists a narrow projection of the space L_p onto a subspace isometric to L_p,
(B) the sum of two narrow operators on L_p need not be narrow.

The following positive result on narrowness of operators from L_∞ was also obtained in [13].

Theorem 1.2 ([13, Theorem 5.1]). *Every AM-compact order-to-norm continuous linear operator $T : L_\infty(\mu) \to X$ is narrow for any Banach space X.*

So, the following question naturally arises.

Question 1.3. *Does there exist a Banach space X such that every order-to-norm continuous operator $T \in \mathcal{L}(L_\infty, X)$ is narrow while not every such an operator is AM-compact?*

We are going to give a positive answer to this question in the present paper.

1.5. **Organization of the paper.**

Section 2 is devoted to a study of order-to-norm continuous operators with domain space $L_\infty(\mu)$. In particular, we show that order-to-norm continuous operators have separable "essential domain" and separable range. Besides, we give a short proof of Theorem 1.2. In Section 3 we consider several possible definitions of narrow operator and show that all of them are equivalent for order-to-norm continuous operators. Finally, we show in Section 4 that $X = c_0(\Gamma)$ gives a positive answer to Question 1.3.

2. Order convergence in $L_p(\mu)$ and order-to-norm continuous operators from $L_\infty(\mu)$ to a Banach space

The following lemma is probably well known, but having no reference concerning it we give a sketch of its proof.

Lemma 2.1. *A sequence (x_n) in $L_p(\mu)$ with $1 \leqslant p \leqslant \infty$ order converges to an element $x \in L_p(\mu)$ if and only if $x_n \longrightarrow x$ a.e. on Ω and it is order bounded in $L_p(\mu)$.*

Proof. It is a non-difficult technical exercise to show that for a decreasing sequence $y_n \downarrow$ in $L_p(\mu)$, the conditions $\inf\limits_n y_n = 0$ and $y_n \longrightarrow 0$ a.e. are equivalent. So, the condition $|x_n - x| \leqslant y_n \downarrow 0$ implies the order boundedness of (x_n) and that $x_n \longrightarrow x$ a.e. Let now (x_n) be order bounded and $x_n \longrightarrow x$ a.e. We set $y_n(\omega) = \sup\limits_{m \geqslant n} |x_m(\omega) - x(\omega)|$. Then $|x_n - x| \leqslant y_n$ and $y_n \downarrow$. By the above, $y_n \downarrow 0$. □

Corollary 2.2. *Let $1 \leqslant p < r \leqslant \infty$ and $x_n, x \in L_r(\mu)$. If $x_n \xrightarrow{\text{o}} x$ in $L_r(\mu)$ then $x_n \xrightarrow{\text{o}} x$ in $L_p(\mu)$, but the converse is not true.*

As an example of the fact that the implication above does not reverse, one can consider the sequence $x_n(t) = t^{-1/r} \mathbf{1}_{(\frac{1}{n+1}, \frac{1}{n}]}(t) \in L_r$. By Lemma 2.1, $x_n \xrightarrow{\text{o}} 0$ in L_p but not in L_r, because (x_n) is not order bounded in L_r.

It is worth mentioning that the order boundedness and the norm boundedness for sets in $L_\infty(\mu)$ coincide. Thus, speaking of bounded sequences in $L_\infty(\mu)$, we need not specify the kind of boundedness we mean.

One more application of Lemma 2.1 is the following short proof of Theorem 1.2 (the proof given in [13] uses much more involved background in the setting when the domain space is a general vector lattice).

Proof of Theorem 1.2. Let $T \in \mathcal{L}\big(L_\infty(\mu), X\big)$ be AM-compact and order-to-norm continuous. Fix any $A \in \Sigma$ and $\varepsilon > 0$. Consider a Rademacher system (r_n) on $L_\infty(A)$, i.e. a system with the properties $|r_n| = \mathbf{1}_A$, $\displaystyle\int_\Omega r_n d\mu = 0$ and if $n \neq m$ then the function $r_n - r_m$ takes values -2 and 2 on some subsets of A of measure $\mu(A)/4$ and vanishing outside these subsets. Since (r_n) is order bounded, (Tr_n) is relatively compact in X. Hence there are indices $n \neq m$ such that for $x_1 = (r_n - r_m)/2$ one has $|x_1| = \mathbf{1}_{B_1}$, $B_1 \subset A$, $\mu(B_1) = \mu(A)/2$, $\displaystyle\int_\Omega x_1 d\mu = 0$ and $\|Tx_1\| < \varepsilon/2$. Setting $A_1 = A \setminus B_1$ we do the same with the set A_1 instead of A to find $x_2 \in L_\infty(\mu)$ with $|x_2| = \mathbf{1}_{B_2}$, $B_2 \subset A_1$, $\mu(B_2) = \mu(A_1)/2 = \mu(A)/4$, $\displaystyle\int_\Omega x_2 d\mu = 0$ and $\|Tx_2\| < \varepsilon/4$. Continuing the procedure in the obvious manner, we construct a sequence (x_n) in $L_\infty(\mu)$ such that $|x_n| = \mathbf{1}_{B_n}$, $A = \bigsqcup\limits_{n=1}^\infty B_n$ (up to a measure null set), $\displaystyle\int_\Omega x_n d\mu = 0$ and $\|Tx_n\| < \varepsilon/2^n$. Now, we set $x(\omega) = x_n(\omega)$ for $\omega \in B_n$ and $x(\omega) = 0$ for $\omega \in \Omega \setminus A$, and observe that $|x| = \mathbf{1}_A$, $\displaystyle\int_\Omega x d\mu = 0$ and $\sum\limits_{k=1}^n x_k \xrightarrow{\text{o}} x$ in $L_\infty(\mu)$ in view of Lemma 2.1. By the order-to-norm continuity of T, the last condition implies that $\lim\limits_{n \longrightarrow \infty} \Big\| T\Big(\sum\limits_{k=1}^n x_k\Big)\Big\| = \|Tx\|$. And since $\Big\| T\Big(\sum\limits_{k=1}^n x_k\Big)\Big\| \leqslant \sum\limits_{k=1}^n \|Tx_k\| < \varepsilon$, we obtain $\|Tx\| \leqslant \varepsilon$. □

Our next goal is to give some characterizations of order continuity for operators with domain $L_\infty(\mu)$.

Theorem 2.3. *Let X be a Banach space. For any operator $T \in \mathcal{L}\big(L_\infty(\mu), X\big)$ the following conditions are equivalent.*

(*i*) *T is order-to-norm continuous.*
(*ii*) *T is order-to-norm σ-continuous.*
(*iii*) *For any bounded sequence (x_n) in $L_\infty(\mu)$ tending to zero in measure, one has that $\|Tx_n\| \longrightarrow 0$.*
(*iv*) *Let $p \in [1,\infty)$. For any bounded sequence (x_n) in $L_\infty(\mu)$, the condition $\|x_n\|_{L_p(\mu)} \longrightarrow 0$ implies that $\|Tx_n\| \longrightarrow 0$.*

Proof. $(i) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (iii)$ Let (x_n) be bounded a sequence in $L_\infty(\mu)$ tending to zero in measure. Suppose, for the sake of contradiction, that $\|Tx_n\| \geqslant \delta > 0$ for infinitely many $n \in \mathbb{N}$. Without loss of generality we assume that this is true for all $n \in \mathbb{N}$. Passing to a subsequence, we obtain that $x_{n_k} \longrightarrow 0$ a.e. and still $\|Tx_{n_k}\| \geqslant \delta > 0$, what contradicts (ii).

$(iii) \Rightarrow (iv)$ It is enough to note that $\|x_n\|_{L_p(\mu)} \longrightarrow 0$ implies that $x_n \longrightarrow 0$ in measure.

$(iv) \Rightarrow (ii)$ Suppose that $x_n \xrightarrow{\text{o}} 0$. Then $|x_n| \leqslant y_n \downarrow 0$ for some sequence (y_n) in $L_\infty(\mu)$. By Lemma 2.1, $y_n \longrightarrow 0$ a.e., from what we deduce that $\|y_n\|_{L_p(\mu)} \longrightarrow 0$. Taking into account that $\|x_n\|_{L_p(\mu)} \leqslant \|y_n\|_{L_p(\mu)}$ for each n, we obtain that $\|x_n\|_{L_p(\mu)} \longrightarrow 0$. By (iv), $\|Tx_n\| \longrightarrow 0$.

$(ii) \Rightarrow (i)$ First we prove the following statement.

Claim. Let $x_\alpha \downarrow 0$ in $L_\infty(\mu)$. Then there exists a strictly increasing sequence of indices (α_n) such that $\inf\limits_n x_{\beta_n} = 0$ for any sequence of indices $\beta_n \geqslant \alpha_n$.

Indeed, since $0 \leqslant x_{\beta_n} \leqslant x_{\alpha_n}$, it is enough to show that $\inf\limits_n x_{\alpha_n} = 0$. Set $t_\alpha = \displaystyle\int_\Omega x_\alpha \, d\mu$ and observe that there exists $t_0 = \lim\limits_\alpha t_\alpha$ because (t_α) is decreasing and is bounded from below by 0. Now, choose a strictly increasing sequence of indices (α_n) so that $\lim\limits_{n\to\infty} t_{\alpha_n} = t_0$. Since the sequence (x_{α_n}) decreases and is bounded from below by 0, $z(\omega) = \lim\limits_{n\to\infty} x_{\alpha_n}(\omega) \geqslant 0$ exists a.e. Observe that $z = \inf\limits_n x_{\alpha_n}$ and, by the Lebesgue theorem, $t_0 = \displaystyle\int_\Omega z \, d\mu$. To prove the claim, it is sufficient to show that $t_0 = 0$. Suppose otherwise that $t_0 > 0$. Since $\inf\limits_\alpha x_\alpha = 0$ and $z \neq 0$, there exists an index β such that $z \wedge x_\beta < z$. Now, for every $n \in \mathbb{N}$ we choose an index γ_n so that $\gamma_n \geqslant \beta$ and $\gamma_n \geqslant \alpha_n$. Then $y = \inf\limits_n x_{\gamma_n} < z$ and $\displaystyle\int_\Omega y \, d\mu = \lim\limits_{n\to\infty} t_{\gamma_n} < t_0$, that contradicts the choice of t_0. Thus, the claim is proved.

Let T be order-to-norm σ-continuous. It is enough to prove that T is order-to-norm continuous at zero. Suppose that a net (x_α) order converges to 0, i.e. there is a net (u_α) such that $|x_\alpha| \leqslant u_\alpha \downarrow 0$. Using the claim, choose a strictly increasing sequence of indices (α_n) with $\inf\limits_n u_{\alpha_n} = 0$.

Now, fix any $\varepsilon > 0$ and consider the index set $A_\varepsilon = \{\alpha : \|Tx_\alpha\| \geqslant \varepsilon\}$. We show that the set A_ε is bounded from above, that is, there exists a β such that $\alpha < \beta$ for each $\alpha \in A_\varepsilon$.

Indeed, supposing the contrary, we obtain that there exists a sequence (β_n) of indices $\beta_n \in A_\varepsilon$ such that $\beta_n > \alpha_n$ for each n. Then $|x_{\beta_n}| \leqslant u_{\beta_n} \leqslant u_{\alpha_n}$ and $\inf_n u_{\alpha_n} = 0$, which implies that $x_{\beta_n} \xrightarrow{\text{o}} 0$. However, we have that $\|Tx_{\beta_n}\| \geqslant \varepsilon$ which contradicts the order-to-norm σ-continuity of T at zero, a contradiction. Therefore, the set A_ε is bounded from above by some β and hence, $\alpha \notin A_\varepsilon$ (equivalently, $\|Tx_\alpha\| < \varepsilon$) for every $\alpha \geqslant \beta$. This means that T is order-to-norm continuous at zero. □

By Theorem 2.3, if an operator $T \in \mathcal{L}(L_\infty(\mu), X)$ can be continuously extended to $L_p(\mu)$ with some $p < \infty$, then T is order-to-norm continuous. Nevertheless, in Section 4 we show that not every order-to-norm continuous operator of $\mathcal{L}\big(L_\infty(\mu), c_0(\Gamma)\big)$ can be extended to some $L_p(\mu)$ with $1 \leqslant p < \infty$.

The following characterization of order-to-norm continuity for operators will be used in the sequel.

Lemma 2.4. *Let X be a Banach space. An operator $T \in \mathcal{L}\big(L_\infty(\mu), X\big)$ is order-to-norm continuous if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for each $x \in L_\infty(\mu)$ with $\|x\| \leqslant 1$ and $\mu(\operatorname{supp} x) < \delta$ one has $\|Tx\| < \varepsilon$.*

Proof. The "ONLY IF" part. Suppose to the contrary that for some $\varepsilon > 0$ there exists a sequence $x_n \in L_\infty(\mu)$, $\|x_n\| \leqslant 1$ such that $\mu(\operatorname{supp} x_n) \longrightarrow 0$ as $n \longrightarrow \infty$ and $\|Tx_n\| \geqslant \varepsilon$ for each $n \in \mathbb{N}$. Then $x_n \longrightarrow 0$ a.e. and in view of Lemma 2.1, $x_n \xrightarrow{\text{o}} 0$. This contradicts the order-to-norm continuity of T.

The "IF" part. Let (x_α) be an order converging to zero net and (u_α) be a net with $|x_\alpha| \leqslant u_\alpha \downarrow 0$. Besides, we assume that $\|u_\alpha\| \leqslant 1$. Fix $\varepsilon > 0$ and choose $\delta > 0$ so that for every $x \in L_\infty(\mu)$ with $\|x\| \leqslant 1$ and $\mu(\operatorname{supp} x) < \delta$ we have that $\|Tx\| < \frac{\varepsilon}{2}$. Then by the boundedness of T, we choose $\delta_1 > 0$ so that $\|Tx\| < \frac{\varepsilon}{2}$ whenever $\|x\| < \delta_1$.

For each α we set $B_\alpha = \{\omega \in \Omega : u_\alpha(\omega) > \frac{\delta_1}{2}\}$. Since $\int_\Omega u_\alpha d\mu \geqslant \frac{\delta_1}{2}\mu(B_\alpha)$ and $\lim_\alpha \int_\Omega u_\alpha d\mu = 0$, we obtain $\lim_\alpha \mu(B_\alpha) = 0$. Then there exists α_0 such that $\mu(B_\alpha) < \delta$ for every $\alpha \geqslant \alpha_0$. Now, we denote $y_\alpha = x_\alpha - x_\alpha \mathbf{1}_{B_\alpha}$ and $z_\alpha = x_\alpha \mathbf{1}_{B_\alpha}$. The condition $|x_\alpha(\omega)| \leqslant u_\alpha(\omega) \leqslant \frac{\delta_1}{2}$ for each $\omega \in \Omega \setminus B_\alpha$ implies $\|y_\alpha\| \leqslant \frac{\delta_1}{2} < \delta_1$. Hence, $\|Ty_\alpha\| < \frac{\varepsilon}{2}$. On the other hand, since $\operatorname{supp} z_\alpha \subseteq B_\alpha$, we have that $\mu(\operatorname{supp} z_\alpha) < \delta$ for each $\alpha \geqslant \alpha_0$, and $\|z_\alpha\| \leqslant \|x_\alpha\| \leqslant \|u_\alpha\| \leqslant 1$. Therefore, $\|Tz_\alpha\| < \frac{\varepsilon}{2}$ and so,

$$\|Tx_\alpha\| = \|T(y_\alpha + z_\alpha)\| \leqslant \|T(y_\alpha)\| + \|T(z_\alpha)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for each $\alpha \geqslant \alpha_0$. Thus, $\lim_\alpha \|Tx_\alpha\| = 0$ and so T is order-to-norm continuous. □

The following two consequences of the above lemma assert that an order-to-norm continuous operator defined on L_∞ has separable "essential domain" and hence, a separable range.

Proposition 2.5. *Let $(h_n)_{n=1}^\infty$ be the Haar system on $[0,1]$, X a Banach space and $S, T \in \mathcal{L}(L_\infty, X)$ order-to-norm continuous operators. If $Sh_n = Th_n$ for each $n \in \mathbb{N}$ then $S = T$.*

Proof. Observe that for the characteristic function of any dyadic interval $w = \mathbf{1}_{\left[\frac{k-1}{2^n},\frac{k}{2^n}\right)}$ we have that $Sw = Tw$ (since w belongs to the linear span of the Haar system). Fix any $x \in L_\infty$ and any $\varepsilon > 0$. First, choose a simple function $y = \sum_{k=1}^{n} a_k \mathbf{1}_{A_k}$, $[0,1] = \bigsqcup_{k=1}^{n} A_k$ with $\|x - y\| < \varepsilon/\big(2\|S\| + 2\|T\|\big)$. Second, using Lemma 2.4, choose a $\delta > 0$ so that for any $u \in L_\infty$, $\|u\| \leqslant 1$ if $\mu(\operatorname{supp} u) < \delta$ then $\|(S-T)\,u\| < \varepsilon/2$. Third, for each $k = 1, \ldots, n$ choose a disjoint union B_k of dyadic intervals so that $B_i \cap B_j = \emptyset$ for $i \neq j$ and $\mu(A_k \triangle B_k) < \delta/n$. Then we obtain $\mu(\operatorname{supp}(y - z)) < \delta$ for $z = \sum_{k=1}^{n} a_k \mathbf{1}_{B_k}$, and hence, $\|(S-T)(y-z)\| < \varepsilon/2$. Besides, by the above argument, $(S-T)z = 0$. Hence,

$$\|Sx - Tx\| \leqslant \|(S-T)(x-y)\| + \|(S-T)(y-z)\|$$
$$\leqslant \|S-T\|\,\|x-y\| + \frac{\varepsilon}{2} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves the proposition by arbitrariness of ε. □

Proposition 2.6. *Let X be a Banach space and $T \in \mathcal{L}(L_\infty, X)$ an order-to-norm continuous operator. Then the range $T(L_\infty)$ is separable.*

Proof. Using the same arguments as in the proof of Proposition 2.5, one can show that the linear span of the set $\{Th_n :\ n \in \mathbb{N}\}$ is dense in $T(L_\infty)$. □

3. Different definitions of narrow operators on $L_\infty(\mu)$

It should be pointed out that Theorem 1.2 is proved for another definition of narrow operator in a more general setting when the domain space is a vector lattice. Since in a vector lattice there are no analogue for both conditions $|x| = \mathbf{1}_A$ and $\int_\Omega x\,d\mu = 0$, the definition should be different. We are not going to recall here this definition in the most general setting, referring the interested reader to [13], because our goal is to study operators defined on $L_\infty(\mu)$.

Let X be a Banach space. For convenience, we consider the following properties of an operator $T \in \mathcal{L}\big(L_\infty(\mu), X\big)$ which mean different types of narrowness.

(*i*) For every $A \in \Sigma$ and every $\varepsilon > 0$ there exists $x \in L_\infty(\mu)$ such that $|x| = \mathbf{1}_A$, $\int_\Omega x\,d\mu = 0$ and $\|Tx\| < \varepsilon$.

(*ii*) For every $A \in \Sigma$ and every $\varepsilon > 0$ there exists $x \in L_\infty(\mu)$ such that $|x| = \mathbf{1}_A$ and $\|Tx\| < \varepsilon$.

(*iii*) For every $y \in L_\infty(\mu)^+$ and every $\varepsilon > 0$ there exists $x \in L_\infty(\mu)$ such that $|x| = y$ and $\|Tx\| < \varepsilon$.

(*iv*) For every $y \in L_\infty(\mu)^+$ and every $\varepsilon > 0$ there exists $x \in L_\infty(\mu)$ such that $|x| = y$, $\int_\Omega x\,d\mu = 0$ and $\|Tx\| < \varepsilon$.

Let us give some comments on the definitions above. Property (i) means our Definition 1.1; Property (iii) exactly means the definition of narrow operator when the domain space is a vector lattice, and for which Theorem 1.2 was proved in [13]; Properties (ii) and (iv) are their weakest and strongest form respectively. So, each property of $(i), (iii)$ implies (ii) and Property (iv) implies all the rest. Equivalence of (iii) and (i) was proved in [13] for operators defined on a Köthe function space with absolutely continuous norm. Our goal is to show that the four definitions are equivalent when the operator is order-to-norm continuous.

Theorem 3.1. *Let X be a Banach space. For an order-to-norm continuous operator $T \in \mathcal{L}\big(L_\infty(\mu), X\big)$ all definitions of narrow operator $(i) - (iv)$ are equivalent.*

Proof. By the above remarks, it is enough to prove the implication $(ii) \Rightarrow (iv)$ only, so let us do it. Let $T \neq 0$. Fix any $y \in L_\infty(\mu)^+$ and $\varepsilon > 0$. Without loss of generality we assume that $0 < \|y\| \leqslant 1$. Choose a simple function $u = \sum\limits_{i=1}^{m} a_i \mathbf{1}_{A_i} \neq 0$ with $a_i \in \mathbb{R}^+$ and pair-wise disjoint elements $A_i \in \Sigma$ so that $A_i \subseteq \operatorname{supp} y = \operatorname{supp} u$ and

$$\|y - u\| < \frac{\varepsilon}{2\,\|T\|}. \tag{3.1}$$

Using Lemma 2.4, find $\delta > 0$ so that for any $z \in L_\infty(\mu)$, $\|z\| \leqslant 1$, if $\mu(\operatorname{supp} z) < \delta$ then

$$\|Tz\| < \frac{\varepsilon}{4m\|u\|}. \tag{3.2}$$

Consider on Σ the measure μ_y generated by y, i.e. $\mu_y(A) = \displaystyle\int_A y\, d\mu$ for any $A \in \Sigma$. We shall use the following simple fact which proof we omit:

Claim: *there exists $n \in \mathbb{N}$ such that for any $A \in \Sigma$ with $A \subseteq \operatorname{supp} y$, if $\mu_y(A) < \mu(\Omega)/n$ then $\mu(A) < \delta$.*

We pick such an n, and for each $i = 1, \ldots, m$ divide A_i into $n+1$ parts of equal μ_y-measure

$$A_i = \bigsqcup_{k=1}^{n+1} A_{i,k}, \quad \mu_y\big(A_{i,k}\big) = \frac{\mu_y(A_i)}{n+1}.$$

Now, for every $i = 1, \ldots, m$ and $k = 1, \ldots, n$, we use Property (ii) to find $x_{i,k} \in L_\infty(\mu)$ so that $\big|x_{i,k}\big| = \mathbf{1}_{A_{i,k}}$ and

$$\big\|Tx_{i,k}\big\| < \frac{\varepsilon}{4nm\|u\|}. \tag{3.3}$$

Then we put $\beta_{i,k} = \displaystyle\int_\Omega y x_{i,k}\, d\mu$ for $i = 1, \ldots, m$ and $k = 1, \ldots, n$, and observe that

$$\big|\beta_{i,k}\big| \leqslant \mu_y\big(A_{i,k}\big) = \frac{\mu_y(A_i)}{n+1} \tag{3.4}$$

for each $i = 1, \ldots, m$ and $k = 1, \ldots, n$.

Fix now any i. Using (3.4) and induction on $\ell = 1, \dots, n$, it can be easily shown that there are signs $\theta_{i,1}, \dots, \theta_{i,\ell} \in \{-1, 1\}$ such that

$$\Big|\sum_{j=1}^{\ell} \theta_{i,j}\beta_{i,j}\Big| \leqslant \frac{\mu_y(A_i)}{n+1}.$$

So, we choose such signs for $\ell = n$. Then pick $x_{i,n+1} \in L_\infty(\mu)$ satisfying $\big|x_{i,n+1}\big| = \mathbf{1}_{A_{i,n+1}}$ and

$$\int_\Omega y x_{i,n+1}\, d\mu = -\sum_{k=1}^{n} \theta_{i,k}\beta_{i,k}. \tag{3.5}$$

By the Claim, since

$$\mu_y\big(A_{i,n+1}\big) = \frac{\mu_y(A_i)}{n+1} < \frac{\mu(\Omega)}{n},$$

one has that $\mu\big(A_{i,n+1}\big) < \delta$ and hence by (3.2)

$$\big\|Tx_{i,n+1}\big\| < \frac{\varepsilon}{4m\|u\|}. \tag{3.6}$$

We finally set $x_i = \sum\limits_{k=1}^{n} \theta_{i,k}x_{i,k} + x_{i,n+1}$ for $i = 1, \dots, m$. From (3.3) and (3.6) we deduce

$$\|Tx_i\| \leqslant \sum_{k=1}^{n}\big\|Tx_{i,k}\big\| + \big\|Tx_{i,n+1}\big\| < n\,\frac{\varepsilon}{4nm\|u\|} + \frac{\varepsilon}{4m\|u\|} = \frac{\varepsilon}{2m\|u\|}. \tag{3.7}$$

Besides, by the above construction, $|x_i| = \mathbf{1}_{A_i}$ and $\Big|\sum\limits_{i=1}^{m} x_i\Big| = \mathbf{1}_{\operatorname{supp} y}$.

We finally set $x = y\sum\limits_{i=1}^{m} x_i$ and $v = \sum\limits_{i=1}^{m} a_i x_i$. Observe that $|x - v| = |y - u|$ a.e. on Ω and hence, $\|x - v\| = \|y - u\|$. Obviously, $|x| = y$. By (3.5),

$$\int_\Omega x d\mu = \sum_{i=1}^{m}\int_\Omega y x_i d\mu = \sum_{i=1}^{m}\left(\sum_{k=1}^{n}\theta_{i,k}\int_\Omega y x_{i,k} d\mu + \int_\Omega y x_{i,n+1} d\mu\right) = 0.$$

Finally, taking into account that $\|x - v\| = \|y - u\|$ and $|a_i| \leqslant \|u\|$, using (3.7) and (3.1) we obtain

$$\|Tx\| \leqslant \|Tv\| + \|T\|\,\|x - v\| \leqslant \sum_{i=1}^{m}|a_i|\,\|Tx_i\| + \frac{\varepsilon}{2} < m\|u\|\frac{\varepsilon}{2m\|u\|} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

The following problem remains unsolved.

Problem 3.2. *Does Property (ii) imply Property (iv) for every Banach space X and every operator T of $\mathcal{L}(L_\infty, X)$?*

4. Order-to-norm continuous operators from $L_\infty(\mu)$ to $c_0(\Gamma)$

The main result of this section asserts that every order-to-norm continuous operator from $L_\infty(\mu)$ to $c_0(\Gamma)$ is narrow, while not every order-to-norm continuous operator from $L_\infty(\mu)$ to $c_0(\Gamma)$ is AM-compact (here and, in what follows, Γ is any infinite set). On the other hand, we construct an example of an order-to-norm continuous operator from L_∞ to c_0 which cannot be extended to a continuous linear operator on any of the spaces L_p with $1 \leqslant p < \infty$. So, our main result cannot be deduced from the known fact that every operator $T \in \mathcal{L}(L_p, c_0)$ is narrow for any p, $1 \leqslant p < \infty$, due to Kadets and Popov [7]. Nevertheless, in our proof we follow the ideas of the already mentioned paper.

Theorem 4.1. *Every order-to-norm continuous operator from $L_\infty(\mu)$ to $c_0(\Gamma)$ is narrow but not every such an operator is AM-compact.*

Proof. Fix any $\varepsilon > 0$ and $A \in \Sigma$ with $\mu(A) > 0$, and consider the set

$$K_{\varepsilon,A} = \Big\{ x \in B_{L_\infty(\mu)} : \ \|Tx\| \leqslant \varepsilon \quad \text{and} \quad \int_\Omega x\, d\mu = 0 \Big\}.$$

We claim that $K_{\varepsilon,A}$ is a convex and weakly compact subset of $L_2(\mu)$. The convexity is simply verified. The only thing that should be explained here is that $K_{\varepsilon,A}$ is weakly closed. By convexity, it is enough to prove that it is norm closed in $L_2(\mu)$. Let $x_n \in K_{\varepsilon,A}$ and $\|x_n - x\|_{L_2(\mu)} \longrightarrow 0$ as $n \longrightarrow \infty$. Then, obviously, $\|x\|_{L_\infty(\mu)} \leqslant 1$ and $\displaystyle\int_\Omega x\, d\mu = 0$. By Theorem 2.3, $\|T(x_n - x)\| \longrightarrow 0$ as $n \longrightarrow \infty$ which implies that $\|Tx\| \leqslant \varepsilon$.

Then, by the Krein-Milman theorem, there exists an extreme point $x_0 \in K_{\varepsilon,A}$. We show that $|x_0| = \mathbf{1}_A$. Suppose, to the contrary, that there exist $\delta > 0$ and a subset $B \subseteq A$ with $\mu(B) > 0$ such that $|x_0(\omega)| \leqslant 1 - \delta$ for each $\omega \in B$. Denote by $(e_\gamma)_{\gamma\in\Gamma}$ the unit vector basis for $c_0(\Gamma)$ and by $(e^*_\gamma)_{\gamma\in\Gamma}$ its biorthogonal functionals. Now, choose a finite set $\Gamma_0 \subset \Gamma$ so that $|e^*_\gamma(Tx_0)| < \varepsilon/2$ for each $\gamma \in \Gamma \setminus \Gamma_0$.

We shall use the following elementary facts.

(1) Let $S : X \to Y$ be a linear operator acting between linear spaces. If a linear subspace $Y_0 \subseteq Y$ has finite codimension in Y, say, m then $X_0 = T^{-1}Y_0$ has finite codimension in X (indeed, for any $x_1, \ldots, x_{m+1} \in X$ there exists a non-trivial linear combination $\alpha_1 Tx_1 + \ldots + \alpha_{m+1}Tx_{m+1} \in Y_0$, and hence, $\alpha_1 x_1 + \ldots + \alpha_{m+1}x_{m+1} \in X_0$).

(2) If Z is an infinite dimensional subspace of a linear space X and Y is a linear subspace of finite codimension in X then $Y \cap Z \neq \{0\}$.

So, since $Y_0 = [e_\gamma]_{\gamma\in\Gamma\setminus\Gamma_0}$ has finite codimension in $c_0(\Gamma)$, the subspace

$$X_0 = T^{-1}Y_0 \cap \Big\{ x \in L_\infty(\mu) : \ \int_\Omega x\, d\mu = 0 \Big\}$$

has finite codimension in $L_\infty(\mu)$. Since $\dim L_\infty(B) = \infty$, there exists $y_0 \in L_\infty(B) \cap X_0$ with $y_0 \neq 0$. Then choose $\alpha \neq 0$ so that $\|\alpha y_0\|_{L_\infty(\mu)} \leqslant \delta$ and $\|T(\alpha y_0)\| < \varepsilon/2$. Thus, $x_0 \pm \alpha y_0 \in K_{\varepsilon,A}$, a contradiction.

This gives that T is narrow by Corollary 3.1.

Finally, we construct an example of an order-to-norm continuous operator $S \in \mathcal{L}(L_\infty, c_0)$ which is not AM-compact. This operator can be used in the obvious manner to obtain an operator of $\mathcal{L}\big(L_\infty(\mu), c_0(\Gamma)\big)$ with the same properties. Denote by (r_n) the Rademacher system on $[0,1]$ and for every $x \in L_\infty$ set $Sx = (\xi_1, \xi_2, \ldots)$ where $\xi_n = \displaystyle\int_\Omega x r_n \, d\mu$ for each $n \in \mathbb{N}$. Since S can be extended to a continuous linear operator $\hat{S} \in \mathcal{L}(L_1, c_0)$, it is order-to-norm continuous by Theorem 2.3. Since the Rademacher system is an order bounded set in L_∞ which is sent by S to a non relatively compact subset of c_0, the operator S is not AM-compact. □

Let us now discuss possible extensions of the above results for operators from L_∞ to ℓ_p.

Remarks 4.2.

(a) *For $1 \leqslant p < 2$, every operator $T \in \mathcal{L}\big(L_\infty(\mu), \ell_p\big)$ is compact.* Indeed, every operator $T \in \mathcal{L}\big(L_\infty(\mu), \ell_p\big)$ factors through a Hilbert space ([10], Corollary 1 in p. 285 and Corollary 2 in p. 291) and hence is compact by Pitt's theorem (see [1, Theorem 2.1.4]).

(b) Since compact operators are AM-compact, the above observation and Theorem 1.2 give that *for $1 \leqslant p < 2$ every order-to-norm continuous operator $T \in \mathcal{L}\big(L_\infty(\mu), \ell_p\big)$ is narrow.*

(c) *For $2 \leqslant p < \infty$, there exists an order-to-norm continuous operator $T \in \mathcal{L}(L_\infty, \ell_p)$ which is not AM-compact.* Indeed, the same operator T generated by the Rademacher system as in the last part of the proof of Theorem 4.1 maps L_∞ to ℓ_2, it is not AM-compact but it is extendable to an operator from L_2 to ℓ_2, so it is order-to-norm continuous. For $p \geqslant 2$, ℓ_2 is continuously embedded in ℓ_p and so the same example works.

(d) Let us also comment that the existence of non compact operators from L_∞ to ℓ_p with $2 \leqslant p < \infty$ follows immediately from the fact that L_1 contains subspaces isomorphic to ℓ_q for $1 < q \leqslant 2$ (see [1, Theorem 6.4.18]) and so, L_∞ contains quotient spaces isomorphic to ℓ_p for $p \geqslant 2$.

The following question remains open.

Problem 4.3. *Let $2 \leqslant p < \infty$. Is every order-to-norm continuous operator $T \in \mathcal{L}(L_\infty, \ell_p)$ narrow?*

Now, we show that not every order-to-norm continuous operator from L_∞ to c_0 can be extended to L_p with some $p < \infty$. Therefore, Theorem 4.1 cannot be deduced from results of [7].

Example 4.4. There exists an order-to-norm continuous operator $T \in \mathcal{L}(L_\infty, c_0)$ which cannot be extended to L_p for any $p < \infty$.

Proof. First observe that it is sufficient for each given p, $1 \leqslant p < \infty$, to construct an order-to-norm continuous operator $T = T_p \in \mathcal{L}(L_\infty, c_0)$ that cannot be extended to L_p, because then the desired operator can be easily obtained as the direct sum of such operators for any sequence of p_n's tending to infinity.

Therefore, we fix $p \in [1, \infty)$ and we first make a general observation.

Let (A_n) be any sequence of disjoint members of Σ and $g \in L_1$. Then, for every $x \in L_\infty$, we set

$$Tx = (\xi_1, \xi_2, \ldots), \quad \text{where} \quad \xi_n = \int_{A_n} gx \, d\mu.$$

Since $gx \in L_1$ and $\mu(A_n) \longrightarrow 0$, we have that $Tx \in c_0$ by the absolute continuity of the Lebesgue integral. Therefore, $T : L_\infty \to c_0$ is a linear operator. Furthermore, given any $x \in L_\infty$, one has that $\displaystyle\int_{A_n} |g||x| \, d\mu \leqslant \|g\|_{L_1} \|x\|_{L_\infty}$ for every $n \in \mathbb{N}$, hence T is bounded with $\|T\| \leqslant \|g\|_{L_1}$.

To show that T is order-to-norm continuous, we consider any sequence (x_n) in L_∞ order converging to zero (i.e. $|x_n| \leqslant y_n \downarrow 0$ for some sequence (y_n) in L_∞). By Lemma 2.1, (y_n) tends to zero a.e. on $[0,1]$. Thus, the sequence $(|g|y_n)$ is decreasing and tends to zero a.e. By the Lebesgue theorem, $G_n = \displaystyle\int_\Omega |g| y_n \, d\mu \longrightarrow 0$ as $n \longrightarrow \infty$. On the other hand, for each $n, m \in \mathbb{N}$ one has that

$$\Bigl| \int_{A_m} g x_n \, d\mu \Bigr| \leqslant \int_\Omega |g||x_n| \, d\mu \leqslant \int_\Omega |g| y_n \, d\mu = G_n,$$

whence we deduce that $\|Tx_n\| \leqslant G_n \longrightarrow 0$ as $n \longrightarrow \infty$. Thus, order-to-norm continuity of T is established by Theorem 2.3.

Now, we choose a suitable sequence (A_n) and $g \in L_1$ as follows. Let (A_n) be any disjoint sequence in Σ with

$$\mu(A_n) = \frac{1}{\alpha\, n^{3p}}, \quad \text{where} \quad \alpha = \sum_{k=1}^{\infty} \frac{1}{n^{3p}}$$

for each $n \in \mathbb{N}$ and $g = \sum\limits_{n=1}^{\infty} n^{3p-2} \mathbf{1}_{A_n}$. Note that

$$\int_\Omega g \, d\mu = \sum_{n=1}^{\infty} n^{3p-2} \mu(A_n) = \frac{1}{\alpha} \sum_{n=1}^{\infty} n^{3p-2} n^{-3p} = \frac{1}{\alpha} \sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6\alpha} < \infty.$$

We show that T cannot be extended continuously to L_p in this case. Indeed, putting $x_n = \bigl(\mu(A_n)\bigr)^{-1/p} \mathbf{1}_{A_n}$, one has $\|x_n\|_{L_p} = 1$ and

$$\begin{aligned} \int_{A_n} g x_n \, d\mu &= n^{3p-2} \bigl(\mu(A_n)\bigr)^{-\frac{1}{p}} \mu(A_n) = n^{3p-2} \bigl(\mu(A_n)\bigr)^{1-\frac{1}{p}} \\ &= n^{3p-2} \frac{1}{\bigl(\alpha\, n^{3p}\bigr)^{1-\frac{1}{p}}} = \alpha^{\frac{1}{p}-1} n \longrightarrow \infty \end{aligned}$$

as $n \longrightarrow \infty$. □

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