A NOTE ON THE NUMERICAL INDEX OF THE L_p SPACE OF DIMENSION TWO

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ABSTRACT. We give a lower bound for the numerical index of the two-dimensional real L_p space.

The numerical index of a Banach space is a constant relating the norm and the numerical radius of the (bounded linear) operators on the space. Let us start by recalling the relevant definitions. Given a Banach space X, we will write X^* for its topological dual and $L(X)$ for the Banach algebra of all (bounded linear) operators on X. For an operator $T \in L(X)$, its *numerical radius* is defined as

$$v(T) := \sup\{|x^*(Tx)| \ : \ x^* \in X^*,\ x \in X,\ \|x^*\| = \|x\| = x^*(x) = 1\}$$

Finally, the *numerical index* of the Banach space X is the constant defined by

$$n(X) := \inf\{v(T) \ : \ T \in L(X),\ \|T\| = 1\}.$$

Obviously, $n(X)$ is the greatest constant $k \geqslant 0$ such that $k\|T\| \leqslant v(T)$ for every $T \in L(X)$. The concept of numerical index was introduced by G. Lumer in 1968 and it appeared for the first time in the 1970 paper [3] where it was deeply studied. Since then, the theory of numerical index has been widely developed. Classical references here are the aforementioned paper [3] and the monographs by F. Bonsall and J. Duncan [1, 2] from the seventies. The reader will find the state-of-the-art on the subject in the recent survey paper [6] and references therein. We refer to all these references for background.

Let us comment on some results regarding the numerical index which will be relevant in the sequel. First, it is clear that $0 \leqslant n(X) \leqslant 1$ for every Banach space X. In the real case, these inequalities are the best possible. In the complex case one has $1/\mathrm{e} \leqslant n(X) \leqslant 1$ and all of these values are possible. Let us also mention that $n(X) \leqslant n(X^*)$, and that the reverse inequality does not always hold. Anyhow, when X is a reflexive space (in particular when X is finite-dimensional), one gets $n(X) = n(X^*)$. Second, there are some classical Banach spaces for which the numerical index has been calculated. For instance, the numerical index of $L_1(\mu)$ is 1, and this property is shared by any of its isometric preduals. In particular, $C(K)$ has numerical index 1 for every compact space and the same is true for all finite-codimensional subspaces of $C[0,1]$. For any real Hilbert space H of dimension greater than one it is known that $n(H) = 0$ (actually, there is $T \in L(H)$ with $\|T\| = 1$ and $v(T) = 0$). In the complex case, $n(H) = 1/2$ for every Hilbert space of dimension greater that one. Finally, the numerical indices of those real two dimensional spaces whose unit balls are regular polygons have been recently achieved and can be expressed in terms of the number of vertices.

Let us comment that the computation of the numerical index of a concrete Banach space is not an easy task. Indeed, the question of computing the numerical index of the classical L_p spaces for

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$p \neq 1, 2, \infty$ goes back to the seventies [3], and it is still an important open problem of the theory of numerical ranges. For instance, it is shown in the aforementioned paper that

$$\left\{ n(\ell_p^{(2)}) \ : \ 1 \leqslant p \leqslant \infty \right\} = [0,1]$$

in the real case, but this was achieved by making use of a connectedness argument and the exact value of $n(\ell_p^{(2)})$ is unknown for $p \neq 1, 2, \infty$. Here we use the notation $\ell_p^{(m)}$ for the m-dimensional L_p space.

Very recently, E. Ed-dari and M. Khamsi [4, 5] have thrown some light on the problem of computing the numerical index of L_p spaces. Their results can be summarized as follows. Let $1 < p < \infty$ be fixed. Then,

(a) $n\big(L_p[0,1]\big) = n(\ell_p) = \lim\limits_m \big\{ n(\ell_p^{(m)}) \ : \ m \in \mathbb{N} \big\}$, and the sequence $\{n(\ell_p^{(m)})\}_{m\in\mathbb{N}}$ is decreasing.

(b) In the real case,

$$\frac{1}{2} \sup_{t\in[0,1]} \frac{|t^{p-1} - t|}{1 + t^p} \leqslant n(\ell_p^{(2)}) \leqslant \sup_{t\in[0,1]} \frac{|t^{p-1} - t|}{1 + t^p}.$$

In view of item (a) above, it is clear that the computation of $n(\ell_p^{(m)})$ acquires more importance, and the first step could be to improve the knowledge of $n(\ell_p^{(2)})$ given in item (b) above. The aim of this short note is to give such an improvement.

Theorem 1. *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, in the real case,*

$$\max\left\{ \frac{1}{2^{1/p}}, \frac{1}{2^{1/q}} \right\} \max_{t\in[0,1]} \frac{|t^{p-1} - t|}{1 + t^p} \leqslant n(\ell_p^{(2)})\,.$$

We need the following way to compute the numerical radius of operators in $L(\ell_p^{(2)})$ which can be easily deduced from [3, Lemma 3.2].

Lemma 2. *Let $1 < p < \infty$, and $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an operator in $L(\ell_p^2)$. Then,*

$$v(T) = \max\left\{ \max_{t\in[0,1]} \frac{|a + d\,t^p| + \big|b\,t + c\,t^{p-1}\big|}{1 + t^p},\ \max_{t\in[0,1]} \frac{|d + a\,t^p| + \big|c\,t + b\,t^{p-1}\big|}{1 + t^p} \right\}.$$

In particular, $v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \max\limits_{t\in[0,1]} \dfrac{|t^{p-1} - t|}{1 + t^p}.$

Proof of the Theorem. Let us start with the case in which $1 < p \leqslant 2$. We fix an operator $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L(\ell_p^{(2)})$ and we are going to show that

$$\|T\| \leqslant 2^{1/q} \max\{|a| + |c|,\, |b| + |d|\}$$

and

$$v(T) \geqslant \max\{|a| + |c|,\, |b| + |d|\} \max_{t\in[0,1]} \frac{|t^{p-1} - t|}{1 + t^p},$$

which clearly gives the desired result. In order to estimate $\|T\|$, we observe that

$$\begin{aligned}
\|T\| &= \sup_{(x,y)\in S_{\ell_p^2}} \|(ax+by,\, cx+dy)\|_p = \sup_{(x,y)\in S_{\ell_p^2}} \left(|ax+by|^p + |cx+dy|^p\right)^{1/p} \\
&\leqslant \sup_{(x,y)\in S_{\ell_p^2}} |ax+by| + |cx+dy| \leqslant \sup_{(x,y)\in S_{\ell_p^2}} (|a|+|c|)|x| + (|b|+|d|)|y| \\
&= \big\|(|a|+|c|, |b|+|d|)\big\|_q = \left[(|a|+|c|)^q + (|b|+|d|)^q\right]^{1/q} \\
&\leqslant 2^{1/q} \max\{|a|+|c|,\ |b|+|d|\}.
\end{aligned}$$

To state the lower estimation for $v(T)$ we distinguish two cases. We assume first that $|a|+|c| \geqslant |b|+|d|$ and, by using Lemma 2, we obtain the following for each $t \in [0,1]$:

$$\begin{aligned}
v(T) &\geqslant \frac{|a| - |d|t^p + |c|t^{p-1} - |b|t}{1+t^p} \\
&= (|a|+|c|)\,\frac{t^{p-1}-t}{1+t^p} + |a|\,\frac{1-t^{p-1}}{1+t^p} + \frac{|a|t + |c|t - |b|t - |d|t^p}{1+t^p} \\
&\geqslant (|a|+|c|)\,\frac{t^{p-1}-t}{1+t^p} + |a|\,\frac{1-t^{p-1}}{1+t^p} + (|a|+|c|-|b|-|d|)\,\frac{t}{1+t^p} \\
&\geqslant (|a|+|c|)\,\frac{t^{p-1}-t}{1+t^p} = \max\{|a|+|c|,\ |b|+|d|\}\,\frac{t^{p-1}-t}{1+t^p}.
\end{aligned}$$

If otherwise $|b|+|d| \geqslant |a|+|c|$ then, for each $t \in [0,1]$, we get

$$\begin{aligned}
v(T) &\geqslant \frac{|d| - |a|t^p + |b|t^{p-1} - |c|t}{1+t^p} \\
&= (|b|+|d|)\,\frac{t^{p-1}-t}{1+t^p} + |d|\,\frac{1-t^{p-1}}{1+t^p} + \frac{|b|t + |d|t - |c|t - |a|t^p}{1+t^p} \\
&\geqslant (|b|+|d|)\,\frac{t^{p-1}-t}{1+t^p} + |d|\,\frac{1-t^{p-1}}{1+t^p} + (|b|+|d|-|a|-|c|)\,\frac{t}{1+t^p} \\
&\geqslant (|b|+|d|)\,\frac{t^{p-1}-t}{1+t^p} = \max\{|a|+|c|,\ |b|+|d|\}\,\frac{t^{p-1}-t}{1+t^p}.
\end{aligned}$$

In both cases we can take maximum with $t \in [0,1]$ to obtain

$$v(T) \geqslant \max\{|a|+|c|,\ |b|+|d|\} \max_{t\in[0,1]} \frac{|t^{p-1}-t|}{1+t^p}\,,$$

which finishes the proof when $1 < p \leqslant 2$.

If $2 \leqslant p < \infty$, we observe that then $1 < q \leqslant 2$ and

$$n(\ell_p^{(2)}) = n\big(\big[\ell_p^{(2)}\big]^*\big) = n(\ell_q^{(2)}).$$

By the result already proved, we have

$$\frac{1}{2^{1/p}} \max_{t\in[0,1]} \frac{|t^{q-1}-t|}{1+t^q} \leqslant n(\ell_q^{(2)})\,.$$

Finally, the substitution $s = t^{q-1}$ gives

$$\max_{t\in[0,1]} \frac{|t^{q-1}-t|}{1+t^q} = \max_{s\in[0,1]} \frac{|s^{p-1}-s|}{1+s^p}. \qquad \square$$

Remark 3. The above proof shows that, for every operator $T \in L(\ell_p^{(2)})$, $1 < p \leqslant 2$, such that $\|T\| \leqslant \max\{|a| + |c|, |b| + |d|\}$, one has
$$\max_{t \in [0,1]} \frac{|t^{p-1} - t|}{1 + t^p} \leqslant \frac{v(T)}{\|T\|}.$$

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