Slicely Countably Determined Banach spaces

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Abstract

A (separable) Banach space X is slicely countably determined if for every convex bounded subset A of X there is a sequence of slices (S_n) such that each slice of A contains one of the S_n. SCD-spaces form a joint generalization of spaces not containing ℓ_1 and those having the Radon-Nikodým property. We present many examples and several properties of this class. We give some applications to Banach spaces with the Daugavet and the alternative Daugavet properties, lush spaces and Banach spaces with numerical index 1. *To cite this article: A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

Résumé

Espaces de Banach dénombrablement déterminés par rapport aux tranches. Un espace de Banach (séparable) X est appelé un espace SCD si pour tout sous-ensemble A de X qui soit convexe et borné, il existe une suite S_n de tranches de A (une tranche est l'intersection non-vide de A avec un demi-espace ouvert de X) telle que chaque tranche de A contient une des tranches S_n. Les espaces SCD sont une généralisation des espaces qui ne contiennent pas ℓ_1 et aussi des espaces avec la propriété de Radon-Nikodým. On présente beacoup d'examples et diverses propriétés de cette classe. On donne aussi quelques applications aux espaces de Banach avec la propriété de Daugavet et la propriété alternative de Daugavet, aux espaces luxuriants (lush spaces), et aux espaces avec indice numérique 1. *Pour citer cet article : A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

1. Introduction

The aim of this Note is to present the concept of slicely countably determined Banach spaces, which is a joint generalization of spaces not containing ℓ_1 and those having the Radon-Nikodým property. We present the main examples, review several properties of this class of spaces and, finally, show some applications. We refer to our manuscript [1] for a detailed account of all the material here.

Given a real or complex Banach space, we write S_X for its unit sphere and B_X for its closed unit ball. The dual space of X is X^* and $L(X)$ is the space of all bounded linear operators from X to itself. A *slice*

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of a convex subset A of X is a nonempty intersection of A with an open half-space. Finally, $\overline{\text{conv}}(\cdot)$ stands for the closed convex hull. $\mathbb{T}$ is the set of modulus one scalars.

2. Slicely countably determined sets

Definition 2.1 *Let X be a Banach space and let A be a convex bounded subset of X. The set A is said to be* slicely countably determined *(*SCD set *in short) if there is a countable family $\{S_n : n \in \mathbb{N}\}$ of slices of A satisfying one of the following equivalent conditions:*

(i) every slice of A contains one of the S_n,

(ii) $A \subseteq \overline{\text{conv}}(B)$ for every $B \subseteq A$ intersecting all the sets S_n.

The basic examples related to Definition 2.1 are the following. Separable Radon-Nikodým sets and separable Asplund sets are SCD, whereas the unit balls of $C[0,1]$ and $L_1[0,1]$ are not and, actually, if X is a separable Banach space with the so-called Daugavet property [4,5], then B_X is not SCD.

With the help of a lemma by J. Bourgain (that every weakly open subset of a bounded convex set contains a convex combination of slices), it is straightforward to get the following reformulation of the SCD property:

Proposition 2.2 *In the definition of SCD sets, we may take a family (S_n) of relatively weakly open subsets.*

This result is the key ingredient to be able to present two important families of SCD sets which extend Radon-Nikodým sets and Asplund sets.

Theorem 2.3 *Strongly regular separable bounded convex sets (in particular CPCP sets) are SCD.*

The following result is an easy consequence of Proposition 2.2. By a π-*base* of a topology τ on a set T we understand a family of nonempty τ-open subsets of T such that every nonempty τ-open subset O of T contains one of the elements of the family.

Proposition 2.4 *Let X be a Banach space and let A be a convex bounded subset of X. If A has a countable π-base of the weak topology, then A is an SCD set.*

To get the main consequence of the above proposition we need the following result which uses deep results by S. Todorčević and by H. Rosenthal:

Theorem 2.5 *Let X be a Banach space and let A be a separable convex bounded subset of X which contains no ℓ_1-sequences. Then, A has a countable π-base for the weak topology.*

Corollary 2.6 *Separable convex bounded subsets containing no ℓ_1-sequences are SCD.*

3. Slicely Countably Determined spaces

Definition 3.1 *A separable Banach space X is said to be* slicely countably determined *(*SCD space *in short) if every convex bounded subset of X is an SCD set.*

Examples 3.2

(a) If X is a separable strongly regular space, then X is SCD. In particular, separable Radon-Nikodým spaces (more generally, separable CPCP spaces) are SCD.

(b) Separable spaces which do not contain copies of ℓ_1 are SCD.

(c) $c_0(\ell_1)$ and $\ell_1(c_0)$ are SCD.

(d) $C[0,1]$, $L_1[0,1]$ *and, in general, Banach spaces with the Daugavet property, are not SCD spaces.*

Let us present some stability results for SCD property.

Remark 3.3 Every subspace of an SCD space is SCD. $C[0,1]$ *is a non-SCD quotient of the SCD space* ℓ_1.

Theorem 3.4 *Let* X *be a Banach space with a subspace* Z *such that* Z *and* $Y = X/Z$ *are SCD spaces. Then,* X *is also an SCD space.*

Two immediate consequences of the last result are the following:

Corollary 3.5

(a) Let X *be a separable Banach space which is not SCD. Then, for every* ℓ_1 *subspace* Y_1 *of* X*, there is another* ℓ_1 *subspace* Y_2 *such that* $\overline{Y_1 + Y_2} = Y_1 + Y_2 = Y_1 \oplus Y_2$.

(b) Let $X_1, \ldots, X_n$ *be SCD Banach spaces. Then,* $X_1 \oplus \cdots \oplus X_n$ *is SCD.*

4. Applications to the Daugavet and the alternative Daugavet properties

A Banach space X has the *Daugavet property* [4,5] if $\|\mathrm{Id} + T\| = 1 + \|T\|$ for every rank-one operator $T \in L(X)$. In this case, all operators on X not fixing copies of ℓ_1 satisfy the same equality. If every rank-one operator $T \in L(X)$ satisfies

$$\max_{\theta \in \mathbb{T}} \|\mathrm{Id} + \theta\, T\| = 1 + \|T\| \tag{aDE}$$

X has the *alternative Daugavet property* [7] and then all weakly compact operators on X also satisfy (aDE).

Our goal here is to present the concept of SCD-operator and to show the relation to the Daugavet and the alternative Daugavet equations. We start with the main definition:

Definition 4.1 *Let* X *and* Y *be Banach spaces. A bounded linear operator* $T: X \longrightarrow Y$ *is said to be an* SCD-operator *if* $T(B_X)$ *is an SCD set.* T *is said to be a* hereditary-SCD-operator *if every convex subset of* $T(B_X)$ *is SCD.*

Examples 4.2 Let X *and* Y *be Banach spaces and let* $T: X \longrightarrow Y$ *be a bounded linear operator such that* $T(X)$ *is separable. Then the following two conditions are sufficient for* T *to be a hereditary-SCD-operator.*

(a) If $T(B_X)$ *is strongly regular, in particular, if* $T(B_X)$ *is a Radon-Nikodým set.*

(b) If $T(B_X)$ *does not contain* ℓ_1*-sequences, in particular, if* T *does not fix copies of* ℓ_1.

We start with the best result we can get for the alternative Daugavet property:

Theorem 4.3 *Let* X *be a Banach space with the alternative Daugavet property and let* $T \in L(X)$ *be an SCD-operator. Then,* T *satisfies* (aDE).

SCD-operators have separable rank, but for some applications the separability condition can be removed.

Corollary 4.4 *Let* X *be a Banach space with the alternative Daugavet property and* $T \in L(X)$*. If* $T(B_X)$ *is strongly regular or* T *does not fix copies of* ℓ_1*, then* T *satisfies* (aDE).

It is possible to show an analogous result to Theorem 4.3 for spaces with the Daugavet property. Even more, a stronger result holds true in this case. We need some notation. A bounded linear operator $T: X \longrightarrow Y$ between two Banach spaces X and Y is said to be *narrow* [6, §3 and §4] if for every $x, y \in S_X$, every $\varepsilon > 0$, and every slice S of B_X containing y, there is an element $z \in S$ such that $\|x+z\| \geqslant 2-\varepsilon$ and $\|Ty-Tz\| < \varepsilon$. It is known that strong Radon-Nikodým operators and operators which do not fix copies of ℓ_1 from a Banach space with the Daugavet property are narrow. It is possible to extend this to hereditary-SCD-operators:

Theorem 4.5 *Let X be a Banach space with the Daugavet property and $T : X \longrightarrow Y$ be a hereditary-SCD-operator. Then, T is narrow.*

5. Applications to lush spaces and to Banach spaces with numerical index 1

A Banach space is said to have *numerical index 1* [3] if every $T \in L(X)$ satisfies (aDE). It follows clearly that spaces with numerical index 1 have the alternative Daugavet property, being false the converse result in general. It follows from Theorem 4.3 that SCD spaces with the alternative Daugavet property have numerical index 1. Actually, it is true that SCD spaces with the alternative Daugavet property fulfill a geometrical property stronger than that to have numerical index 1 called lushness. A Banach space X is said to be *lush* [2] if for every $x, y \in S_X$ and every $\varepsilon > 0$, there is a slice S of B_X such that $x \in S$ and $\operatorname{dist}(y, \operatorname{conv}(\mathbb{T}S)) < \varepsilon$. Lush spaces have numerical index 1, being the converse result false in general.

Theorem 5.1 *Every Banach space X with the alternative Daugavet property whose unit ball is an SCD set is lush. In particular, every SCD space with the alternative Daugavet property is lush.*

Concerning some particular cases of this result, the separability assumption (implicit with the SCD hypothesis) can be removed.

Corollary 5.2 *Let X be a Banach space with the alternative Daugavet property. If X is strongly regular (in particular, CPCP) or X does not contain ℓ_1, then X is lush.*

It has been recently proved that the dual of an infinite-dimensional real lush space contains ℓ_1. The above corollary extends this to the alternative Daugavet property, giving the main application of SCD property.

Corollary 5.3 *let X be an infinite-dimensional real Banach space with the alternative Daugavet property (in particular, if $n(X) = 1$). Then, X^* contains ℓ_1.*

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