NUMERICAL INDEX OF BANACH SPACES OF WEAKLY OR WEAKLY-STAR CONTINUOUS FUNCTIONS

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Abstract. We show that the space of weakly continuous functions from a compact Hausdorff space into a Banach space has the same numerical index as the range space. We also establish some inequalities for the numerical index of the space of weakly-star continuous functions from a compact Hausdorff space into the dual of a Banach space.

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The numerical index of a Banach space is a constant relating the norm and the numerical radius of operators on the space. Let us present the relevant definitions. For a Banach space X, we write B_X for the closed unit ball, S_X for the unit sphere, X^* for the dual space, and $\Pi(X)$ for the subset of $X \times X^*$ given by

$$\Pi(X) = \{(x, x^*) \in X \times X^* \ : \ x^*(x) = \|x^*\| = \|x\| = 1\}.$$

For a bounded linear operator T on X, we define its *numerical radius* by

$$v(T) = \sup\{|x^*(Tx)| \ : \ (x, x^*) \in \Pi(X)\}.$$

The *numerical index* of the space X is then given by

$$\begin{aligned} n(X) &= \max\{k \geqslant 0 \ : k\,\|T\| \leqslant v(T) \ \ \forall\, T \in L(X)\} \\ &= \inf\{v(T) \ : \ T \in L(X), \ \|T\| = 1\}, \end{aligned}$$

where $L(X)$ stands for the space of all bounded linear operators on X.

The concept of numerical index was first suggested by G. Lumer in 1968. At that time, it was known that a Hilbert space of dimension greater than 1 has numerical index 1/2 in the complex case, and 0 in the real case. Two years later, J. Duncan, C. McGregor, J. Pryce, and A. White proved that L-spaces and M-spaces have numerical index 1. They also determined the range of values of the numerical index. More precisely, for a real Banach space X, $n(X)$ can be any number in the interval $[0, 1]$, while $\{n(X) \ : \ X \text{ complex Banach space}\} = [1/\mathrm{e}, 1]$. The remarkable result that $n(X) \geqslant 1/e$ for every complex Banach space X goes back to H. Bohnenblust and S. Karlin. The disk algebra is another example of a Banach space with numerical index 1. For the above results and background, we refer the reader to the books [2, 3] and to the expository paper [13]. More recent results can be found in [7, 10, 12, 14, 15, 16, 17, 18] and the references therein.

Let us mention here a couple of facts concerning the numerical index which are relevant to our discussion. First, one has $v(T^*) = v(T)$ for every $T \in L(X)$, where T^* is the adjoint operator of T (see [2, § 9]) and it clearly follows that $n(X^*) \leqslant n(X)$ for every Banach space X. The question if this is actually an equality seems to be open. Second, to calculate the numerical radius of an operator $T \in L(X)$, it is not needed to use the whole set $\Pi(X)$, but a subset Γ of $\Pi(X)$ whose first projection is dense in S_X [2, Theorem 9.3].

In [16, Theorem 5], it is proved that $n(C(K, X)) = n(X)$ for every compact Hausdorff space K and every Banach space X. The first aim of this paper is to adapt the arguments given there to prove that the same equality holds when $C(K, X)$ is replaced by $C_\omega(K, X)$, the Banach space of all weakly continuous functions from K into X, equipped with the supremum norm. We will need the following lemma, whose first part appears in [8, Page 1905]. We include its proof for the sake of completeness. The second part gives us a dense subset of the unit sphere of $C_\omega(K, X)$.

Lemma 1. *Let K be a compact Hausdorff space and X a Banach space.*

(a) *For every $f \in C_\omega(K, X)$, the set*

$$\{t \in K \ : \ f \ \ \textit{is norm continuous at } t\}$$

is dense in K.

(b) *The set*

$$\left\{f \in S_{C_\omega(K,X)} \ : \ f \ \ \textit{attains its norm}\right\}$$

is dense in $S_{C_\omega(K,X)}$.

Proof. (a). For each $n \in \mathbb{N}$, let

$$O_n = \{t \in K \ : \ \exists \text{ an open set } V \text{ s.t. } t \in V \text{ and } \operatorname{diam} f(V) < 1/n\}.$$

Note that each O_n is open in K and

$$\bigcap_{n\in\mathbb{N}} O_n = \{t \in K \ : \ f \text{ is norm continuous at } t\}.$$

To get the result, it suffices to prove that each O_n is dense in K, and then Baire's Theorem applies. Indeed, for an open subset V of K and $n \in \mathbb{N}$, since $f(V)$ is dentable (see [5, Proposition VI.2.2]), there exists a weakly open set U in X with $U \cap f(V)$ nonempty and $\operatorname{diam}\big(U \cap f(V)\big) < 1/n$. Then, the set $W = f^{-1}(U) \cap V$ is nonempty, open and a subset of $V \cap O_n$.

(b). For $f \in S_{C_\omega(K,X)}$ and $0 < \varepsilon < 1$, by (a), we may find $t_0 \in K$ such that $\|f(t_0)\| > 1 - \varepsilon$ and f is norm continuous at t_0. Therefore, writing $x_0 = \dfrac{f(t_0)}{\|f(t_0)\|}$, the point t_0 does not belong to the closure of the set $\{t \in K \ : \ \|f(t) - x_0\| \geqslant \varepsilon\}$, so it is possible to find a continuous function $\varphi : K \longrightarrow [0,1]$ such that

$$\varphi(t_0) = 1 \qquad \text{and} \qquad \varphi(t) = 0 \quad \text{if } \|f(t) - x_0\| \geqslant \varepsilon.$$

Now, defining

$$g(t) = (1 - \varphi(t))\, f(t) + \varphi(t)\, x_0 \qquad (t \in K),$$

we have $g \in C_\omega(K,X)$ satisfying

$$\|g\| = \|g(t_0)\| = 1 \qquad \text{and} \qquad \|f - g\| < \varepsilon,$$

as desired. □

We can now state and prove the promised result.

Theorem 2. *Let K be a compact Hausdorff space and let X be a Banach space. Then,*

$$n\left(C_\omega(K,X)\right) = n(X).$$

Proof. To show that $n(C_\omega(K,X)) \geqslant n(X)$, we fix $T \in L(C_\omega(K,X))$ with $\|T\| = 1$ and prove that $v(T) \geqslant n(X)$. Given $\varepsilon > 0$, we may find $f_0 \in C_\omega(K,X)$ with $\|f_0\| = 1$ and, by using Lemma 1.a, we may also find $t_0 \in K$ such that f_0 is norm continuous at t_0 and

$$\|[Tf_0](t_0)\| > 1 - \varepsilon. \tag{1}$$

Denoting $y_0 = f_0(t_0)$, the continuity of f_0 at t_0 gives us that t_0 does not belong to the closure of the set $\{t \in K \ : \ \|f_0(t) - y_0\| \geqslant \varepsilon\}$, so it is possible to find a continuous function $\varphi : K \to [0,1]$ such that

$$\varphi(t_0) = 1 \qquad \text{and} \qquad \varphi(t) = 0 \ \text{ if } \|f_0(t) - y_0\| \geqslant \varepsilon.$$

Now, we write $y_0 = \lambda x_1 + (1-\lambda)x_2$ with $0 \leqslant \lambda \leqslant 1$, $x_1, x_2 \in S_X$, and we consider the functions

$$f_j = (1-\varphi)f_0 + \varphi x_j \in C_\omega(K,X) \qquad (j = 1,2).$$

Then, $\|\varphi f_0 - \varphi y_0\| < \varepsilon$, meaning that

$$\|f_0 - (\lambda f_1 + (1-\lambda) f_2)\| < \varepsilon,$$

and, using (1), we must have

$$\|[Tf_1](t_0)\| > 1 - 2\varepsilon \qquad \text{or} \qquad \|[Tf_2](t_0)\| > 1 - 2\varepsilon.$$

By making the right choice of $x_0 = x_1$ or $x_0 = x_2$ we get $x_0 \in S_X$ such that

$$\|\left[T\left((1-\varphi)f_0 + \varphi x_0\right)\right](t_0)\| > 1 - 2\varepsilon. \tag{2}$$

Next we fix $x_0^* \in S_{X^*}$ with $x_0^*(x_0) = 1$, denote

$$\Phi(x) = x_0^*(x)(1-\varphi)f_0 + \varphi x \in C_\omega(K,X) \qquad (x \in X),$$

and consider the operator $S \in L(X)$ given by

$$Sx = [T(\Phi(x))](t_0) \qquad (x \in X).$$

Since, by (2),

$$\|S\| \geqslant \|Sx_0\| > 1 - 2\varepsilon,$$

we may find $x \in S_X$, $x^* \in S_{X^*}$ such that

$$x^*(x) = 1 \qquad \text{and} \qquad |x^*(Sx)| \geqslant n(X)[1 - 2\varepsilon].$$

Now, define $g \in S_{C_\omega(K,X)}$ by $g = \Phi(x)$, for this x, and consider the functional $g^* \in S_{C_\omega(K,X)^*}$ given by

$$g^*(h) = x^*(h(t_0)) \qquad (h \in C_\omega(K,X)).$$

Since $g(t_0) = x$, we have $g^*(g) = 1$ and

$$|g^*(Tg)| = |x^*(Sx)| \geqslant n(X)[1 - 2\varepsilon].$$

Hence $v(T) \geqslant n(X)$, as required.

For the reverse inequality, take an operator $S \in L(X)$ with $\|S\| = 1$, and define $T \in L(C_\omega(K,X))$ by

$$[T(f)](t) = S(f(t)) \qquad (t \in K,\ f \in C_\omega(K,X)).$$

Then $\|T\| = 1$, so $v(T) \geqslant n(C_\omega(K,X))$. By Lemma 1.b and [2, Theorem 9.3], the numerical radius of T is given by

$$v(T) = \sup\left\{\left|x^*\big([Tf](t)\big)\right| \ :\ f \in S_{C_\omega(K,X)},\ t \in K,\ x^* \in S_{X^*},\ x^*(f(t)) = 1\right\}.$$

Therefore, given $\varepsilon > 0$, we may find $f \in S_{C_\omega(K,X)}$, $x^* \in S_{X^*}$, and $t \in K$, such that $x^*(f(t)) = 1$ and

$$n(C_\omega(K,X)) - \varepsilon < \left|x^*\big([Tf](t)\big)\right| = |x^*(S(f(t)))|.$$

It clearly follows that $v(S) \geqslant n(C_\omega(K,X))$, so $n(X) \geqslant n(C_\omega(K,X))$. □

It is well known (see [6, Theorem VI.7.1]) that the space $C_\omega(K,X^*)$ can be identified with $W(X,C(K))$, the space of weakly-compact operators from the Banach space X into $C(K)$. Therefore, the above result also reads as

Corollary 3. *Let K be a compact Hausdorff space and let X be a Banach space. Then,*

$$n\big(W(X,C(K))\big) = n(X^*).$$

Our next aim is to study the numerical index of $C_{\omega^*}(K,X^*)$, the Banach space of weakly-star continuous functions from a compact Hausdorff space K into the dual of a Banach space X. Unfortunately, we are not able to completely determine such numerical index, but some partial results can be established.

To state the first partial result, we need the following lemma, which gives us a dense subset of the unit sphere of $C_{\omega^*}(K,X^*)$. Let us first recall the well known identification between $C_{\omega^*}(K,X^*)$ and $L(X,C(K))$ (see [6, Theorem VI.7.1]): the mapping $\Phi : L(X,C(K)) \longrightarrow C_{\omega^*}(K,X^*)$ given by

$$[\Phi(T)](t) = T^*(\delta_t) \qquad \big(t \in K,\ T \in L(X,C(K))\big)$$

is an isometric isomorphism. By δ_t we denote the evaluation at t, which is a linear continuous functional on $C(K)$.

Lemma 4. *Let K be a compact Hausdorff space and let X be a Banach space. Then, the set*

$$\mathcal{A} = \left\{f \in S_{C_{\omega^*}(K,X^*)} \ :\ f \ \text{ attains its norm}\right\}$$

is dense in $S_{C_{\omega^*}(K,X^*)}$.

Proof. By using the identification Φ given before, the set $\mathcal{A}$ coincides with the set of those norm-one operators $T \in L(X, C(K))$ whose adjoints attain their norm at some δ_t. By a result of Zizler [20, Proposition 4], the set of norm-one operators whose adjoints attain their norms is always dense in the set of all norm-one operators. By a result of T. Johannesen [11, Theorem 5.8] (see the proof of [1, Theorem 2]), when the adjoint of an operator attains its norm, then it does at an extreme point of the dual ball. Finally, the set of extreme points of the unit ball of $C(K)^*$ coincides with $\{\lambda\, \delta_t \ :\ t \in K,\ |\lambda| = 1\}$. ☐

Proposition 5. *Let K be a compact Hausdorff space and X a Banach space. Then,*

$$n\left(C_{\omega^*}(K, X^*)\right) \leqslant n(X).$$

Proof. We take an operator $S \in L(X)$ with $\|S\| = 1$, and define the operator $T : C_{\omega^*}(K, X^*) \longrightarrow C_{\omega^*}(K, X^*)$ by

$$[T(f)](t) = S^*(f(t)) \qquad (t \in K,\ f \in C_{\omega^*}(K, X^*)).$$

It is clear that $T \in L(C_{\omega^*}(K, X^*))$ with $\|T\| = 1$, so $v(T) \geqslant n(C_{\omega^*}(K, X^*))$. By Lemma 4 and [2, Theorem 9.3], the numerical radius of T is equal to

$$\sup\left\{\left|x^{**}\big([Tf](t)\big)\right| \ :\ f \in S_{C_{\omega^*}(K,X^*)},\ t \in K,\ x^{**} \in S_{X^{**}},\ x^{**}(f(t)) = 1\right\}.$$

Therefore, given $\varepsilon > 0$, we may find $f \in S_{C_{\omega^*}(K,X^*)}$, $x^{**} \in S_{X^{**}}$, and $t \in K$, such that $x^{**}(f(t)) = 1$ and

$$n(C_{\omega^*}(K, X^*)) - \varepsilon < \left|x^{**}\big([Tf](t)\big)\right| = |x^{**}(S^*(f(t)))| \leqslant v(S^*) = v(S).$$

Hence, $n(X) \geqslant n(C_{\omega^*}(K, X^*))$. ☐

We are not able to obtain a lower bound for $n\left(C_{\omega^*}(K, X^*)\right)$ in the general case. Our next partial result gives sufficient conditions on K or X to obtain the desired bound. First we need another lemma, similar to Lemma 1.a.

Lemma 6. *Let K be a compact Hausdorff space and X an Asplund space. Then, for every $f \in C_{\omega^*}(K, X^*)$ the set*

$$\{t \in K \ :\ f \quad \textit{is norm continuous at } t\}$$

is dense in K.

Proof. Since X is Asplund, each bounded subset of X^* is w^*-dentable (see [4, Theorem 4.4.1]). Now, we can obviously adapt the proof of Lemma 1.a. ☐

Proposition 7. *Let K be a compact Hausdorff space and X a Banach space. If X is Asplund or K has a dense subset of isolated points then,*

$$n(X^*) \leqslant n\left(C_{\omega^*}(K, X^*)\right).$$

Proof. If X is Asplund, the first part of the proof of Theorem 2 can be straightforwardly adapted, using Lemma 6 instead of Lemma 1.a.

If K has a dense subset of isolated points the thesis of Lemma 6 remains trivially true, and the above argument applies. ☐

Using the identification between $C_{\omega^*}(K, X^*)$ and $L(X, C(K))$, Propositions 5 and 7 read as

Corollary 8. *Let K be a compact Hausdorff space and X Banach space. Then,*

$$n\big(L(X, C(K))\big) \leqslant n(X).$$

If, in addition, X is an Asplund space or K has a dense subset of isolated points, then

$$n(X^*) \leqslant n\big(L(X, C(K))\big).$$

In view of Corollaries 3 and 8, one may wonder if they might be special cases of a general result giving $n\big(W(X,Y)\big)$ and $n\big(L(X,Y)\big)$ as a function of $n(X)$ and $n(Y)$. The Example 10 of [16] says that this is not the case.

To finish the paper, let us give a stability result for the so-called Daugavet property. A Banach space X is said to have the *Daugavet property* [9] if the equality

$$\text{(DE)} \qquad \|Id + T\| = 1 + \|T\|$$

holds for every rank-one operator $T \in L(X)$; in such a case, all weakly compact operators also satisfy (DE). For a compact Hausdorff space K, the space $C_\omega(K,X)$ has the Daugavet property whenever K is perfect, for every Banach space X (see [19]). The next remark is another criterion for determining that the space of X-valued weakly continuous functions has the Daugavet property.

Remark 9. The first part of the proof of Theorem 2 can be easily adapted to get that $C_\omega(K,X)$ has the Daugavet property whenever X does, for every K. Indeed, just assume that $T \in L(C_\omega(K,X))$ with $\|T\| = 1$ is a rank-one operator, build $S \in L(X)$ exactly as in the above proof, and note that S is a rank-one operator as well. The Daugavet property of X gives an $x \in S_X$ which satisfies

$$\|x + Sx\| > 1 + \|S\| \ - \varepsilon > 2 - 3\varepsilon.$$

Now, we define the function $g \in S_{C_\omega(K,X)}$ as in the proof of the theorem, and we note that

$$\|Id + T\| \geqslant \big\|\big[(Id + T)(g)\big](t_0)\big\| = \|x + Sx\| > 2 - 3\varepsilon.$$

It is worth mentioning that the above argument is analogous to the one given for $C(K,X)$ in [16, Remark 6].

It is natural to ask if a result similar to the above one can be stated for the space $C_{\omega^*}(K, X^*)$, using Proposition 7 instead of Theorem 2. The only obstacle is that, to apply this proposition, we need X to be Asplund or K to have a dense subset of isolated points. In the first case, X cannot have the Daugavet property (see [9, page 857]); in the second case, we obtain the following result: *if K is a Hausdorff compact space with a dense subset of isolated points and X is a Banach space with the Daugavet property, then the space $C_{\omega^*}(K, X^*)$ also has the Daugavet property.*

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