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CONVEXITY AROUND THE UNIT OF A BANACH ALGEBRA*

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Abstract. We estimate the (midpoint) modulus of convexity at the unit $\mathbf{1}$ of a Banach algebra A showing that

$$\inf\left\{\max_{\pm}\|\mathbf{1}\pm x\|-1\ :\ x\in A,\ \|x\|=\varepsilon\right\}\geqslant\frac{\pi}{4\mathrm{e}}\,\varepsilon^{2}+o(\varepsilon^{2})$$

as $\varepsilon\to 0$. We also give a characterization of two-dimensional subspaces of Banach algebras containing the identity in terms of polynomial inequalities.

1. Introduction.

Let A be a unital Banach algebra. It is very well known that the unit $\mathbf{1}$ is an extreme point of the closed unit ball of A. An easy proof of this fact can be found in the classical book by S. Sakai [8, Proposition 1.6.6]. Moreover, it is also known that $\mathbf{1}$ has to be a "strongly extreme

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point" or a point of "midpoint locally uniform convexity" of the unit ball (see [1, Theorem 4.5] or [5, Theorem 18] for the complex case and [3, Proposition 3.3] for the real case). We refer the reader to [4, 6] for more information and background on this concept. We need some common notation. If X is a real or complex Banach space, we denote by B_X and S_X the closed unit ball and the unit sphere of the space. We write X^* to denote the (topological) dual space of X and $L(X)$ will be the space of all (bounded linear) operators on X.

Definition 1.1. *Let X be a Banach space. A point $x_0 \in S_X$ is said to be a* strongly extreme point *or a point of* midpoint locally uniform convexity *(MLUC) if any (and then every) of the following equivalent conditions is satisfied.*

(*i*) *For every $\varepsilon > 0$ there is $\delta > 0$ such that whenever y, z belongs o B_X with $\|y + z - 2x_0\| < \delta$, then $\|y - z\| < \varepsilon$.*

(*ii*) *Given two sequences (y_n), (z_n) in B_X, if $(y_n + z_n) \longrightarrow 2x_0$, then $(y_n - z_n) \longrightarrow 0$ (equivalently, $(y_n) \longrightarrow x_0$ and $(z_n) \longrightarrow x_0$).*

(*iii*) *For every sequence (x_n) in X, if $\|x_0 + x_n\| \longrightarrow 1$ and $\|x_0 - x_n\| \longrightarrow 1$, then $(x_n) \longrightarrow 0$.*

(*iv*) *For every sequence (x_n) in X, if $\max_{\pm} \|x_0 \pm x_n\| \longrightarrow 1$, then $(x_n) \longrightarrow 0$.*

(*v*) *For every $\varepsilon > 0$, the number*

$$\delta_X(x_0; \varepsilon) = \inf \left\{ \max_{\pm} \|x_0 \pm x\| - 1 \; : \; x \in X, \; \|x\| = \varepsilon \right\}$$

is positive.

Item (v) above gives a quantitative version of the concept of strongly extreme point using the so-called *modulus of midpoint local convexity.* The main goal of this note is to give lower bounds for $\delta_X(x_0; \varepsilon)$ when X is a Banach algebra and x_0 is its unit. To do this, given a Banach algebra A with unit $\mathbf{1}$, let us denote

$$\delta_A(\varepsilon) := \inf \left\{ \max_{\pm} \|\mathbf{1} \pm x\| - 1 \; : \; x \in A, \; \|x\| = \varepsilon \right\}$$

and

$$\delta(\varepsilon) := \inf \left\{ \delta_A(\varepsilon) \; : \; A \text{ unital Banach algebra} \right\}.$$

In Section 2 we use the theory of numerical ranges and follow the proof of [1, Theorem 4.5] to show that $\delta(\varepsilon) \geqslant \sqrt{1 + \mathrm{e}^2\varepsilon^2} - 1$. For some particular

Banach algebras, the above lower bound can be improved. For instance, $\delta_A(\varepsilon) \geqslant \sqrt{1+\varepsilon^2} - 1$ when A is a unital Banach subalgebra of $L(H)$ for any Hilbert space H.

In Section 3 we follow a different approach by using polynomial identities to get that

$$\delta(\varepsilon) \geqslant \frac{\pi}{4\mathrm{e}}\varepsilon^2 + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0.$$

Finally, Section 4 is devoted to state a characterization of two-dimensional subspaces of Banach algebras containing the identity in terms of polynomial inequalities for the norm.

2. An application of numerical range. Let us recall some definitions and facts concerning numerical ranges we need. All of them can be found in the classical monographs by F. Bonsall and J. Duncan from the 1970's [1, 2], where we refer for more information and background. Let A be a Banach algebra with the unit $\mathbf{1}$. We denote by $D(A)$ for the set of all *states* of A, i.e.

$$D(A) = \{f \in A^* \;:\; f(\mathbf{1}) = \|f\| = 1\}.$$

For every $x \in A$, $V(x)$ stands for the *numerical range* of x, namely

$$V(x) = \{f(x) \;:\; f \in D(A)\}$$

and the *numerical radius* of x is

$$v(x) = \sup\{|f(x)| \;:\; f \in D(A)\}.$$

Evidently, v is a seminorm with $v(x) \leqslant \|x\|$ for every $x \in A$. There are algebras in which v is actually a norm equivalent to the given norm. To measure this fact it is introduced the *numerical index* of the Banach algebra A as

$$n(A) = \inf\{v(a) \;:\; a \in S_A\} = \max\{k \geqslant 0 \;:\; k\|a\| \leqslant v(a)\ \forall a \in A\}.$$

It is a celebrated result due to H. Bohnenblust and S. Karlin (see [1, Theorem 4.1]) that $n(A) \geqslant 1/\mathrm{e}$ for every complex Banach algebra A or, equivalently, that

$$v(x) \geqslant \frac{1}{\mathrm{e}}\,\|x\|$$

for every $x \in A$. This result is clearly false in the real case. For instance, if we consider $A = \mathbf{C}$ viewed as a real algebra, then $n(A) = 0$. We also recall that any real Banach algebra A can be *complexified* i.e. there is a complex Banach algebra $A_{\mathbf{C}}$ containing A as a (real) subalgebra (see [7, Theorem 1.3.2] for instance).

From the proof of [1, Theorem 4.5], one can easily extract the following estimate of $\delta(\varepsilon)$. We detail the proof for the sake of completeness.

Theorem 2.1. $\delta(\varepsilon) \geqslant (\sqrt{1+\mathrm{e}^{-2}\varepsilon^2}-1)$. *In particular,*

$$\delta(\varepsilon) \geqslant \frac{1}{2\,\mathrm{e}^2}\varepsilon^2 + o(\varepsilon^2)$$

as $\varepsilon \longrightarrow 0$.

Proof. Let A be a unital Banach algebra. Since in the definition of $\delta_A(\varepsilon)$ only the real structure of the algebra A is taken into account, and δ_A does not increase if we enlarge the algebra, we may suppose that A is a complex algebra (otherwise we use $A_{\mathbf{C}}$).

Then, for every $x \in A$ with $\|x\| = \varepsilon$ and for every $f \in D(A)$ we have

$$\max_{\pm} \|\mathbf{1} \pm x\|^2 \geqslant \max_{\pm}\big|f(\mathbf{1} \pm x)\big|^2 \geqslant \frac{1}{2}\left(\big|f(\mathbf{1}+x)\big|^2 + \big|f(\mathbf{1}-x)\big|^2\right) = 1 + |f(x)|^2.$$

Taking supremum over $f \in D(A)$ and applying that $n(A) \geqslant 1/\mathrm{e}$, we get the required estimate:

$$\max_{\pm} \|\mathbf{1} \pm x\|^2 \geqslant 1 + |v(x)|^2 \geqslant 1 + n(A)^2\varepsilon^2 \geqslant 1 + \frac{1}{\mathrm{e}^2}\varepsilon^2. \qquad \square$$

Let us observe that the quadratic estimate for $\delta(\varepsilon)$ from below is the best possible:

$$\delta(\varepsilon) \leqslant \delta_{\mathbb{C}}(\varepsilon) = \sqrt{1+\varepsilon^2} - 1.$$

For some particular algebras the estimate above can be essentially improved. The following two results give examples in this line.

Examples 2.2.

(a) *If A is a complex Banach algebra, then*

$$\delta_A(\varepsilon) \geqslant \sqrt{1 + n(A)^2\,\varepsilon^2} - 1.$$

Indeed, just follow the proof of the above theorem.

(b) *If A is a real Banach algebra, then*

$$\delta_A(\varepsilon) \geqslant n(A)\,\varepsilon.$$

Indeed, for $x \in A$ with $\|x\| = \varepsilon > 0$ and $f \in D(A)$ we have

$$\max_{\pm} \|\mathbf{1} \pm x\| \geqslant \max_{\pm} \big|f(\mathbf{1} \pm x)\big| = \max_{\pm} \big|1 \pm f(x)\big| = 1 + |f(x)|.$$

Taking supremum over $f \in D(A)$ we get

$$\max_{\pm} \|\mathbf{1} \pm x\| \geqslant 1 + n(A)\,\varepsilon.$$

Proposition 2.3. *Let H be a Hilbert space and let A be a closed unital subalgebra of $L(H)$. Then $\delta_A(\varepsilon) \geqslant \sqrt{1+\varepsilon^2} - 1$. In particular, this happens for any (real or complex) C^*-algebra.*

Proof. Of course, it is enough to show the result for $A = L(H)$. Let $T \in L(H)$, $\|T\| > \varepsilon$. Select an element $x \in S_H$ for which $\|Tx\| > \varepsilon$. Then

$$\begin{aligned} \max_{\pm} \|\mathrm{Id} \pm T\|^2 &\geqslant \max_{\pm} \|x \pm Tx\|^2 \\ &\geqslant \frac{1}{2}\left(\|x + Tx\|^2 + \|x - Tx\|^2\right) = 1 + \|Tx\|^2 > 1 + \varepsilon^2. \qquad \square \end{aligned}$$

A sight to the above result gives us to the following conjecture:

Conjecture 2.4. $\delta(\varepsilon) = \sqrt{1+\varepsilon^2} - 1$.

The following simple result can be regarded as the first step toward the proof of the conjecture:

Remark 2.5. $\delta(\varepsilon) = \sqrt{1+\varepsilon^2} - 1$ for $\varepsilon = 1$. Indeed, for every Banach algebra A and for every $x \in A$ we have

$$x = \frac{1}{4}\big((\mathbf{1} + x)^2 - (\mathbf{1} - x)^2\big),$$

so

$$\|x\| \leqslant \frac{1}{4}\big(\|\mathbf{1} + x\|^2 + \|\mathbf{1} - x\|^2\big).$$

Then $(\delta_A(\varepsilon) + 1)^2 \geqslant \inf\left\{\frac{1}{2}\left(\|\mathbf{1} + x\|^2 + \|\mathbf{1} - x\|^2\right) : x \in A, \|x\| = \varepsilon\right\} \geqslant 2\varepsilon$, so $\delta(\varepsilon) \geqslant \sqrt{2\varepsilon} - 1$, which gives for $\varepsilon = 1$ the estimate $\delta(1) \geqslant \sqrt{2} - 1$ we need.

3. An approach using polynomial identities. Theorem 2.1 shows that there is an absolute constant $c > 0$ such that

$$\delta(\varepsilon) \geqslant c\,\varepsilon^2 + o(\varepsilon^2)$$

for small $\varepsilon > 0$. The main result of this paper which we present in this section shows that the "easy" estimate of c given in that Theorem ($c \geqslant \frac{1}{2\mathrm{e}^2} \simeq .067668$) is not optimal. We prove that $c \geqslant \frac{\pi}{4\mathrm{e}} \simeq .28893$.

Theorem 3.1. *The inverse to $\delta(\varepsilon)$ function $\varepsilon(\delta)$ can be estimated as follows:*

$$\varepsilon(\delta) \leqslant 2\sqrt{\frac{\mathrm{e}\,\log(1+\delta)}{\pi}} \cdot (1+\delta)^6. \tag{1}$$

Consequently $\delta(\varepsilon) \geqslant \frac{\pi}{4\mathrm{e}}\varepsilon^2 + o(\varepsilon^2)$ as $\varepsilon \to 0$.

We start with some algebraic formulas. In all the text below C_n^k stands for binomial coefficients: $C_n^k = \frac{n!}{k!(n-k)!}$, $0 \le k \le n$. Let us agree that C_n^k is defined also for $k < 0$ and takes value 0.

Lemma 3.2. *For all $m \in \mathbf{N}$ the following two identities hold true*

$$\sum_{k=0}^{m} C_{2m+1}^k (2m+1-2k) = (m+1) C_{2m+1}^m \tag{2}$$

$$\sum_{k=0}^{m-1} C_{2m}^k (2m-2k) = m C_{2m}^m. \tag{3}$$

Proof. We will prove both identities simultaneously by induction on m. For $m = 1$ both (2) and (3) obviously hold.

Suppose (3) is true for $m = p$. Let us prove that (2) is true for $m = p$. We have

$$\begin{aligned}\sum_{k=0}^{p}(2p+1-2k)C_{2p+1}^k &= \sum_{k=0}^{p}(2p+1-2k)(C_{2p}^k + C_{2p}^{k-1}) \\ &= C_{2p}^p + \sum_{k=0}^{p-1} C_{2p}^k\big((2p+1-2k) + (2p+1-2(k+1))\big) \\ &= C_{2p}^p + \sum_{k=0}^{p-1} C_{2p}^k(4p-4k) = (2p+1)C_{2p}^p = (p+1)C_{2p+1}^p.\end{aligned}$$

Suppose (2) is true for $m = p$. Let us prove that (3) is true for $m = p+1$. We have

$$\sum_{k=0}^{p}(2p+2-2k)C_{2p+2}^{k} = \sum_{k=0}^{p}(2p+2-2k)(C_{2p+1}^{k} + C_{2p+1}^{k-1})$$

$$= 2C_{2p+1}^{p} + 2\sum_{k=0}^{p-1} C_{2p+1}^{k}(2p+1-2k) = (2p+2)C_{2p+1}^{p} = (p+1)C_{2p+2}^{p+1}.$$

The lemma is proved. $\square$

P r o o f o f T h e o r e m 3.1. The system of polynomials $\{B_n(x,k) = (1+x)^{n-k}(1-x)^k\}_{k=0}^{n}$ forms a basis in the linear space $P_n = \{P(x) = \sum_{k=0}^{n} a_k x^k : a_k \in \mathbf{R}\}$. Let us find the coefficients b_k such that

$$x = \sum_{k=0}^{n} b_k (1+x)^{n-k}(1-x)^k. \tag{4}$$

We have

$$\frac{x}{(1+x)^n} = \sum_{k=0}^{n} b_k \left(\frac{1-x}{1+x}\right)^k. \tag{5}$$

Denote by $y = \dfrac{1-x}{1+x}$, then $x = \dfrac{1-y}{1+y}$, $1+x = \dfrac{2}{1+y}$, and we have

$$\frac{(1-y)(1+y)^{n-1}}{2^n} = \sum_{k=0}^{n} b_k y^k. \tag{6}$$

So

$$b_0 = \frac{1}{2^n},\ b_n = -\frac{1}{2^n}, \quad b_k = \frac{C_{n-1}^{k} - C_{n-1}^{k-1}}{2^n}, 1 \le k \le n-1. \tag{7}$$

Since $C_{n-1}^{k} - C_{n-1}^{k-1} = C_n^k \cdot \dfrac{n-2k}{n}$, we finally obtain that for every $n \in \mathbf{N}$ the following identity holds

$$x = \sum_{k=0}^{n} \frac{C_n^k}{2^n} \cdot \frac{n-2k}{n}(1+x)^{n-k}(1-x)^k. \tag{8}$$

This identity can be applied to an element x of a Banach algebra. So, if $\|x\| = \varepsilon$, and $\max\{\|1+x\|, \|1-x\|\} = 1+\delta$, then we derive

$$\varepsilon \leqslant \frac{(1+\delta)^n}{n2^n} \sum_{k=0}^{n} C_n^k |n-2k|. \tag{9}$$

Suppose $n = 2m$, $m \in \mathbb{N}$. Then

$$\begin{aligned}
\sum_{k=0}^{n} C_n^k |n-2k| &= \sum_{k=0}^{2m} C_{2m}^k |2m-2k| \\
&= \sum_{k=0}^{m-1} C_{2m}^k (2m-2k) + \sum_{k=m+1}^{2m} C_{2m}^k (2k-2m) \\
&= 2\sum_{k=0}^{m-1} C_{2m}^k (2m-2k)
\end{aligned}$$

(to get the last equality we have substituted into the second sum $k = 2m-p$ and applied equality $C_{2m}^{2m-p} = C_{2m}^p$). Now substituting this into (9) and applying (3) we obtain

$$\varepsilon \leqslant \frac{(1+\delta)^{2m}}{(2m)2^{2m-1}} m C_{2m}^m = \left(\frac{1+\delta}{2}\right)^{2m} C_{2m}^m. \tag{10}$$

We will use the following Stierling formula with the estimation of the remainder term:

$$\sqrt{2\pi n}\; n^n\, \mathrm{e}^{-n} < n! < \sqrt{2\pi n}\; n^n\, \mathrm{e}^{-n}\, \mathrm{e}^{\frac{1}{4n}}. \tag{11}$$

Using this formula we obtain

$$C_{2m}^m = \frac{(2m)!}{(m!)^2} \leqslant \frac{\sqrt{4\pi m}\; (2m)^{2m}\, \mathrm{e}^{-2m}\, \mathrm{e}^{\frac{1}{8m}}}{2\pi m\; m^{2m}\, \mathrm{e}^{-2m}} = \frac{2^{2m}\, \mathrm{e}^{\frac{1}{8m}}}{\sqrt{\pi m}}.$$

Substituting in (10) we have for any $m \in \mathbf{N}$

$$\varepsilon \leqslant \frac{(1+\delta)^{2m}}{\sqrt{\pi m}}\, \mathrm{e}^{\frac{1}{8m}}.$$

Finally, for $\delta > 0$ we put $m = 1 + \left\lfloor \dfrac{1}{4\log(1+\delta)} \right\rfloor$ and the desired estimate (1) follows from the above. $\square$

4. A characterization of two-dimensional subspaces of Banach algebras containing the unit. The problem of $\delta(\varepsilon)$ estimate, studied in the previous two sections, can be considered as a particular case of the following problem: what can be said about the geometry of 2-dimensional real subspaces of a Banach algebra, which contain the unit of the algebra? The theorem 4.2 below says that the approach through polynomial equalities and inequalities, used in the previous section, in principle can give all the information about such subspaces.

Definition 4.1. *A pair (X, e), where X is a two-dimensional normed space and $\|e\| = 1$ is called a chest. A chest (X, e) is said to be algebraic if there is a linear isometric embedding of X into a Banach algebra, which maps e into the unit element.*

Theorem 4.2. *A chest (X, e) is algebraic if and only if for every $N \in \mathbf{N}$, for every collection of naturals $\{n_j\}_{j=1}^{N}$ and for all collections of scalars $\{a_{k,j}\}_{k=1}^{n_j}$, $\{b_{k,j}\}_{k=1}^{n_j}$ if the polynomial equality*

$$t \equiv \sum_{j=1}^{N} \prod_{k=1}^{n_j} (a_{k,j} + b_{k,j}\, t) \tag{12}$$

takes place, then the following estimate has to be true for all $x \in X$:

$$\|x\| \leqslant \sum_{j=1}^{N} \prod_{k=1}^{n_j} \|a_{k,j}\, e + b_{k,j}\, x\|. \tag{13}$$

P r o o f. Necessity of the condition evidently follows from the triangle inequality and the multiplicative triangle inequality for the norm of a Banach algebra. Let us prove the sufficiency. Fix any $x_0 \in S_X \setminus \mathrm{Lin}\{e\}$. Consider the algebra $\mathcal{P}$ of all polynomials and introduce the following seminorm q on $\mathcal{P}$: for every $p \in \mathcal{P}$ put

$$q(p) = \inf \left\{ \sum_{j=1}^{N} \prod_{k=1}^{n_j} \big\|a_{k,j}\, e + b_{k,j}\, x_0\big\| \;:\; p(t) \equiv \sum_{j=1}^{N} \prod_{k=1}^{n_j} \big(a_{k,j} + b_{k,j}\, t\big) \right\}.$$

For every $x \in X$, $x = a\,e + b\,x_0$, define $Tx := a + bt$. It is easy to check that T embeds X into $(\mathcal{P}, q)$ isometrically, and $Te = \mathbf{1}$. Evidently the seminorm q is algebraic, so its kernel Y is an ideal, and we can quotient out this kernel to get a normed algebra $A = \mathcal{P}/Y$. The norms of equivalence classes are the same as seminorms of their representatives, so the map $x \longmapsto Tx$ is still an isometry. $\square$

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