# ON THE POLYNOMIAL NUMERICAL INDEX OF THE REAL SPACES $c_0$, $\ell_1$ AND $\ell_\infty$


SUNG GUEN KIM,

Department of Mathematics
Kyungpook National University
Daegu 702-701, South Korea
*E-mail:* sgk317@knu.ac.kr

MIGUEL MARTÍN, AND JAVIER MERÍ

Departamento de Análisis Matemático
Facultad de Ciencias
Universidad de Granada
18071 Granada, Spain
*E-mail:* mmartins@ugr.es, jmeri@ugr.es



ABSTRACT. We give a lower bound for the polynomial numerical index of order $k$ for real lush spaces. We use this bound to compute the polynomial numerical index of order 2 of the real spaces $c_0$, $\ell_1$ and $\ell_\infty$. Finally, we present an example of a real Banach space $X$ whose polynomial numerical indices are positive while the ones of its bidual are zero.


## 1. INTRODUCTION

The polynomial numerical indices of a Banach space are constants relating the norm and the numerical radius of homogeneous polynomials on the space. Let us present the relevant definitions. For a Banach space $X$, we write $B_X$ for the closed unit ball, $S_X$ for the unit sphere, $X^*$ for the dual space, and $\Pi(X)$ for the subset of $X \times X^*$ given by

$$\Pi(X) = \{(x, x^*) \in S_X \times S_{X^*} \ : \ x^*(x) = 1\} .$$

For $k \in \mathbb{N}$ we denote by $\mathcal{P}\left({}^k X; X\right)$ the space of all $k$-homogeneous polynomials from $X$ into $X$ endowed with the norm

$$\|P\| = \sup\{\|P(x)\| \ : \ x \in B_X\}.$$

Given $P \in \mathcal{P}\left({}^k X; X\right)$, the *numerical range* of $P$ is the subset of the scalar field given by

$$V(P) = \{x^*(Px) \ : \ (x, x^*) \in \Pi(X)\},$$

and the *numerical radius* of $P$ is

$$v(P) = \sup\{|x^*(Px)| \ : \ (x, x^*) \in \Pi(X)\}.$$

The concept of numerical range was originally introduced in the sixties for linear operators by F. Bauer and G. Lumer, extending the classical Toeplitz's field of values of a matrix. Still in the sixties, F. Bonsall, B. Cain, and H. Schneider extended the numerical range to arbitrary continuous functions from the unit sphere of a real or complex Banach space into the space. We refer the reader to the books [2, 3] for background. In the seventies, L. Harris [9] studied numerical ranges for holomorphic functions and polynomials on complex Banach spaces. Since then, a deep interest has been posed on the study of the numerical range and the numerical radius of homogeneous


*Date*: February 21st, 2007.

polynomials on Banach spaces, particularly in questions related to numerical radius attaining polynomials (see [1] and references therein, for instance).

Recently, Y.S. Choi, D. García, S.G. Kim and M. Maestre [5] have introduced the *polynomial numerical index of order k* of a Banach space $X$ as the constant $n^{(k)}(X)$ defined by

$$\begin{aligned} n^{(k)}(X) &= \max\left\{c \geqslant 0 \ : \ c\,\|P\| \leqslant v(P) \ \ \forall\, P \in \mathcal{P}\left({}^k X; X\right)\right\} \\ &= \inf\left\{v(P) \ : \ P \in \mathcal{P}\left({}^k X; X\right),\ \|P\| = 1\right\} \end{aligned}$$

for every $k \in \mathbb{N}$. This concept is a generalization of the *numerical index* of a Banach space (recovered for $k = 1$) which was first suggested by G. Lumer in 1968. At that time, it was known that a Hilbert space of dimension greater than 1 has numerical index $1/2$ in the complex case, and 0 in the real case. Two years later, J. Duncan, C. McGregor, J. Pryce, and A. White proved that $L$-spaces and $M$-spaces have numerical index 1. They also determined the range of values of the numerical index proving that

$$\begin{aligned} \{n(X) \ : \ X \text{ complex Banach space }\} &= [\mathrm{e}^{-1}, 1], \\ \{n(X) \ : \ X \text{ real Banach space }\} &= [0, 1]. \end{aligned}$$

The remarkable result that $n(X) \geqslant 1/\mathrm{e}$ for every complex Banach space $X$ goes back to H. Bohnenblust and S. Karlin. The disk algebra and the finite codimensional subspaces of $C[0,1]$ are more examples of Banach spaces with numerical index 1. The computation of the numerical index of those two-dimensional real normed spaces whose unit balls are regular polygons has been recently done. Specially interesting values of the numerical index of a Banach space are 0 and 1, since the spaces having these values of the numerical index can be characterized in the finite-dimensional case. For background, more information and recent results on numerical index, we refer the reader to the recent expository paper [10] and references therein.

Contrary to the linear case, the general theory of polynomial numerical indices is still in its infancy. However, in the already mentioned paper [5] and in [6] some results are given. Let us present some of them. The easiest examples are $n^{(k)}(\mathbb{R}) = 1$ and $n^{(k)}(\mathbb{C}) = 1$. For every complex Banach space $X$ and every $k \geqslant 2$, one has $n^{(k)}(X) \geqslant \exp\left(\frac{k\log(k)}{1-k}\right)$. This is not true in the real case, since real Hilbert spaces of dimension greater than one have polynomial numerical index of order $k$ equal to 0 for every $k$. For every complex Banach space $X$ such that $X^*$ is isometrically isomorphic to an $L_1(\mu)$ space, $n^{(k)}(X) = 1$ for every $k \in \mathbb{N}$; in particular, this happens for complex $C(K)$ spaces. This result does not hold in the real case; for instance, the polynomial numerical indices of order 2 of the real spaces $c_0$, $c$, $\ell_\infty$ are all smaller than $1/2$. Also, $n^{(2)}(\ell_1) \leqslant 1/2$ both in the real and in the complex cases. Let us comment that the above examples show that the polynomial numerical indices distinguish between $L$-spaces and $M$-spaces in the complex case, and this is not possible if we only use the usual (linear) numerical index. The first author [11] defined two more kinds of numerical indices, namely, the *multilinear numerical index of order k of X* by

$$n_m^{(k)}(X) = \inf\{v(A) \ : \ A \in L({}^k X; X),\ \|A\| = 1\}$$

and the *symmetric multilinear numerical index of order k of X* by

$$n_s^{(k)}(X) = \inf\{v(A) \ : \ A \in L_s({}^k X; X),\ \|A\| = 1\},$$

where $L_s({}^k X; X)$ is the subspace of all symmetric continuous $k$-linear mappings in $L({}^k X; X)$. Clearly $0 \leqslant n_m^{(k)}(X) \leqslant 1, 0 \leqslant n_s^{(k)}(X) \leqslant 1$. The relationship between these two indices and the polynomial numerical indices is studied in the cited paper. Also, in another recent paper, the first author [12] gives some results concerning the polynomial numerical indices for $L_p$ spaces which generalize the corresponding ones for the (linear) numerical index (see [8]). Namely, for $1 < p < \infty$ and $k \in \mathbb{N}$, one has $n^{(k)}(\ell_p) = \inf\{n^{(k)}(\ell_p^m) \ : \ m \in \mathbb{N}\}$, that the sequence

$\{n^{(k)}(\ell_p^m)\}_{m\in\mathbb{N}}$ is decreasing, that $\inf\{n^{(k)}(\ell_p) : k \in \mathbb{N}\} = \inf\{n^{(k)}(\ell_p^m) : k \in \mathbb{N}\} = 0$ for every $m \geqslant 2$ and, finally, that $n^{(k)}(L_p(\mu)) \geqslant n^{(k)}(\ell_p)$ for every positive measure $\mu$. A very curious result has been very recently given by H. J. Lee [13, Theorem 2.7]: the only finite-dimensional real Banach space $X$ with $n^{(2)}(X) = 1$ is $X = \mathbb{R}$.

Other results dealing with polynomial numerical indices which will be interesting to our discussion are the following. For every real or complex Banach space $X$ and every $k \in \mathbb{N}$, one has

$$n^{(k+1)}(X) \leqslant n^{(k)}(X) \qquad \text{and} \qquad n^{(k)}(X^{**}) \leqslant n^{(k)}(X).$$

The first inequality may be strict, just take $X = \ell_1$ and $k = 1$. It has been proved very recently that the second inequality may be also strict for $k = 1$ [4]. We will show that the same is true for every $k \geqslant 2$ (Example 2.6). Finally, the polynomial numerical indices of an $L$-summand or $M$-summand of a Banach space are greater or equal than the corresponding ones of the whole space.

As shown by the above paragraphs, there are few Banach spaces for which the polynomial numerical indices are known and, in the real case, the lack of examples is even bigger. The aim of this paper is to give a tool to estimate the numerical index of higher order of many real spaces like $c_0$, $\ell_1$, and $\ell_\infty$. The idea is to estimate the polynomial numerical indices of the so-called lush spaces, a wide class of Banach spaces having linear numerical index 1. Following [4], we say that a real Banach space $X$ is *lush* if for every $x, y \in S_X$ and every $\varepsilon > 0$, there exists $x^* \in S_{X^*}$ such that $y \in S(B_X, x^*, \varepsilon)$ and

$$\operatorname{dist}\left(x, \operatorname{co}\left(S(B_X, x^*, \varepsilon) \cup -S(B_X, x^*, \varepsilon)\right)\right) < \varepsilon$$

where $S(B_X, x^*, \varepsilon) = \{x \in B_X : \operatorname{Re} x^*(x) > 1 - \varepsilon\}$ is a *slice* of the unit ball of $X$. This property is the weakest known geometric property implying that the space has numerical index 1. For instance, $C(K)$ spaces, $L_1(\mu)$ spaces and their isometric preduals are lush, and the same is true for finite-dimensional subspaces of $C[0,1]$.

We give a lower bound for the numerical indices of higher order of real lush spaces. In particular, we show that $n^{(2)}(X) \geqslant \frac{1}{2}$ for every real lush space $X$ and that this inequality is sharp. Indeed, we will prove that

$$n^{(2)}(c_0) = n^{(2)}(\ell_1) = n^{(2)}(\ell_\infty) = \frac{1}{2}.$$

Also, we give an example of a real Banach space $X$ such that

$$n^{(k)}(X) > 0 \quad \text{and} \quad n^{(k)}(X^{**}) = 0$$

for every $k \in \mathbb{N}$.

## 2. The results

We are ready to state and prove the main result of the paper, which gives a lower bound for the polynomial numerical indices of lush spaces.

**Theorem 2.1.** *Let $X$ be a real lush Banach space. Then, for $k \geqslant 1$ we have*

$$n^{(k)}(X) \geqslant \frac{2^k}{2 + M_k(2^k - 2)},$$

*where* $M_k = \displaystyle\sum_{j=1}^{k} \frac{j^k}{j!\,(k-j)!}$.

In order to prove the above result we will need a couple of lemmas. The first one is an almost immediate consequence of the binomial theorem. The second follows from the Bishop-Phelps-Bollobás Theorem.

**Lemma 2.2** ([11]). *Let $M > 1$ and $k \geqslant 1$. Then,*

$$\max_{t\in[0,1]} \left[ t^k + (1-t)^k + M \sum_{j=1}^{k-1} \binom{k}{j} t^j (1-t)^{k-j} \right] = \frac{2 + M(2^k - 2)}{2^k}.$$

We recall that for a real Banach space $X$ and $P \in \mathcal{P}\left({}^k X; X\right)$, $\check{P}$ denotes the unique continuous symmetric $k$-linear map from $X^k$ into $X$ associated with $P$. Given $x, y \in X$ and $\ell \in \{1, \dots, k\}$ we write

$$\check{P}(x^\ell, y^{k-\ell}) = \check{P}(x, \overset{(\ell)}{\dots}, x, y, \overset{(k-\ell)}{\dots}, y)$$

and in the extreme case in which $\ell = k$, we understand that $\check{P}(x^k, y^0) = P(x)$.

**Lemma 2.3.** *Let $k \geqslant 1$, $X$ a Banach space, $P \in \mathcal{P}\left({}^k X; X\right)$ with $\|P\| \leqslant 1$, and $\varepsilon > 0$. Then, for every $x^* \in S_{X^*}$ and $x \in X$ satisfying $x \in S\left(B_X, x^*, \frac{\varepsilon^2}{4}\right)$ the following holds*

$$|x^*(P(x))| \leqslant v(P) + \varepsilon + k\,\|\check{P}\|\,\varepsilon.$$

*Proof.* Under the hypothesis we can apply Bishop-Phelps-Bollobás Theorem (see [3, §16], for instance) to find $(y, y^*) \in \Pi(X)$ so that

$$\|x - y\| < \varepsilon \qquad \text{and} \qquad \|x^* - y^*\| < \varepsilon.$$

Therefore, we have

$$\begin{aligned} |x^*(P(x))| &\leqslant |y^*(P(y))| + |x^*(P(y)) - y^*(P(y))| + |x^*(P(x)) - x^*(P(y))| \\ &\leqslant v(P) + \varepsilon + \|P(x) - P(y)\| \\ &\leqslant v(P) + \varepsilon + \sum_{j=1}^{k} \left\|\check{P}\big(x^{k-j+1}, y^{j-1}\big) - \check{P}\big(x^{k-j}, y^j\big)\right\| \\ &\leqslant v(P) + \varepsilon + k\|\check{P}\|\,\varepsilon. \end{aligned}$$

□

*Proof of Theorem 2.1.* For $P \in \mathcal{P}\left({}^k X; X\right)$ with $\|P\| = 1$ and $0 < \varepsilon < 1$ fixed, we take $x_0 \in S_X$ such that $\|P(x_0)\| > 1 - \varepsilon$, and we apply the definition of lushness to $x_0$ and $\frac{P(x_0)}{\|P(x_0)\|}$ to find $x^* \in S_{X^*}$ with $\frac{P(x_0)}{\|P(x_0)\|} \in S\left(B_X, x^*, \frac{\varepsilon^2}{4}\right)$, $\lambda \in [0,1]$, and $x_1, x_2 \in S\left(B_X, x^*, \frac{\varepsilon^2}{4}\right)$ so that

$$\|x_0 - (\lambda x_1 - (1-\lambda)x_2)\| < \frac{\varepsilon^2}{4}.$$

From this it is easy to deduce that

$$\|P(x_0) - P(\lambda x_1 - (1-\lambda)x_2)\| < k\,\|\check{P}\|\,\frac{\varepsilon^2}{4}$$

and, therefore,

$$\begin{aligned} (1) \qquad \left|x^*\big(P(\lambda x_1 - (1-\lambda)x_2)\big)\right| &\geqslant |x^*(P(x_0))| - \left|x^*(P(x_0)) - x^*\big(P(\lambda x_1 - (1-\lambda)x_2)\big)\right| \\ &\geqslant \left(1 - \frac{\varepsilon^2}{4}\right)(1-\varepsilon) - k\,\|\check{P}\|\,\frac{\varepsilon^2}{4}. \end{aligned}$$

On the other hand, we have that

$$
\begin{aligned}
(2)\quad & \big|x^*\big(P(\lambda x_1-(1-\lambda)x_2)\big)\big| \\
&= \left|\lambda^k x^*(P(x_1)) + (1-\lambda)^k x^*(P(-x_2)) + \sum_{\ell=1}^{k-1}\binom{k}{\ell}\lambda^\ell(1-\lambda)^{k-\ell}x^*\left(\check{P}(x_1^\ell,(-x_2)^{k-\ell})\right)\right| \\
&\leqslant \lambda^k\big|x^*(P(x_1))\big| + (1-\lambda)^k\big|x^*(P(x_2))\big| + \sum_{\ell=1}^{k-1}\binom{k}{\ell}\lambda^\ell(1-\lambda)^{k-\ell}\left|x^*\left(\check{P}(x_1^\ell,x_2^{k-\ell})\right)\right|
\end{aligned}
$$

and, using Lemma 2.3,

$$
\leqslant \big(\lambda^k+(1-\lambda)^k\big)\big(v(P)+\varepsilon+k\|\check{P}\|\,\varepsilon\big) + \sum_{\ell=1}^{k-1}\binom{k}{\ell}\lambda^\ell(1-\lambda)^{k-\ell}\left|x^*\left(\check{P}(x_1^\ell,x_2^{k-\ell})\right)\right|.
$$

Our next goal is to estimate $\big|x^*(\check{P}(x_1^\ell,x_2^{k-\ell}))\big|$ for $\ell\in\{1,\dots,k-1\}$. To do so, we write $B=\{1,\dots,k\}$, $y_i=x_1$ for $i\in\{1,\dots,\ell\}$, and $y_i=x_2$ for $i\in\{\ell+1,\dots,k\}$, and we use [7, Lemma 3.4] to get

$$
\begin{aligned}
& k!\big|x^*(\check{P}(x_1^\ell,x_2^{k-\ell}))\big| \\
&\quad\leqslant \big|\,x^*(P(\ell x_1+(k-\ell)x_2))\big| + \sum_{\{i_1,\dots,i_{k-1}\}\subset B}\big|\,x^*(P(y_{i_1}+\cdots+y_{i_{k-1}}))\big| \\
&\qquad + \sum_{\{i_1,\dots,i_{k-2}\}\subset B}\big|\,x^*(P(y_{i_1}+\cdots+y_{i_{k-2}}))\big| + \cdots + \ell|x^*(P(x_1))| + (k-\ell)|x^*(P(x_2))| \\
&\quad\leqslant k^k\left|x^*\left(P\left(\tfrac{\ell x_1+(k-\ell)x_2}{k}\right)\right)\right| + (k-1)^k\sum_{\{i_1,\dots,i_{k-1}\}\subset B}\left|x^*\left(P\left(\tfrac{y_{i_1}+\cdots+y_{i_{k-1}}}{k-1}\right)\right)\right| \\
&\qquad + (k-2)^k\sum_{\{i_1,\dots,i_{k-2}\}\subset B}\left|x^*\left(P\left(\tfrac{y_{i_1}+\cdots+y_{i_{k-2}}}{k-2}\right)\right)\right| \\
&\qquad + \cdots + \ell|x^*(P(x_1))| + (k-\ell)|x^*(P(x_2))|\,.
\end{aligned}
$$

Since $x_1,x_2\in S\left(B_X,x^*,\frac{\varepsilon^2}{4}\right)$, so does $\mu x_1+(1-\mu)x_2$ for every $\mu\in[0,1]$. Therefore, we can use Lemma 2.3 a number of times to obtain

$$
\begin{aligned}
& k!\big|x^*(\check{P}(x_1^\ell,x_2^{k-\ell}))\big| \\
&\quad\leqslant k^k\big(v(P)+\varepsilon+k\|\check{P}\|\,\varepsilon\big) + (k-1)^k\binom{k}{k-1}\big(v(P)+\varepsilon+k\|\check{P}\|\,\varepsilon\big) \\
&\qquad + (k-2)^k\binom{k}{k-2}\big(v(P)+\varepsilon+k\|\check{P}\|\,\varepsilon\big) + \cdots + k\big(v(P)+\varepsilon+k\|\check{P}\|\,\varepsilon\big) \\
&\quad= \sum_{j=1}^{k} j^k\frac{k!}{j!\,(k-j)!}\big(v(P)+\varepsilon+k\|\check{P}\|\,\varepsilon\big)
\end{aligned}
$$

from which we deduce that

$$
\big|x^*(\check{P}(x_1^\ell,x_2^{k-\ell}))\big| \leqslant M_k\big(v(P)+\varepsilon+k\|\check{P}\|\,\varepsilon\big).
$$

This, together with (2), tells us that

$$
\begin{aligned}
\big|x^*\big(P(\lambda x_1-(1-\lambda)x_2)\big)\big| &\leqslant \big(\lambda^k+(1-\lambda)^k\big)\big(v(P)+\varepsilon+k\|\check{P}\|\,\varepsilon\big) \\
&\qquad + M_k\big(v(P)+\varepsilon+k\|\check{P}\|\,\varepsilon\big)\sum_{\ell=1}^{k-1}\binom{k}{\ell}\lambda^\ell(1-\lambda)^{k-\ell},
\end{aligned}
$$

and Lemma 2.2 yields

$$\left|x^*\big(P(\lambda x_1-(1-\lambda)x_2)\big)\right| \leqslant \frac{2+M_k(2^k-2)}{2^k}\,\big(v(P)+\varepsilon+k\|\check{P}\|\,\varepsilon\big).$$

Finally, using (1), we obtain

$$\left(1-\frac{\varepsilon^2}{4}\right)(1-\varepsilon)-k\|\check{P}\|\,\frac{\varepsilon^2}{4} \leqslant \frac{2+M_k(2^k-2)}{2^k}\big(v(P)+\varepsilon+k\|\check{P}\|\,\varepsilon\big)$$

and the arbitrariness of $\varepsilon$ allows us to write

$$\frac{2^k}{2+M_k(2^k-2)} \leqslant v(P),$$

which finishes the proof. □

For $k=1$ the above result shows that $n(X)=1$ for lush spaces, which is [4, Proposition 2.2]. Now, we would like to particularize Theorem 2.1 to some classes of lush spaces. Let us give the necessary definitions. A real Banach space $X$ is said to be a *CL-space* if $B_X$ is the absolutely closed convex hull of every maximal proper face of $S_X$. Examples of real CL-spaces are $L_1(\mu)$ for an arbitrary measure $\mu$, and its isometric preduals, in particular $C(K)$, where $K$ is a compact Hausdorff space. We refer to [14, §3], [15], and [17] for more information and background. Real CL-spaces are clearly lush, but the reverse result is not true [4, Example 3.4]. Let us comment that the proof of the above theorem simplifies is it is done only for CL-spaces. Another family of lush spaces is the one of C-rich subspaces of $C(K)$ spaces. Following [4], we say that a closed subspace $X$ of a $C(K)$ space is *C-rich* if for every nonempty open subset $U$ of $K$ and every $\varepsilon>0$, there is a positive function $h$ of norm 1 with support inside $U$ such that the distance from $h$ to $X$ is less than $\varepsilon$. Examples of C-rich subspaces are the finite-codimensional subspaces of $C(K)$ when $K$ is perfect.

**Corollary 2.4.** *Let $X$ be a real CL-space or a C-rich subspace of a $C(K)$ space. Then, for $k\geqslant 2$ we have*

$$n^{(k)}(X) \geqslant \frac{2^k}{2+M_k(2^k-2)}\,,$$

*where $M_k=\displaystyle\sum_{j=1}^{k}\frac{j^k}{j!\,(k-j)!}$. In particular, this applies to $L_1(\mu)$ spaces and their isometric preduals, in particular to $C(K)$ spaces, and to finite-codimensional subspaces of $C(K)$.*

The above corollary applies to the real spaces $\ell_\infty^m$, $\ell_1^m$ ($m\geqslant 2$), $c_0$, $\ell_1$, and $\ell_\infty$. Actually, for these spaces an upper bound for their polynomial numerical indices can be given. In particular, the second order polynomial numerical index of those spaces is calculated.

**Corollary 2.5.** *Let $X$ denote any of the real spaces $\ell_\infty^m$, $\ell_1^m$ ($m\geqslant 2$), $c_0$, $\ell_1$, and $\ell_\infty$. Then,*

$$n^{(2)}(X)=\frac{1}{2} \qquad \text{and} \qquad \frac{2^k}{2+M_k(2^k-2)} \leqslant n^{(k)}(X) \leqslant \frac{2}{k}\left(\frac{k-2}{k}\right)^{\frac{k-2}{2}} \qquad (k\geqslant 3).$$

*Proof.* The inequality

$$\frac{2^k}{2+M_k(2^k-2)} \leqslant n^{(k)}(X) \qquad (k\geqslant 2).$$

follows from Corollary 2.4. When $k=2$, we get $M_2=3$ so we have that $n^{(2)}(X)\geqslant\frac{1}{2}$. To prove the reverse inequality, we observe that $\ell_\infty^2\equiv\ell_1^2$ is either a $L$-summand or a $M$-summand of $X$.

Thus, in view of [5, Proposition 2.8] it suffices to show that $n^{(k)}(\ell_1^2) \leqslant \frac{1}{2}$, which is clear since the polynomial $P \in \mathcal{P}\left({}^k\ell_1^2; \ell_1^2\right)$ (shown in [7, pp. 141]) given by

$$P(x,y) = \left(\frac{1}{2}x^2 + 2xy, -\frac{1}{2}y^2 - xy\right) \qquad \left((x,y) \in \ell_1^2\right)$$

satisfies that $\|P\| = 1$ and $v(P) \leqslant \frac{1}{2}$.

To get the upper bound for $k \geqslant 3$, we consider the polynomial $P \in \mathcal{P}\left({}^k\ell_\infty^2; \ell_\infty^2\right)$ given by

$$P(x,y) = (x^2y^{k-2} - y^k, 0) \qquad \left((x,y) \in \ell_\infty^2\right).$$

On the one hand, it is straightforward to show that $\|P\| = 1$. On the other hand, by just using the definition of numerical radius and the fact that $P$ is homogeneous, we have

$$\begin{aligned} v(P) &= \max\left\{ \sup_{t\in[-1,1]} |(1,0)P(1,t)|, \sup_{t\in[-1,1]} |(0,1)P(t,1)|, \sup_{s\in[0,1]} |(s,1-s)P(1,1)| \right\} \\ &= \sup_{t\in[-1,1]} |(1,0)P(1,t)| = \sup_{t\in[-1,1]} |t^{k-2} - t^k| = \frac{2}{k}\left(\frac{k-2}{k}\right)^{\frac{k-2}{2}}. \end{aligned}$$

□

As we mentioned in the introduction, it is proved in [5, Corollary 2.15] that $n^{(k)}(X^{**}) \leqslant n^{(k)}(X)$ for every $k \in \mathbb{N}$. The next example shows that this inequality can be strict.

**Example 2.6.** *There exists a real Banach space $X$ such that*

$$n^{(k)}(X) > 0 \quad \textit{and} \quad n^{(k)}(X^{**}) = 0$$

*for every* $k \in \mathbb{N}$. Indeed, given a separable Banach space $E$, it is constructed in [16, Theorem 3.3] a C-rich subspace $X(E)$ of $C[0,1]$ such that $E^*$ is an $L$-summand of $X(E)^*$. Let us consider the above space for $E = \ell_2$. On the one hand, since $X(\ell_2)$ is a C-rich subspace of $C(K)$, Corollary 2.4 gives that

$$n^{(k)}\big(X(\ell_2)\big) > 0$$

for every $k \in \mathbb{N}$. On the other hand, $X(\ell_2)^{**} = \ell_2 \oplus_\infty Z$ for a suitable space $Z$ and, by [5, Proposition 2.8], we get

$$n^{(k)}\big(X(\ell_2)^{**}\big) \leqslant n^{(k)}(\ell_2) = 0$$

for every $k \in \mathbb{N}$.

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