THE POLYNOMIAL NUMERICAL INDEX FOR SOME COMPLEX VECTOR-VALUED FUNCTION SPACES

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Abstract. We study the relation between the polynomial numerical indices of a complex vector-valued function space and the ones of its range space. It is proved that the spaces $C(K,X)$, and $L_\infty(\mu,X)$ have the same polynomial numerical index as the complex Banach space X for every compact Hausdorff space K and every σ-finite measure μ, which does not hold any more in the real case. We give an example of a complex Banach space X such that, for every $k \geqslant 2$, the polynomial numerical index of order k of X is the greatest possible, namely 1, while the one of X^{**} is the least possible, namely $k^{\frac{k}{1-k}}$. We also give new examples of Banach spaces with the polynomial Daugavet property, namely $L_\infty(\mu,X)$ when μ is atomless, and $C_w(K,X)$, $C_{w^*}(K,X^*)$ when K is perfect.

1. Introduction and Preliminaries

The concept of the numerical range was first considered by O. Toeplitz in 1918 for matrices. In the 1960's, F. Bauer and G. Lumer gave a theory of the numerical range of bounded linear operators on Banach spaces, extended to arbitrary continuous functions from the unit sphere of a Banach space into the space by F. Bonsall, B. Cain, and H. Schneider. We refer the reader to the 1970's monographs by F. Bonsall and J. Duncan [4, 5] for background.

Given a Banach space X over $\mathbb{K}$ ($=\mathbb{R}$ or $\mathbb{C}$), by B_X we denote the closed unit ball and by S_X the unit sphere of X. We use the notation $\mathbb{T} = S_{\mathbb{K}}$ and $\mathbb{D} = B_{\mathbb{K}}$. By $\Pi(X)$ we will denote the set

$$\Pi(X) = \{(x,x^*) \ : \ x \in S_X,\ x^* \in S_{X^*},\ \text{and } x^*(x) = 1\}.$$

If $f : S_X \longrightarrow X$ is a bounded function, the *numerical range of* f is the set of scalars

$$V(f) = \{x^*(f(x)) : (x,x^*) \in \Pi(X)\}.$$

and the *numerical radius of* f is $v(f) = \sup\{|x^*(f(x))| : (x,x^*) \in \Pi(X)\}$.

The notion of the *numerical index* of a Banach space was first introduced by G. Lumer in 1968 (see [13]), and it is the greatest constant of equivalence between the numerical radius and the usual norm in the Banach algebra $L(X)$ of all bounded linear operators on the space. It was extended to the polynomial case in [7] by considering homogeneous polynomials instead of linear operators. That is, for $k \in \mathbb{N}$, we define

$$n^{(k)}(X) = \inf\{v(P) \ : \ P \in \mathcal{P}(^kX;X),\ \|P\| = 1\},$$

where $\mathcal{P}(^kX;X)$ is the space of all continuous k-homogeneous polynomials from X into X, and call it the *polynomial numerical index of order k of X*. For more information and background, we refer the reader to the recent papers [17] (a survey) and to [7, 9, 19, 20] and references therein.

Let us present some examples and known results about (linear) numerical index and polynomial numerical indices which will be relevant for our discussion. For the linear case, it is known that

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M-spaces and L-spaces have numerical index 1, a property shared by the disk algebra and by finite-codimensional subspaces of $C[0,1]$. The numerical index of a Hilbert space of dimension greater than one is $1/2$ in the complex case and 0 in the real case. Every complex Banach space X satisfies $n(X) \geqslant 1/\mathrm{e}$. The numerical index of the dual of a Banach space is smaller than or equal to the one of the space, and it has been recently shown that this inequality can be strict.

Let us go to the polynomial case. The easiest examples are $n^{(k)}(\mathbb{R}) = 1$ and $n^{(k)}(\mathbb{C}) = 1$ for every $k \in \mathbb{N}$. One has $n^{(k)}(C(K)) = 1$ for every $k \in \mathbb{N}$, while the situation for $L_1(\mu)$ spaces is different since, for instance, $n^{(2)}(\ell_1) \leqslant \frac{1}{2}$ in the complex case. This shows that the polynomial numerical index distinguishes between L-spaces and M-spaces in the complex case, and this is not possible with the (linear) numerical index. In the real case, the situation is even more different from the linear case. For instance, the only finite-dimensional real Banach space X with $n^{(2)}(X) = 1$ is $X = \mathbb{R}$. The real spaces c_0, ℓ_1 and ℓ_∞ have polynomial numerical index of order 2 equal to $1/2$. Other results dealing with polynomial numerical indices which will be interesting to our discussion are the following. For every real or complex Banach space X and every $P \in \mathcal{P}(X;X)$, one has $\|P\| = \|\overline{P}\|$ and $v(P) = v(\overline{P})$, where $\mathcal{P}(X;X)$ is the space of all finite sums of homogeneous polynomials on X and $\overline{P} \in \mathcal{P}(X^{**};X^{**})$ is the Aron-Berner extension of P (see [8, Theorem 3.1]). It follows that $n^{(k)}(X^{**}) \leqslant n^{(k)}(X)$ for every $k \in \mathbb{N}$. The inequality $n^{(k+1)}(X) \leqslant n^{(k)}(X)$ holds for every real or complex Banach space X and every $k \in \mathbb{N}$. This gives, in particular, that $n^{(k)}(H) = 0$ for every $k \in \mathbb{N}$ in the real case. Finally, it follows from an old result by Harris [15, Theorem 1] that $1 \geqslant n^{(k)}(X) \geqslant k^{\frac{k}{1-k}}$ for every complex Banach space X and every $k \geqslant 2$.

Given a set Γ and a Banach space X, we write $B(\Gamma, X)$ to denote the Banach space of all bounded functions from Γ into X, endowed with the supremum norm. We will also consider some of its closed subspaces. Given a compact Hausdorff space K, we denote by $C(K,X)$ (resp. $C_w(K,X)$) the space of all continuous function from K into X (resp. weakly continuous functions), and $C_{w^*}(K,X^*)$ is the space of all weakly* continuous functions from K into X^*. For a locally compact Hausdorff space L, we denote by $C_0(L,X)$ the Banach space of all continuous functions from L into X vanishing at infinity. Finally, for a completely regular space Ω, we write $C_b(\Omega, X)$ for the Banach space of all bounded continuous functions from Ω into X. If Z is any of the above subspaces of $B(\Gamma, X)$ and $t \in \Gamma$, we write $\delta_t(f) = f(t)$ for every $f \in Z$. Also, given a σ-finite measure space (Ω, Σ, μ), we denote by $L_\infty(\mu, X)$ the Banach space of all (equivalence classes of) essentially bounded Böchner-measurable functions from Ω into X, endowed with the essential supremum norm.

We can see in [21, 24, 25] that the numerical index of a real or complex Banach space X is well preserved in some of the above mentioned X-valued function spaces, concretely

$$n\big(C_w(K,X)\big) = n\big(C(K,X)\big) = n\big(L_\infty(\mu,X)\big) = n(X).$$

The proofs depend upon some convexity arguments, which causes difficulties in extending those to the polynomial numerical index. Actually, in the real case, they are not true (for instance, $n^{(2)}(\mathbb{R}) = 1$ while $n^{(2)}(c_0) = n^{(2)}(c) = n^{(2)}(\ell_\infty) = 1/2$). However, as we will see in this paper, in the complex case we may replace the convexity arguments by the maximum modulus theorem to extend some of the above equalities to the polynomial setting. It is also true that $n\big(L_1(\mu,X)\big) = n(X)$, where $L_1(\mu, X)$ is the space of all (equivalent classes of) Böchner-integrable functions from a σ-finite measure space (Ω, Σ, μ) into a Banach space X. This result does not pass to the polynomial case, since $n^{(2)}(\ell_1) \leqslant 1/2$ in both the real and the complex case, while $n^{(2)}(\mathbb{K}) = 1$.

The last part of the paper is devoted to a concept related to the numerical range, the so-called Daugavet equation. In 1963, I. Daugavet [10] showed that every compact linear operator T on $C[0,1]$ satisfies $\|\mathrm{Id} + T\| = 1 + \|T\|$, a norm equality which has currently become known as the Daugavet equation. Over the years, the validity of the above equality has been established for many classes of operators on many Banach spaces. For instance, weakly compact linear operators on $C(K)$, K perfect, and $L_1(\mu)$, μ atomless, satisfy Daugavet equation (see [27] for an elementary approach). We refer the reader to the books [1, 2] and the papers [18, 28] for more information and background. Very recently [9], the study of the Daugavet equation has been extended to polynomials and, more generally, to bounded functions from the unit sphere of a Banach space into the space. Let us recall the relevant definitions. Let X be a real or complex Banach space. A function $\Phi \in B(B_X, X)$ is said to satisfy the *Daugavet equation* if the norm equality

$$\|\mathrm{Id} + \Phi\| = 1 + \|\Phi\| \tag{DE}$$

holds. If this happens for all weakly compact polynomials, we say that X has the *polynomial Daugavet property*. In the complex case, this implies that every weakly compact $\Phi \in \mathcal{A}_u(B_X, X)$ satisfies the Daugavet equation, where $\mathcal{A}_u(B_X, X)$ stands the Banach space of all uniformly continuous mappings from B_X into X which are holomorphic on the open unit ball, endowed with the supremum norm. Examples of Banach spaces having the polynomial Daugavet property are the real or complex spaces $C_b(\Omega, X)$ when Ω is perfect (X any Banach space) and their finite-codimensional subspaces.

The paper is organized as follows. In section 2 we show that for every $k \in \mathbb{N}$ the polynomial numerical index of order k of a c_0- or ℓ_∞-sum of complex Banach spaces is equal to the infimum of those indices of the summands. In sections 3 and 4, we study the polynomial numerical index for the complex spaces $C(K, X)$, $C_w(K, X)$, $C_0(L, X)$, $C_b(\Omega, X)$ and $L_\infty(\mu, X)$. Section 5 is devoted to give some remarks about the analytic numerical index, i.e. the best constant of equivalence between the numerical radius and the norm in the complex space $\mathcal{A}_u(B_X, X)$. Finally, in section 6 we give new examples of Banach spaces with the polynomial Daugavet property, namely $L_\infty(\mu, X)$ when μ is atomless, and $C_w(K, X)$, $C_{w^*}(K, X^*)$ when K is perfect, and we discuss its behavior with respect to c_0 and ℓ_∞ sums.

2. Discrete results

In [7, Proposition 2.8], it is shown that for every $k \in \mathbb{N}$ the polynomial numerical index of order k of a c_0-, ℓ_1-, or ℓ_∞-sum of real or complex Banach spaces is less than or equal to the infimum of those indices of the summands. This result is also true for ℓ_p-sums, and also for absolute sums. A closed subspace Y of a Banach space X is said to be an *absolute summand* of X, if there exists another closed subspace Z such that $X = Y \oplus Z$ and for every $y \in Y$ and $z \in Z$, the norm of $y + z$ depends only on $\|y\|$ and $\|z\|$. We also say that X is an *absolute sum* of Y and Z. The typical examples of absolute sums are the ℓ_p-sums of Banach spaces for $1 \leqslant p \leqslant \infty$. Given an arbitrary family $\{X_\lambda : \lambda \in \Lambda\}$ of Banach spaces and $1 \leqslant p \leqslant \infty$, we denote by $\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{\ell_p}$ the ℓ_p-sum of the family, and by $\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{c_0}$ the c_0-sum. In case where Λ has only two elements, we use the simpler notation $X \oplus_p Y$.

The proof of the following proposition is based on the fact that every polynomial on Y can be canonically extended to a polynomial on X with the same norm and same numerical radius. The proof for the cases $p = 1, \infty$ appears in [7, Proposition 2.8]. That proof is easy to extend to an arbitrary absolute sum.

Proposition 2.1. *Let X be a real or complex Banach space, and let Y be an absolute summand of X. Then, $n^{(k)}(X) \leqslant n^{(k)}(Y)$ for every $k \in \mathbb{N}$.*

In particular, each summand of a c_0- or ℓ_p-sum ($1 \leqslant p \leqslant \infty$) of Banach spaces is absolute, which gives the following inequalities.

Corollary 2.2. *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a non-empty family of real or complex Banach spaces. Then*

$$n^{(k)}\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{c_0}\Big) \leqslant \inf_\lambda n^{(k)}(X_\lambda), \qquad n^{(k)}\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{\ell_p}\Big) \leqslant \inf_\lambda n^{(k)}(X_\lambda)$$

for all $1 \leqslant p \leqslant \infty$ and every $k \in \mathbb{N}$.

For $1 < p < \infty$, the equality in the above inequalities does not hold in general. In fact, $n^{(k)}(\ell_p) \leqslant n(\ell_p) < 1$ while $n^{(k)}(\mathbb{K}) = 1$ for every $k \in \mathbb{N}$. For $p = 1$, the equality holds in the case $k = 1$ [24, Proposition 1], but does not hold for $k \geqslant 2$. Notice that $n^{(k)}(\ell_1) \leqslant n^{(2)}(\ell_1) \leqslant 1/2$ [19], while $n^{(k)}(\mathbb{K}) = 1$ for $k \geqslant 2$. However, we will show that the polynomial numerical index of a c_0- or ℓ_∞-sum of complex Banach spaces coincides with the infimum of those indices of the summands. In the linear case, this result appears in [24, Proposition 1], and our proof is based on it.

Proposition 2.3. *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a non-empty family of complex Banach spaces. Then*
$$n^{(k)}\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{c_0}\Big) = n^{(k)}\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{\ell_\infty}\Big) = \inf_\lambda n^{(k)}(X_\lambda),$$
for all $k \in \mathbb{N}$.

Proof. We will prove the ℓ_∞ case only. Let $X = \big[\bigoplus_{\lambda\in\Lambda} X_\lambda\big]_{\ell_\infty}$ and $P \in \mathcal{P}(^k X; X)$ with $\|P\| = 1$. We can write $P = (P_\lambda)_{\lambda\in\Lambda}$, where $P_\lambda \in \mathcal{P}(^k X; X_\lambda)$ for all $\lambda \in \Lambda$. Given $0 < \varepsilon < 1$ there exists $\lambda_0 \in \Lambda$ such that $\|P_{\lambda_0}\| > 1 - \varepsilon$. Now we write $X = X_{\lambda_0} \oplus_\infty Y$, where $Y = \Big[\bigoplus_{\lambda\neq\lambda_0} X_\lambda\Big]_{\ell_\infty}$. Choose $(x_0, y_0) \in S_X$ so that $x_0 \in X_{\lambda_0}$, $y_0 \in Y$ and $\|P_{\lambda_0}(x_0, y_0)\| > 1 - \varepsilon$. Let us show that we may suppose that $\|x_0\| = 1$. Indeed, we take $x_1 \in S_X$ such that $\|x_0\|\, x_1 = x_0$ and we take $x^*_{\lambda_0} \in S_{X^*_{\lambda_0}}$ such that
$$\big|x^*_{\lambda_0}(P_{\lambda_0}(x_0, y_0))\big| > 1 - \varepsilon.$$
Now, we define the entire function $g : \mathbb{C} \longrightarrow \mathbb{C}$ by
$$g(z) = x^*_{\lambda_0}\big(P_{\lambda_0}(zx_1, y_0)\big) \qquad \big(z \in \mathbb{C}\big).$$
By the maximum modulus theorem, we have
$$1 - \varepsilon < \big|x^*_{\lambda_0}(P_{\lambda_0}(x_0, y_0))\big| = |g(\|x_0\|)| \leqslant |g(z_0)|,$$
for some $z_0 \in \mathbb{C}$ with $|z_0| = 1$. So, replacing x_0 by $z_0 x_1$, the claim is done.

Next, we choose $x^*_0 \in X^*_{\lambda_0}$ so that $\|x^*_0\| = 1$ and $x^*_0(x_0) = 1$. We define $Q \in \mathcal{P}(^k X_{\lambda_0}; X_{\lambda_0})$ by
$$Q(u) = P_{\lambda_0}(u, x^*_0(u)y_0) \qquad \big(u \in X_{\lambda_0}\big).$$
We have
$$1 \geqslant \|Q\| \geqslant \|Q(x_0)\| = \|P_{\lambda_0}(x_0, y_0)\| > 1 - \varepsilon.$$
Hence we may find $(u_0, u^*_0) \in \Pi(X_{\lambda_0})$ such that
$$|u^*_0(Q(u_0))| > \Big(n^{(k)}(X_{\lambda_0}) - \varepsilon\Big)\,(1 - \varepsilon).$$
Set $x = (u_0, x^*_0(u_0)y_0) \in S_X$ and $x^* = (u^*_0, 0) \in S_{X^*}$. Then $(x, x^*) \in \Pi(X)$ and
$$v(P) \geqslant |x^*(P(x))| = |u^*_0\big(P_{\lambda_0}(u_0, x^*_0(u_0)y_0)\big)| = |u^*_0(Q(u_0))| > \Big(n^{(k)}(X_{\lambda_0}) - \varepsilon\Big)\,(1 - \varepsilon),$$
which implies that $n^{(k)}(X) \geqslant \inf_\lambda n^{(k)}(X_\lambda)$. The reversed inequality is given in Corollary 2.2. ☐

Remark 2.4. *The above proposition is not true in the real case.* We note that $n^{(2)}(c_0) = 1/2$ and $n^{(2)}(\ell_\infty) = 1/2$ in the real case (see [19]), while $n^{(2)}(\mathbb{R}) = 1$.

3. Spaces of vector-valued bounded functions

The next lemma contains an abstract condition valid to estimate the polynomial numerical indices of some closed subspaces of a space of vector-valued bounded functions. We will show next that it could be applied to many concrete situation.

Lemma 3.1. *Let X be a complex Banach space, Γ a non-empty set, and let Y be a closed subspace of $B(\Gamma, X)$. Suppose that for every $f_0 \in Y$ there is a subset U_0 of Γ which is norming for Y such that for every $t_0 \in U_0$ and every $\delta > 0$, there is a function $\varphi : \Gamma \longrightarrow [0, 1]$ with $\varphi(t_0) = 1$ such that*
$$\Psi(x) = (1 - \varphi)f_0 + \varphi\, x \ \in Y \qquad \big(x \in X\big)$$
and there is $x_0 \in B_X$ such that $\|f_0 - \Psi(x_0)\| < \delta$. Then $n^{(k)}(Y) \geqslant n^{(k)}(X)$ for all $k \in \mathbb{N}$.

Proof. Let $P \in \mathcal{P}(^kY;Y)$ with $\|P\|=1$. Given $0<\varepsilon<1$, we find $f_0 \in S_Y$ and $t_0 \in U_0$ such that

$$\big\|\big[P(f_0)\big](t_0)\big\| > 1-\varepsilon/2.$$

Since P is continuous at f_0, there exists $\delta>0$ such that

$$\|P(f_0)-P(g)\| < \varepsilon/2 \qquad \text{for all } g\in Y \text{ with } \|f_0-g\|<\delta.$$

We use the hypothesis to find $x_0 \in B_X$ with $\|f_0-\Psi(x_0)\|<\delta$, so

$$\big\|\big[P(\Psi(x_0))\big](t_0)\big\| > 1-\varepsilon$$

and we may find $x_0^* \in S_{X^*}$ such that

$$x_0^*\big(\big[P(\Psi(x_0))\big](t_0)\big) > 1-\varepsilon.$$

If we write $x_0 = z_0\,\widetilde{x_0}$ for suitable $z_0\in\mathbb{D}$ and $\widetilde{x_0}\in S_X$, since the function

$$z \longmapsto x_0^*\big(\big[P(\Psi(z\,\widetilde{x_0}))\big](t_0)\big) \qquad (z\in\mathbb{C})$$

is entire, we may apply the maximum modulus theorem to get $z_1\in\mathbb{T}$ such that

$$\big\|\big[P(\Psi(z_1\,\widetilde{x_0}))\big](t_0)\big\| \geqslant x_0^*\big(\big[P(\Psi(z_1\,\widetilde{x_0}))\big](t_0)\big) \geqslant x_0^*\big(\big[P(\Psi(z_0\,\widetilde{x_0}))\big](t_0)\big) > 1-\varepsilon.$$

Finally, let $x_1 = z_1\,\widetilde{x_0}\in S_X$, consider $x_1^*\in S_{X^*}$ such that $x_1^*(x_1)=1$ and define

$$\Phi(x) = x_1^*(x)(1-\varphi)f_0 + \varphi\, x \in Y \qquad (x\in X).$$

We observe that $\Phi(x_1)=\Psi(z_1\,\widetilde{x_0})$. Hence

$$\big\|\big[P(\Phi(x_1))\big](t_0)\big\| > 1-\varepsilon.$$

Now, we define a continuous k-homogeneous polynomial $Q: X\longrightarrow X$ by

$$Q(x) = \big[P(\Phi(x))\big](t_0) \qquad (x\in X) \tag{1}$$

which satisfies that

$$\|Q\| \geqslant \|Q(x_1)\| = \big\|\big[P(\Phi(x_1))\big](t_0)\big\| > 1-\varepsilon.$$

We choose $(x_2,x_2^*)\in\Pi(X)$ so that

$$|x_2^*(Q(x_2))| \geqslant n^{(k)}(X)(1-\varepsilon).$$

Then, $\big(\Phi(x_2), x_2^*\circ\delta_{t_0}\big)\in\Pi(Y)$ and

$$v(P) \geqslant \big|\big(x_2^*\circ\delta_{t_0}\big)\big(P(\Phi(x_2))\big)\big| = |x_2^*(Q(x_2))| \geqslant n^{(k)}(X)(1-\varepsilon).$$

It follows that $n^{(k)}(Y)\geqslant n^{(k)}(X)$. □

We are now able to state the main result of the section. Let us say that everything was previously known for the linear case (i.e. $k=1$) and it can be found in [6, 21, 22, 24].

Theorem 3.2. *Let X be a complex Banach space and $k\in\mathbb{N}$.*

- (a) *If K is a compact Hausdorff space, then $n^{(k)}(C_w(K,X)) \geqslant n^{(k)}(X)$.*
- (b) *Let K be a compact Hausdorff space. If X is an Asplund space or K has a dense subset of isolated points, then $n^{(k)}(C_{w^*}(K,X^*)) \geqslant n^{(k)}(X^*)$.*
- (c) *If L is a locally compact Hausdorff space, then $n^{(k)}(C_0(L,X)) \geqslant n^{(k)}(X)$.*
- (d) *If Ω is a completely regular Hausdorff space, then $n^{(k)}(C_b(\Omega,X)) \geqslant n^{(k)}(X)$.*
- (e) *If K is a compact Hausdorff space, then $n^{(k)}(C(K,X)) \geqslant n^{(k)}(X)$.*
- (f) *For a compact Hausdorff space K and an open and dense subset U of K, we write*

$$Y = \big\{f\in C(K) \ : \ f(K\setminus U)=0\big\}.$$

Then, for every subspace X of $C(K)$ containing Y, one has $n^{(k)}(X)=1$ for every $k\in\mathbb{N}$.

Proof. We just show that in every case the conditions of Lemma 3.1 are satisfied.

(a). Given $f_0 \in C_w(K,X)$, the set $U_0 = \{t \in K : f \text{ is norm continuous at } t\}$ is dense in K (see [16, p. 1905] or [21, Lemma 1]). Now, given $t_0 \in U_0$, the set $W = \overline{\{t \in K : \|f_0(t) - f_0(t_0)\| \geqslant \delta\}}$ does not contain t_0, so Urysohn's lemma gives us a continuous function $\varphi : K \longrightarrow [0,1]$ such that $\varphi(t_0) = 1$ and $\varphi(W) = 0$. Now,

$$\Psi(x) = (1-\varphi)f_0 + \varphi\, x \in C_w(K,X) \qquad (x \in X)$$

and, writing $x_0 = f_0(t_0) \in B_X$, we have

$$\big\|f_0 - \Psi(x_0)\big\| = \sup_{t\in K}\big\|f_0(t) - \big((1-\varphi(t))f_0(t) + \varphi(t)\, f_0(t_0)\big)\big\| = \sup_{t\in K}\varphi(t)\|f_0(t) - f_0(t_0)\| < \delta.$$

(b). If X is an Asplund space, the above proof can be repeated by just using that for every $f \in C_{w^*}(K,X^*)$, the set $\{t \in K : f \text{ is norm continuous at t}\}$ is dense in K (see [21, Lemma 6]). If K contains a dense subset of isolated points, the above observation is trivially true.

(c) and (d) also follow as (a), and (e) is a direct consequence of any of these two items.

(f). Everything works fine if we follow the proof of (a) with $X = \mathbb{C}$, $U_0 = U$, and taking care of the fact that we may choose all Urysohn functions with support inside U. □

In [7, Proposition 2.10], it was shown that $n^{(k)}(C(K,X)) \leqslant n^{(k)}(X)$ for real or complex spaces. This, together with the above result gives the equality in the complex case. Actually, the same can be proved for more complex vector-valued function spaces.

Corollary 3.3. *If X is a complex Banach space, L is a locally compact Hausdorff space, Ω is a completely regular Hausdorff space, and K is a compact Hausdorff space, then*

$$n^{(k)}(C_0(L,X)) = n^{(k)}(C_b(\Omega,X)) = n^{(k)}(C(K,X)) = n^{(k)}(X)$$

for every $k \in \mathbb{N}$.

Proof. We shall consider just the space $C_b(\Omega,X)$. Given $Q \in \mathcal{P}(^kX;X)$ with $\|Q\| = 1$, we define a polynomial $P \in \mathcal{P}\big(^kC_b(\Omega,X);C_b(\Omega,X)\big)$ by

$$[P(f)](t) = Q(f(t)) \qquad (t \in K,\ f \in C_b(\Omega,X)). \tag{2}$$

Then $\|P\| = 1$, so $v(P) \geqslant n^{(k)}(C_b(\Omega,X))$. It is not difficult to show that the set of functions in the unit sphere of $C_b(\Omega,X)$ attaining their norm is dense, so [26, Theorem 2.5] gives us that the numerical radius of P is given by

$$v(P) = \sup\left\{\big|x^*\big(P(f)(t)\big)\big| \ :\ f \in S_{C_b(\Omega,X)},\ t \in K,\ x^* \in S_{X^*},\ x^*(f(t)) = 1\right\}.$$

Therefore, given $\varepsilon > 0$, we may find $f \in S_{C_b(\Omega,X)}$, $x^* \in S_{X^*}$, and $t \in K$ such that $x^*(f(t)) = 1$ and

$$n^{(k)}(C_b(\Omega,X)) - \varepsilon < \big|x^*\big(P(f)(t)\big)\big| = |x^*(Q(f(t)))| \leqslant v(Q),$$

which implies that $n^{(k)}(X) \geqslant n^{(k)}(C_b(\Omega,X))$. □

Remark 3.4. Contrary to the linear case, the norm-continuous polynomials on a Banach space are not necessarily weak-continuous. For this reason, *the proof of the inequality $n^{(k)}\big(C_b(\Omega,X)\big) \leqslant n^{(k)}(X)$ given above, which is apparently the "easy one", cannot be adapted to get the analogous one for the space $C_w(K,X)$.* Indeed, the definition of the polynomial P given in (2) is not valid if we replace $C_b(\Omega,X)$ by $C_w(K,X)$, because P needs to be weakly continuous to assure that $P(f)$ belongs to $C_w(K,X)$ for every $f \in C_w(K,X)$. Nevertheless, *if K has at least one isolated point (say k_0), then $n^{(k)}\big(C_w(K,X)\big) = n^{(k)}(X)$.* Indeed, in this case $C_w(K,X)$ is isometrically isomorphic to $C_w(\{k_0\},X) \oplus_\infty C_w(K \setminus \{k_0\},X)$ and $C_w(\{k_0\},X)$ is isometrically isomorphic to X, so $n^{(k)}\big(C_w(K,X)\big) \leqslant n^{(k)}(X)$ by Proposition 2.3. *We do not know if the equality holds whenever K is a perfect compact space.*

It is well known (see [14, Theorem VI.7.1]) that for any compact Hausdorff space K and every Banach space X, the spaces $C(K,X^*)$, $C_w(K,X^*)$ and $C_{w^*}(K,X^*)$ can be identified isometrically with $K(X,C(K))$, $W(X,C(K))$ and $L(X,C(K))$ respectively, where $K(X,C(K))$ (resp. $W(X,C(K))$) denotes the Banach space of all compact (resp. weakly compact) operators from X into $C(K)$. Therefore, Theorem 3.2 and Corollary 3.3 also read as follows.

Corollary 3.5. *Let X be a complex Banach space and K a compact Hausdorff space. Then,*

$$n^{(k)}\big(W(X,C(K))\big) \geqslant n^{(k)}(X^*) = n^{(k)}\big(K(X,C(K))\big)$$

for every $k\in\mathbb{N}$. If X is an Asplund space or K contains a dense subset of isolated points, then

$$n^{(k)}\big(L(X,C(K))\big) \geqslant n^{(k)}(X^*)\ ,$$

for every $k\in\mathbb{N}$.

Differently from the linear case, there is no general inequality between the polynomial numerical index of a Banach space and that of its dual space. For instance, $n^{(k)}(c_0)=1$, $n^{(k)}(\ell_1)\leqslant 1/2$, $n^{(k)}(\ell_\infty)=1$ for $k\geqslant 2$ in the complex case. Nevertheless, as it was commented in the introduction, $n^{(k)}(X^{**})\leqslant n^{(k)}(X)$ for every Banach space X and every $k\in\mathbb{N}$ [7, Corollary 2.15]. Our next example shows that the above inequality may be strict. In the linear case, this was an old latent open question, which has been recently solved by K. Boyko, V. Kadets, M. Martín, and D. Werner [6, Example 3.1]. It is possible to prove that the same example works for the polynomial case. Anyhow, we prefer to give an example using the approach given in the very recent paper [23], which will allow us to present a space showing that the behavior of the polynomial numerical index with respect to the biduality is the worst possible. We will use Theorem 3.3 of [23], actually part of its proof. We include most of the details for completeness. A similar example for the real case can be found in [19].

Example 3.6. *There exists a complex Banach space X such that*

$$n^{(k)}(X)=1 \qquad \text{and} \qquad n^{(k)}(X^{**})=k^{\frac{k}{1-k}}$$

for every $k\geqslant 2$, i.e. the polynomial numerical index of X is the greatest possible, while that of X^{**} is the least possible.

Proof. Let E be a separable complex Banach space such that $n^{(k)}(E)=k^{\frac{k}{1-k}}$ for every $k\geqslant 2$ (see [17, Example 9]). By the Banach-Mazur Theorem, we may consider E as a closed subspace of $C(\Delta)$, where Δ denotes the Cantor middle third set viewed as a subspace of $[0,1]$. We write $T: C[0,1]\longrightarrow C(\Delta)$ for the restriction operator, i.e.

$$\big[T(f)\big](t)=f(t) \qquad \big(t\in\Delta,\ f\in C[0,1]\big).$$

We define the closed subspaces $X(E)$ and Y of $C[0,1]$ by

$$X(E)=\{f\in C[0,1]\ :\ T(f)\in E\}\,, \qquad Y=\operatorname{Ker}T.$$

Now, Theorem 3.2.(f) gives us that $n^{(k)}\big(X(E)\big)=1$ for every $k\geqslant 2$. On the other hand, as it is shown in the proof of [23, Theorem 3.3] (see [23, Remark 3.4.b]), $X(E)^*\equiv E^*\oplus_1 Y^*$ so $X(E)^{**}\equiv E^{**}\oplus_\infty Y^{**}$. Then, by Corollary 2.2, $n^{(k)}\big(X(E)^{**}\big)\leqslant n^{(k)}(E^{**})\leqslant n^{(k)}(E)=k^{\frac{k}{1-k}}$, and the other inequality always holds [15]. ☐

Let us finish the section with some remarks about the real case of the above results. Although the equalities proved in this section are not true for real spaces, it is readily verified that one of the inequalities (the one proved in Corollary 3.3) does not depend on the base field. We summarize this in the following result.

Proposition 3.7. *Let X be a real Banach space, K a compact Hausdorff space, L a locally compact Hausdorff space and Ω a completely regular Hausdorff space. If Y is any of the real spaces $C(K,X)$, $C_0(L,X)$ or $C_b(\Omega,X)$, then $n^{(k)}(Y)\leqslant n^{(k)}(X)$ for every $k\in\mathbb{N}$.*

The above proposition allows us to calculate some polynomial numerical index of order 2.

Example 3.8. *Let Δ be the Cantor ternary set. Then, the real space $C(\Delta)$ satisfies*

$$n^{(2)}\big(C(\Delta)\big) = \frac{1}{2}\,, \qquad \frac{2^k}{2+M_k(2^k-2)} \leqslant n^{(k)}\big(C(\Delta)\big) \leqslant \frac{2}{k}\left(\frac{k-2}{k}\right)^{\frac{k-2}{2}} \qquad (k \geqslant 3)$$

where $M_k = \sum_{j=1}^{k} \frac{j^k}{j!\,(k-j)!}$. Indeed, the lower bounds follow from [19, Corollary 2.4] and are actually valid for every predual of an $L_1(\mu)$ space. For the reversed inequalities, since Δ is homeomorphic to the disjoint union of two copies of Δ, it follows that

$$C(\Delta) \equiv C(\Delta) \oplus_\infty C(\Delta) \equiv C(\Delta, \ell_\infty^2).$$

Then,

$$n^{(k)}\big(C(\Delta)\big) = n^{(k)}\big(C(\Delta, \ell_\infty^2)\big) \leqslant n^{(k)}(\ell_\infty^2)$$

by the proposition above. On the other hand,

$$n^{(2)}(\ell_\infty^2) = \frac{1}{2}\,, \qquad n^{(k)}(\ell_\infty^2) \leqslant \frac{2}{k}\left(\frac{k-2}{k}\right)^{\frac{k-2}{2}} \qquad (k \geqslant 3)$$

by [19, Corollary 2.5].

The above argument also applies to the real space $C\big([0,1]\cup[2,3]\big)$, but not to the real space $C[0,1]$. Actually, an interesting problem is to calculate the polynomial numerical indices of this space or, more generally, of the real space $C(K)$ when K is connected.

4. Spaces of vector-valued essentially bounded functions

In [25, Theorem 2.3], M. Martín and A. Villena showed that $n\big(L_\infty(\mu, X)\big) = n(X)$ for every σ-finite measure μ and every real or complex Banach space X. Our aim here is to extend the above result to polynomial numerical indices. In the real case, the equality does not hold as $n^{(2)}(\ell_\infty) = 1/2$ and $n^{(2)}(\mathbb{R}) = 1$, but it holds in the complex case as the following result shows.

Theorem 4.1. *Let (Ω, Σ, μ) be a σ-finite measure space and let X be a complex Banach space. Then*

$$n^{(k)}\left(L_\infty(\mu, X)\right) = n^{(k)}(X)$$

for every $k \in \mathbb{N}$.

Proof. First we prove $n^{(k)}(L_\infty(\mu,X)) \geqslant n^{(k)}(X)$ for every $k \in \mathbb{N}$. Let $P \in \mathcal{P}\big({}^k L_\infty(\mu,X); L_\infty(\mu,X)\big)$ with $\|P\| = 1$. Given $0 < \varepsilon < 1$, we may find $f \in S_{L_\infty(\mu,X)}$ such that

$$\|P(f)\| > 1 - \frac{\varepsilon}{2}.$$

Then there exists $C \in \Sigma$ with $\mu(C) > 0$ and such that

$$\big\|\big[P(f)\big](t)\big\| > 1 - \frac{\varepsilon}{2} \qquad \big(t \in C\big).$$

Since P is continuous at f, there exists $\delta > 0$ such that

$$\|P(f) - P(h)\| < \frac{\varepsilon}{2} \qquad \text{for all } h \in L_\infty(\mu, X) \text{ with } \|f - h\| < \delta.$$

By [25, Lemma 2.2], there exist $y_0 \in X$ such that $0 < \|y_0\| \leqslant 1$ and $A \subseteq C$ with $\mu(A) > 0$ such that $\|(f - y_0)\chi_A\| < \delta$. Thus, by the above two equations, there exists a subset $A_1 \in \Sigma$ of A with $\mu(A \setminus A_1) = 0$ and such that

$$\big\|\big[P(y_0\chi_A + f\chi_{\Omega\setminus A})\big](t)\big\| > 1 - \varepsilon \qquad \big(t \in A_1\big).$$

Now we consider the function $\Theta : \mathbb{C} \longrightarrow (L_\infty(A_1, \mu_{|A_1}, X), \|\cdot\|_{A_1})$ defined as

$$\Theta(z) = \pi \circ P(zy_0\chi_A + f\chi_{\Omega\setminus A}),$$

where $\pi : (L_\infty(\Omega,\mu,X),\|\cdot\|_\Omega) \longrightarrow (L_\infty(A_1,\mu_{|A_1},X),\|\cdot\|_{A_1})$ is the continuous linear mapping defined by $\pi(h) = h_{|A_1}$. Clearly, Θ is an entire function and so, by the maximum modulus theorem,

$$1-\varepsilon < \|P(y_0\chi_A + f\chi_{\Omega\setminus A})\|_{A_1} = \|\Theta(1)\|_{A_1} \leqslant \max_{|z|\leqslant \|y_0\|^{-1}} \|\Theta(z)\|_{A_1}.$$

We put $z_0 \in \mathbb{C}$ with $|z_0| = \|y_0\|^{-1}$ for a value such that

$$\|\Theta(z_0)\|_{A_1} = \max_{|z|\leqslant \|y_0\|^{-1}} \|\Theta(z)\|_{A_1}.$$

Then, $\|z_0y_0\| = 1$ and

$$1-\varepsilon < \|P(z_0y_0\chi_A + f\chi_{\Omega\setminus A})\|_{A_1}.$$

Hence, we can find $A_2 \in \Sigma$ with $A_2 \subset A_1$ such that $0 < \mu(A_2) < \infty$ and

$$1-\varepsilon < \big\|\big[P(z_0y_0\chi_A + f\chi_{\Omega\setminus A})\big](t)\big\| \qquad \big(t \in A_2\big).$$

If we apply now [25, Lemma 2.1] to $P(z_0y_0\chi_A + f\chi_{\Omega\setminus A})$ as an element of $L_\infty\big((A_2,\mu_{|A_2},X),\|\cdot\|_{A_2}\big)$, we can find $B \in \Sigma$ with $0 < \mu(B) < \infty$ and $B \subset A_2$ such that

$$1-\varepsilon < \left\|\frac{1}{\mu(B)}\int_B \big[P(z_0y_0\chi_A + f\chi_{\Omega\setminus A})\big](t)\, d\mu(t)\right\| \tag{3}$$

Next we fix $x_0^* \in S_{X^*}$ with $x_0^*(z_0y_0) = 1$, we write

$$\Phi(x) = x\,\chi_A + x_0^*(x)f\,\chi_{\Omega\setminus A} \in L_\infty(\mu,X) \qquad (x \in X),$$

and we consider the k-homogeneous polynomial $S \in \mathcal{P}(^kX;X)$ given by

$$Sx = \frac{1}{\mu(B)}\int_B \big[P\left(\Phi(x)\right)\big](t)\, d\mu(t) \qquad (x \in X).$$

According to (3), we have $\|S\| \geqslant \|S(z_0y_0)\| > 1-\varepsilon$. So we may find $x \in S_X$ and $x^* \in S_{X^*}$ such that

$$x^*(x) = 1 \qquad \text{and} \qquad |x^*(Sx)| \geqslant n^{(k)}(X)(1-\varepsilon).$$

Set $g = \Phi(x) \in S_{L_\infty(\mu,X)}$ and define the functional $g^* \in S_{L_\infty(\mu,X)^*}$ by

$$g^*(h) = x^*\left(\frac{1}{\mu(B)}\int_B h(t)\, d\mu(t)\right) \qquad (h \in L_\infty(\mu,X)).$$

Since $B \subseteq A$, we have

$$g^*(g) = x^*\left(\frac{1}{\mu(B)}\int_B (x\,\chi_A(t) + x_0^*(x)f(t)\,\chi_{\Omega\setminus A}(t))d\mu(t)\right) = x^*(x) = 1$$

and

$$|g^*(Pg)| = |x^*(Sx)| \geqslant n^{(k)}(X)(1-\varepsilon).$$

Hence, $v(P) \geqslant n^{(k)}(X)$ for all $P \in \mathcal{P}(^k\big(L_\infty(\mu,X);L_\infty(\mu,X)\big)$ and so $n^{(k)}(L_\infty(\mu,X)) \geqslant n^{(k)}(X)$.

To prove the reverse inequality, for $S \in \mathcal{P}(^kX;X)$ with $\|S\| = 1$, we consider the polynomial P in $\mathcal{P}(^k\big(L_\infty(\mu,X);L_\infty(\mu,X)\big)$ by

$$[P(f)](t) = S(f(t)) \qquad (t \in \Omega,\ f \in L_\infty(\mu,X)).$$

Then $\|P\| = 1$ and so $v(P) \geqslant n^{(k)}(L_\infty(\mu,X))$. According to [25, Lemma 2.2] together with [26, Theorem 2.5], given $\varepsilon > 0$ there exist $x \in S_X$, $f \in B_{L_\infty(\mu,X)}$, $A \in \Sigma$ with $0 < \mu(A) < \infty$, and $x^* \in S_{X^*}$ with $x^*(x) = 1$ such that

$$v(P) - \varepsilon < \left|x^*\left(\frac{1}{\mu(A)}\int_A \big[P(x\,\chi_A + f\,\chi_{\Omega\setminus A})\big](t)\, d\mu(t)\right)\right|.$$

Since

$$\big[P(x\,\chi_A + f\,\chi_{\Omega\setminus A})\big](t) = S(x\,\chi_A(t) + f(t)\,\chi_{\Omega\setminus A}(t)) = S(x)$$

for all $t \in A$, we obtain

$$\left|x^*\left(\frac{1}{\mu(A)}\int_A \big[P(x\,\chi_A + f\,\chi_{\Omega\setminus A})\big](t)\, d\mu(t)\right)\right| = |x^*(S(x))|,$$

so $n^{(k)}(X) \geqslant n^{(k)}(L_\infty(\mu, X))$. □

As happened in the previous section for the spaces of vector-valued continuous functions, the proof of the second inequality in the above theorem does not depend on the base field.

Proposition 4.2. *Let (Ω, Σ, μ) be a σ-finite measure space and let X be a real Banach space. Then*

$$n^{(k)}\left(L_\infty(\mu, X)\right) \leqslant n^{(k)}(X)$$

for every $k \in \mathbb{N}$.

We may use this proposition to estimate the polynomial numerical indices of $L_\infty[0,1]$ with the same argument as the one given in Example 3.8.

Example 4.3. *The real space $L_\infty[0,1]$ satisfies*

$$n^{(2)}\big(L_\infty[0,1]\big) = \frac{1}{2}\,, \qquad \frac{2^k}{2 + M_k(2^k - 2)} \leqslant n^{(k)}\big(L_\infty[0,1]\big) \leqslant \frac{2}{k}\left(\frac{k-2}{k}\right)^{\frac{k-2}{2}} \qquad (k \geqslant 3)$$

where $M_k = \sum_{j=1}^{k} \frac{j^k}{j!\,(k-j)!}$.

5. The analytic numerical index

By considering elements of $\mathcal{A}_u(B_X, X)$ instead of continuous k-homogeneous polynomials, we can define, as it has been done in [20], the *analytic numerical index of X* as

$$n_a(X) = \inf\{v(f) \ : \ f \in \mathcal{A}_u(B_X, X),\ \|f\| = 1\}.$$

Since the set $\mathcal{P}(X;X)$ of all continuous polynomials from X into X is dense in $\mathcal{A}_u(B_X, X)$ we have that

$$n_a(X) = \inf\{v(P) \ : \ P \in \mathcal{P}(X;X),\ \|P\| = 1\},$$

i.e. $n_a(X)$ can be defined as the "*non-homogeneous polynomial numerical index of X*". Clearly,

$$n_a(X) \leqslant n^{(k)}(X)$$

for every $k \in \mathbb{N}$.

Observe that in the proof of Proposition 2.3 and Lemma 3.1, the hypothesis of homogeneity of the polynomials is never used, and therefore we have the following.

Corollary 5.1. *The following statements hold:*

(a) $n_a\Big(\big[\bigoplus_{\lambda\in\Lambda} X_\lambda\big]_{c_0}\Big) = n_a\Big(\big[\bigoplus_{\lambda\in\Lambda} X_\lambda\big]_{\ell_\infty}\Big) = \inf_\lambda n_a(X_\lambda)$, *for every non-empty family of complex Banach spaces $\{X_\lambda \,:\, \lambda \in \Lambda\}$.*

(b) *Let K be a compact Hausdorff space and let U be an open dense subset of K. We write*

$$Y = \big\{f \in C(K) \ : \ f(K \setminus U) = 0\big\}.$$

Then, for every subspace X of the complex space $C(K)$ containing Y, one has $n_a(X) = 1$.

(c) *Let X be a complex Banach space, K a compact Hausdorff space, L a locally compact Hausdorff space and Ω a completely regular Hausdorff space. Then,*

$$n_a\big(C(K,X)\big) = n_a\big(C_0(L,X)\big) = n_a\big(C_b(\Omega,X)\big) = n_a(X), \quad n_a\big(C_w(K,X)\big) \geqslant n_a(X).$$

(d) *Let X be a complex Banach space and let K be a compact Hausdorff space. If X is an Asplund space or K has a dense subset of isolated points, then*

$$n_a\big(C_{w^*}(K, X^*)\big) \geqslant n_a(X^*).$$

As we commented in the introduction, the Aron-Berner extension of a polynomial has the same norm and numerical radius as the original polynomial. It clearly follows that $n_a(X^{**}) \leqslant n_a(X)$ for every complex Banach space X. It is easy to get from (b) of the above corollary that for the space given in Example 3.6 this inequality is strict in the worst possible way.

Example 5.2. *There is a complex Banach space X such that*

$$n_a(X) = 1 \qquad \text{and} \qquad n_a(X^{**}) = 0.$$

Indeed, let X be the space $X(E)$ given in Example 3.6. Then, by Corollary 5.1.b, one gets $n_a(X) = 1$. On the other hand, $n_a(X^{**}) \leqslant n^{(k)}(X^{**}) = k^{\frac{k}{1-k}} \longrightarrow 0$.

6. On the polynomial Daugavet Property

We devote this section to give some new examples of Banach spaces with the polynomial Daugavet property and, actually, to characterize this property for some vector-valued function spaces. In the whole section we will use the following characterization, which expresses the polynomial Daugavet property in terms of scalar-valued polynomials.

Lemma 6.1 ([9, Proposition 1.3 and Corollary 2.2])**.** *Let X be a real or complex Banach space. Then, the following are equivalent:*

- (i) *X has the polynomial Daugavet property, i.e. every weakly compact $P \in \mathcal{P}(X;X)$ satisfies the Daugavet equation.*
- (ii) *For every $p \in \mathcal{P}(X)$ with $\|p\| = 1$, every $x_0 \in S_X$, and every $\varepsilon > 0$, there exist $\omega \in \mathbb{T}$ and $y \in B_X$ such that*
$$\operatorname{Re} \omega p(y) > 1 - \varepsilon \qquad \text{and} \qquad \|x_0 + \omega y\| > 2 - \varepsilon.$$
- (iii) *Every weakly compact $P \in \mathcal{P}(X;X)$ satisfies $\sup \operatorname{Re} V(P) = \|P\|$.*

It is proved in [9, Theorem 2.4] that for every completely regular Hausdorff topological space Ω without isolated points and every Banach space X, the real or complex space $C_b(\Omega, X)$ has the polynomial Daugavet property. Actually, this holds for every C_b-rich subspace of $C_b(\Omega, X)$. A subspace $\mathcal{F}$ of $C_b(\Omega, X)$ is *C_b-rich* if for every open subset U of Ω, every $x \in X$ and every $\varepsilon > 0$, there exists a norm one continuous function $\varphi : \Omega \longrightarrow [0, 1]$ with support included in U such that the distance of $\varphi \otimes x$ to $\mathcal{F}$ is less than ε. When K is a perfect Hausdorff compact space, then the spaces $C(K, X)$ and $C_b(K, X)$ are the same. If L is a locally compact Hausdorff topological space without isolated points, then the space $C_0(L, X)$ is clearly C_b-rich in $C_b(L, X)$, and so it has the polynomial Daugavet property. Let us emphasize all these results for the sake of completeness.

Corollary 6.2. *Let K be a perfect compact Hausdorff space, L a locally compact Hausdorff topological space without isolated points, Ω a completely regular Hausdorff topological space without isolated points, and let X be a Banach space. Then, the real or complex spaces $C(K, X)$, $C_0(L, X)$, $C_b(\Omega, X)$ and all of their C_b-rich subspaces, have the polynomial Daugavet property.*

Our main aim in this section is to show that the above result extends to $C_w(K, X)$ and $C_{w^*}(K, X^*)$ when K is perfect, and to $L_\infty(\mu, X)$ when μ is atomless. This will be is a consequence of a general sufficient condition for the polynomial Daugavet property. Given a sequence (z_n) in a Banach space X, we say that the series $\sum_n z_n$ is *weakly unconditionally Cauchy* if $\sum_n |x^*(z_n)| < \infty$ for every $x^* \in X^*$ (see [11, §V] for background). It is clear that if the series $\sum_n z_n$ is wuC, then the sequence (z_n) is weakly-null and, therefore, bounded.

Proposition 6.3. *Let X be a Banach space. Suppose that for every $x, z \in S_X$, $\omega \in \mathbb{T}$, and $\varepsilon > 0$, there exists a sequence (z_n) in X such that $\sum_n z_n$ is weakly unconditionally Cauchy and*

$$\limsup \|z + z_n\| \leqslant 1 \qquad \text{and} \qquad \|x + \omega(z + z_n)\| > 2 - \varepsilon$$

for every $n \in \mathbb{N}$. Then X has the polynomial Daugavet property.

Proof. Let $x \in S_X$, $p \in \mathcal{P}(X)$ with $\|p\| = 1$ and $\varepsilon > 0$ be fixed. We take $z \in S_X$ and $\omega \in \mathbb{T}$ such that

$$\operatorname{Re} \omega p(z) > 1 - \varepsilon$$

and we use the hypothesis to get a sequence (z_n) in X such that

$$\limsup \|z + z_n\| \leqslant 1 \qquad \text{and} \qquad \|x + \omega(z + z_n)\| > 2 - \varepsilon.$$

Now, if $\inf \|z_n\| > 0$ (and being (z_n) weakly-null), the Bessaga-Pelczynski Selection Principle (see [11, p. 42]) allows us to extract a basic subsequence $(z_{\sigma(n)})$ which is equivalent to the unit vector basis of c_0 (see [11, p. 45]). It then follows from the weak continuity of polynomials on bounded subset of real or complex c_0 [3] (see [12, Proposition 1.59]) that

$$\operatorname{Re} \omega p\big(z + z_{\sigma(n)}\big) \longrightarrow \operatorname{Re} \omega p(z). \tag{4}$$

Otherwise, (z_n) has a norm-null subsequence $(z_{\sigma(n)})$ and (4) follows by the norm-continuity of p.

If $\limsup \|z + z_{\sigma(n)}\| < 1$, we take a suitable $n \in \mathbb{N}$, such that $y = z + z_{\sigma(n)} \in B_X$, $\operatorname{Re} \omega p(y) > 1 - \varepsilon$ and we have

$$\|x + \omega y\| > 2 - \varepsilon.$$

Finally we suppose that $\limsup \|z + z_{\sigma(n)}\| = 1$. Since $\|u - \frac{u}{\|u\|}\| = |1 - \|u\||$ for all $u \in X \setminus \{0\}$, and since p is uniformly continuous on $2B_X$, we can choose $n \in \mathbb{N}$ so that $1 - \varepsilon \leqslant \|z + z_{\sigma(n)}\| \leqslant 1 + \varepsilon$ and $y = \frac{z + z_{\sigma(n)}}{\|z + z_{\sigma(n)}\|} \in S_X$ satisfies $\operatorname{Re} \omega p(y) > 1 - \varepsilon$. Thus

$$\|x + \omega y\| > \big\|x + \omega(z + z_{\sigma(n)})\big\| - \varepsilon > 2 - 2\varepsilon,$$

and, in both cases, Lemma 6.1 applies. □

Remark 6.4. Let us observe that if we would like to get only the (linear) Daugavet property, it is enough to require in the above proposition that the sequence (z_n) is weakly null. We do not know if this is actually enough for the polynomial Daugavet property.

We are now ready to give more examples of Banach spaces with the polynomial Daugavet property.

Theorem 6.5. *Let X be a real or complex Banach space.*

(a) *If μ is an atomless σ-finite measure, then $L_\infty(\mu, X)$ has the polynomial Daugavet property.*
(b) *If K is a perfect compact Hausdorff topological space, then $C_w(K, X)$ and $C_{w^*}(K, X^*)$ have the polynomial Daugavet property.*

Proof. (a). We fix $f, g \in L_\infty(\mu, X)$ with $\|f\| = \|g\| = 1$, $\omega \in \mathbb{T}$ and $\varepsilon > 0$. Since, the set

$$V = \{t \in \Omega \ : \ \|f(t)\| > 1 - \varepsilon/2\}$$

has positive measure and μ is atomless, we may find a sequence of pairwise disjoint measurable subsets of V with positive measure (V_n). Now, for every $n \in \mathbb{N}$, we write

$$g_n = \big(\omega^{-1} f - g\big)\chi_{V_n} \in L_\infty(\mu, X).$$

By disjointness of the supports, the series $\sum_n g_n$ is wuC (see [11, Theorem V.6]). Also,

$$\|g + g_n\| = \big\|g\chi_{\Omega \setminus V_n} + \omega^{-1} f \chi_{V_n}\big\| \leqslant 1 \qquad (n \in \mathbb{N})$$

and, for every $t \in V_n$, one has

$$\big\|\big[f + \omega(g + g_n)\big](t)\big\| = \|2f(t)\| > 2(1 - \varepsilon/2) = 2 - \varepsilon,$$

so $\|f + \omega(g + g_n)\| > 2 - \varepsilon$, and Proposition 6.3 applies.

(b). We give the proof for $C_{w^*}(K, X^*)$. The other case is analogous. We fix $f, g \in C_{w^*}(K, X^*)$ with $\|f\| = \|g\| = 1$, $\omega \in \mathbb{T}$ and $\varepsilon > 0$. Since B_{X^*} is w^*-closed, the set

$$V = \{t \in K \ : \ \|f(t)\| > 1 - \varepsilon/2\}$$

is open and, since K is perfect, we may find a sequence (V_n) of pairwise disjoint nonvoid open subsets of V. Now, for every $n \in \mathbb{N}$, we use Urysohn's lemma to find a continuous function $\varphi_n : K \longrightarrow [0, 1]$ whose support is contained in V_n, and $t_n \in V_n$ such that $\varphi_n(t_n) = 1$. Now, we write

$$g_n = \big(\omega^{-1} f - g\big)\varphi_n \in C_{w^*}(K, X^*).$$

By disjointness of the supports, the series $\sum_n g_n$ is weakly unconditionally Cauchy (see [11, Theorem V.6]). Also, for every $n \in \mathbb{N}$

$$\big\|\big[g + g_n\big](t)\big\| = \big\|\big(1 - \varphi_n(t)\big)g(t) + \varphi_n(t)\,\omega^{-1} f\big\| \qquad (t \in K),$$

so $\|g + g_n\| \leqslant 1$ by convexity, and

$$\|f + \omega(g + g_n)\| \geqslant \big\|\big[f + \omega(g + g_n)\big](t_n)\big\| = \|2f(t_n)\| > 2(1 - \varepsilon/2) = 2 - \varepsilon,$$

and Proposition 6.3 applies. □

As a consequence of the above results, by taking into account that the polynomials from Y into Y are dense in $\mathcal{A}_u(B_Y, Y)$ for every complex Banach space Y, we get the following corollary.

Corollary 6.6. *Let X be a complex Banach space. Let Y denote any of the complex spaces $C_w(K, X)$, $C_{w^*}(K, X^*)$ for K compact and perfect, or $C_0(L, X)$ for L locally compact and perfect, or $C_b(\Omega, X)$ for Ω completely regular without isolated points, or $L_\infty(\mu, X)$ with μ an atomless σ-finite measure. Then, every weakly compact $\Phi \in \mathcal{A}_u(B_Y, Y)$ satisfies the Daugavet equation.*

Our final aim in this section is to show that the vector-valued function spaces studied before have the polynomial Daugavet property when the range space does. We first need to state that the polynomial Daugavet property has a good behavior with respect to c_0 and ℓ_∞ sums.

Proposition 6.7. *The polynomial Daugavet property is inherited by M-summands. Conversely, if $\{X_\lambda : \lambda \in \Lambda\}$ is a non-empty family of real or complex Banach spaces with the polynomial Daugavet property, then $\big[\bigoplus_{\lambda\in\Lambda} X_\lambda\big]_{\ell_\infty}$ and $\big[\bigoplus_{\lambda\in\Lambda} X_\lambda\big]_{c_0}$ have the polynomial Daugavet property.*

Proof. Suppose X has the polynomial Daugavet property and $X = Y \oplus_\infty Z$. Given a weakly compact polynomial $P : Y \longrightarrow Y$, $P \neq 0$. We define $Q : X \longrightarrow X$ by

$$Q\big(y, z\big) = \big(P(y), 0\big).$$

Clearly Q is weakly compact with $\|Q\| = \|P\|$. By hypothesis

$$\begin{aligned} 1 < 1 + \|P\| = 1 + \|Q\| = \|\mathrm{Id}_X + Q\| &= \max\left\{\sup_{\|y\|\leqslant 1} \|y + P(y)\|,\ 1\right\} \\ &= \sup_{\|y\|\leqslant 1} \|y + P(y)\| = \|\mathrm{Id}_Y + P\|. \end{aligned}$$

Conversely, assume that every X_λ has the polynomial Daugavet property and put $X = \big[\bigoplus_{\lambda\in\Lambda} X_\lambda\big]_{\ell_\infty}$ or $X = \big[\bigoplus_{\lambda\in\Lambda} X_\lambda\big]_{c_0}$. Let $p : X \longrightarrow \mathbb{K}$ be a continuous polynomial of norm one, $y = (y_\lambda) \in S_X$, and $0 < \varepsilon < 1$. Since $\|y\| = 1$ there exists $\mu \in \Lambda$ such that $1 - \frac{\varepsilon}{2} < \|y_\mu\|$. We take $z = (z_\lambda) \in B_X$ with

$$|p(z)| > \frac{1 - \varepsilon}{1 - \frac{\varepsilon}{2}}\,. \tag{5}$$

We define the polynomial $q : X_\mu \longrightarrow \mathbb{K}$ by

$$q(x_\mu) = p(z + i_\mu(x_\mu - z_\mu)) \qquad \big(x_\mu \in X_\mu\big).$$

We have

$$1 = \|p\| \geqslant \|q\| \geqslant |q(z_\mu)| = |p(z)| > \frac{1 - \varepsilon}{1 - \frac{\varepsilon}{2}}\,.$$

Since X_μ has the polynomial Daugavet property, we use Lemma 6.1, for $q/\|q\|$, $y_\mu/\|y_\mu\|$ and $\varepsilon/2$ to get $\omega \in \mathbb{T}$ and $x_\mu^0 \in B_{X_\mu}$ such that

$$\mathrm{Re}\,\omega \frac{q}{\|q\|}(x_\mu^0) > 1 - \frac{\varepsilon}{2} \qquad \text{and} \qquad \left\|\frac{y_\mu}{\|y_\mu\|} + \omega x_0\right\| > 2 - \frac{\varepsilon}{2}.$$

Now, if we take $x_0 := z + i_\mu(x_\mu^0 - z_\mu) \in B_X$, we have

$$\mathrm{Re}\,\omega p(x_0) = \mathrm{Re}\,\omega q(x_\mu^0) > (1 - \frac{\varepsilon}{2})\|q\| > 1 - \varepsilon$$

and

$$\|y + \omega x_0\| \geqslant \big\|y_\mu + \omega x_\mu^0\big\| \geqslant \left\|\frac{y_\mu}{\|y_\mu\|} + \omega x_0\right\| - \left\|\frac{y_\mu}{\|y_\mu\|} - y_\mu\right\| > 2 - \frac{\varepsilon}{2} - \big(1 - \|y_\mu\|\big) > 2 - \varepsilon.$$

□

Let us comment that we do not know if the polynomial Daugavet property is inherited by M-ideals. This is true in the linear case [18, Proposition 2.10].

Given a Banach space X and a σ-finite measure μ, there exists a countable set J and an atomless σ-finite measure ν such that $L_\infty(\mu, X) \equiv L_\infty(\nu, X) \oplus_\infty B(J, X)$. Hence, Proposition 6.7 and Theorem 6.5 yields the following corollary that was obtained in [25, Theorem 2.5] for the linear case.

Corollary 6.8. *Let (Ω, Σ, μ) be a σ-finite measure space and let X be a real or complex Banach space. Then $L_\infty(\mu, X)$ has the polynomial Daugavet property if and only if X has the polynomial Daugavet property or μ is atomless.*

For the spaces of vector-valued continuous functions the proofs are not so tidy, since it is not possible in general to get a decomposition into the "atomic" and "non-atomic" parts as in the $L_\infty(\mu, X)$ case. We need the following analogous to Lemma 3.1 for the polynomial Daugavet property.

Lemma 6.9. *Let X be a complex Banach space with the polynomial Daugavet property, Γ a non-empty set, and let Y be a closed subspace of $B(\Gamma, X)$. Suppose that for every $f_0 \in Y$ there is a subset U_0 of Γ which is norming for Y such that for every $t_0 \in U_0$ and every $\delta > 0$, there is a function $\varphi : \Gamma \longrightarrow [0, 1]$ with $\varphi(t_0) = 1$ such that*

$$\Psi(x) = (1 - \varphi) f_0 + \varphi\, x \ \in Y \qquad \big(x \in X\big)$$

and there is $x_0 \in B_X$ such that $\|f_0 - \Psi(x_0)\| < \delta$. Then Y has the polynomial Daugavet property.

Proof. We take a (non-homogeneous) weakly compact polynomial $P \in \mathcal{P}\big(Y; Y\big)$ and follow the proof of Lemma 3.1 (it is never used there that P is homogeneous) up to equation (1), taking into account that the polynomial $Q \in \mathcal{P}(X; X)$ that we define there is also weakly compact. Now, we use Lemma 6.1 to get that $\sup \operatorname{Re} V(Q) = \|Q\|$ and we continue following the proof of Lemma 3.1 to show that $\sup \operatorname{Re} V(P) = \|P\|$. Then, Lemma 6.1 gives us that Y has the polynomial Daugavet property. □

As a consequence of all the previous results we get the following.

Proposition 6.10. *Let X be a complex Banach space, K a compact space, L a locally compact space and Ω a completely regular space.*

(a) *$C(K, X)$ has the polynomial Daugavet property if and only if X does or K is perfect.*
(b) *$C_w(K, X)$ has the polynomial Daugavet property if and only if X does or K is perfect.*
(c) *$C_0(L, X)$ has the polynomial Daugavet property if and only if X does or L is perfect.*
(d) *$C_b(\Omega, X)$ has the polynomial Daugavet property if and only if X does or Ω is perfect.*
(e) *If K contains a dense subset of isolated points and X^* has the polynomial Daugavet property, then $C_{w^*}(K, X^*)$ has the polynomial Daugavet property.*

Proof. (a) to (d). If any of the vector-valued function spaces above has the polynomial Daugavet property and the domain space contains an isolated point, then X is an M-summand of the corresponding space and Proposition 6.7 gives us that X also has the polynomial Daugavet property. Conversely, it was shown in the proof of Theorem 3.2 that the spaces $C(K, X)$, $C_w(K, X)$, $C_0(L, X)$, and $C_b(\Omega, X)$ satisfy the hypothesis of Lemma 6.9, so they have the polynomial Daugavet property as soon as X does. Finally, if K, L or Ω is perfect, the result follows from Corollary 6.2 and Theorem 6.5.

(e). If K has a dense subset of isolated points, the argument above also works for $C_{w^*}(K, X^*)$. □

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