

The Alternative Daugavet Property of C^* -algebras and JB^* -triples

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A Banach space X is said to have the alternative Daugavet property if for every (bounded and linear) rank-one operator $T : X \rightarrow X$ there exists a modulus one scalar ω such that

$$\|\text{Id} + \omega T\| = 1 + \|T\|.$$

We give geometric characterizations of this property in the setting of C^* -algebras, JB^* -triples, and of their isometric preduals.

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For a real or complex Banach space X , we write X^* for its topological dual and $L(X)$ for the Banach algebra of bounded linear operators on X . We denote by \mathbb{T} the set of modulus-one scalars.

A Banach space X is said to have the *alternative Daugavet property* if the norm equality

$$\max_{\omega \in \mathbb{T}} \|\text{Id} + \omega T\| = 1 + \|T\| \tag{aDE}$$

holds for every rank-one operator $T \in L(X)$ [38]. In such a case, all weakly compact operators on X also satisfy (aDE) (see [38, Theorem 2.2]). It is clear that a Banach space X has the alternative Daugavet property whenever X^* has, but the reverse result does not hold [38, Example 4.4].

Observe that (aDE) for an operator T just means that there exists a modulus-one scalar ω such that the operator $S = \omega T$ satisfies the well known *Daugavet equation*

$$\|\text{Id} + S\| = 1 + \|S\|. \tag{DE}$$

Therefore, the *Daugavet property* (i.e. every rank-one –equivalently, every weakly compact– operator satisfies (DE) [25, Theorem 2.3]) implies the alternative Daugavet property. Examples of spaces having the Daugavet property are $C(K)$ and $L_1(\mu)$, provided that K is perfect and μ does not have any atoms (see [45] for an elementary approach), and certain function algebras such as the disk algebra $A(\mathbb{D})$ or the algebra of bounded analytic functions H^∞ [46, 48]. A good introduction to the Daugavet equation is given in the books [3, 4] and the state-of-the-art on the subject can be found in the papers [25, 47]. For very recent results we refer the reader to [6, 8, 26, 27, 41] and references therein.

Let us mention that (aDE) appears in several papers from the 80's as [1, 2, 20, 21, 45], where it is proved that this equation is satisfied by all operators $T \in L(X)$ whenever $X = C(K)$ or $X = L_1(\mu)$. Actually, this result appeared in the 1970 paper [16], where (aDE) is related to a constant introduced by G. Lumer in 1968, the numerical index of a Banach space. Let us give the necessary definitions. Given an operator $T \in L(X)$, the *numerical radius* of T is

$$v(T) := \sup\{|x^*(Tx)| : x \in X, x^* \in X^*, \|x^*\| = \|x\| = x^*(x) = 1\}.$$

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The *numerical index* of the space X is

$$n(X) := \max \{k \geq 0 : k\|T\| \leq v(T) \text{ for all } T \in L(X)\}.$$

It has been shown in [16, pp. 483] that

$$\text{an operator } T \in L(X) \text{ satisfies (aDE) if and only if } v(T) = \|T\|,$$

thus, $n(X) = 1$ if and only if (aDE) is satisfied by all operators in $L(X)$. Therefore, the numerical index 1 implies the alternative Daugavet property. Examples of spaces with numerical index 1 are $L_1(\mu)$ and its isometric preduals, for any positive measure μ . For more information and background, we refer the reader to the monographs by F. Bonsall and J. Duncan [9, 10] and to the survey paper [35]. Recent results can be found in [28, 33, 37, 40, 42] and references therein. Let us just mention that one has $v(T^*) = v(T)$ for every $T \in L(X)$, where T^* is the adjoint operator of T (see [9, §9]) and it clearly follows that $n(X^*) \leq n(X)$ for every Banach space X .

The alternative Daugavet property does not imply neither the Daugavet property nor numerical index 1. Indeed, *the space $c_0 \oplus_\infty C([0, 1], \ell_2)$ has the alternative Daugavet property, but it does not have the Daugavet property and it does not have numerical index 1* (see [38, Example 3.2] for the details).

In [38], C^* -algebras having the alternative Daugavet property are characterized as those whose atomic projections are central. This result is deduced from a characterization of JB^* -triples having the alternative Daugavet property in terms of minimal tripotents. It is also proved in [38] that a JBW^* -triple (and so does a von Neumann algebra) has the alternative Daugavet property if (and only if) its predual does. The necessary definitions and basic results on JB^* -triples will be given in section 2 of the present paper.

Here, we give geometric characterizations of the alternative Daugavet property for JB^* -triples and for their isometric preduals. In particular, our results contain the above mentioned algebraic characterizations, but the proofs are independent.

To state the main results of the paper we need to fix notation and recall some definitions.

Let X be a Banach space. The symbols B_X and S_X denote, respectively, the closed unit ball and the unit sphere of X . We write $\text{co}(A)$ for the convex hull of the set A , and we denote by $\text{ext}(B)$ the set of extreme points in the convex set B . Let us fix u in S_X . We define the set $D(X, u)$ of all *states* of X relative to u by

$$D(X, u) := \{f \in B_{X^*} : f(u) = 1\},$$

which is a non-empty w^* -closed face of B_{X^*} . The norm of X is said to be *Fréchet-smooth* or *Fréchet differentiable* at $u \in S_X$ whenever there exists $\lim_{\alpha \rightarrow 0} \frac{\|u + \alpha x\| - 1}{\alpha}$ uniformly for $x \in B_X$. A point $x \in S_X$ is said to be *strongly exposed* if there exists $f \in D(X, x)$ such that $\lim \|x_n - x\| = 0$ for every sequence (x_n) of elements of B_X such that $\lim \text{Re } f(x_n) = 1$. If X is a dual space and f can be taken from the predual, we say that x is a *w^* -strongly exposed point*. It is known that x is strongly exposed if and only if there is a point of Fréchet-smoothness in $D(X, x)$, which is actually a strongly exposing functional for x (see [14, Corollary I.1.5]). Finally, if X and Y are Banach spaces, we write $X \oplus_1 Y$ and $X \oplus_\infty Y$ to denote, respectively, the ℓ_1 -sum and the ℓ_∞ -sum of X and Y .

The main results of the paper are the characterizations of the alternative Daugavet property for JB^* -triples and preduals of JBW^* -triples given in Theorems 2.6 and 2.3 respectively. For a JB^* -triple X , the following are equivalent:

- (i) X has the alternative Daugavet property.
- (ii) $|x^{**}(x^*)| = 1$ for every $x^{**} \in \text{ext}(B_{X^{**}})$ and every w^* -strongly exposed point x^* of B_{X^*} .
- (iii) Each elementary triple ideal of X is equal to \mathbb{C} .
- (iv) There exists a closed triple ideal Y with $c_0(\Gamma) \subseteq Y \subseteq \ell_\infty(\Gamma)$ for convenient index set Γ , and such that X/Y is non-atomic.
- (v) All minimal tripotents of X are diagonalizing.

For the predual X_* of a JBW^* -triple X , the following are equivalent:

- (i) X_* has the alternative Daugavet property,
- (ii) X has the alternative Daugavet property,
- (iii) $|x(x_*)| = 1$ for every $x \in \text{ext}(B_X)$ and every $x_* \in \text{ext}(B_{X_*})$.
- (iv) $X = \ell_\infty(\Gamma) \oplus_\infty \mathcal{N}$ for suitable index set Γ and non-atomic ideal \mathcal{N} .

Let us mention that geometric characterizations of the Daugavet property for C^* -algebras, JB^* -triples, and their isometric preduals, can be found in [6].

The outline of the paper is as follows. In section 1 we include some preliminary results on the alternative Daugavet property and on Banach spaces with numerical index 1. Section 2 is devoted to the above cited characterizations of the alternative Daugavet property for JB^* -triples and their isometric preduals, and we dedicate section 3 to particularize these results to C^* -algebras and von Neumann preduals, and to compile some results on C^* -algebras and von Neumann preduals with numerical index 1.

1 Preliminary results

We devote this section to summarize some results concerning the alternative Daugavet property and Banach spaces with numerical index 1, which we will use along the paper. Most of them appear implicitly in [33, 38], but there are no explicit references. Therefore, we include them here for the sake of completeness.

Our starting point is McGregor's characterization of finite-dimensional spaces with numerical index 1 [34, Theorem 3.1]: *A finite-dimensional space X satisfies $n(X) = 1$ if and only if $|x^*(x)| = 1$ for all extreme points $x \in B_X$ and $x^* \in B_{X^*}$.* For infinite-dimensional X , $\text{ext}(B_X)$ may be empty (e.g. c_0), so the right statement of McGregor's condition in this case should read

$$|x^{**}(x^*)| = 1 \quad \text{for every } x^* \in \text{ext}(B_{X^*}) \text{ and every } x^{**} \in \text{ext}(B_{X^{**}}). \quad (1)$$

One can easily show that this condition is sufficient to ensure $n(X) = 1$. Actually, a bit more general result can be also proved easily.

Lemma 1.1 *Let X be a Banach space and let A be a subset of S_{X^*} such that $B_{X^*} = \overline{\text{co}(A)}^{w^*}$. If*

$$|x^{**}(a^*)| = 1 \quad \text{for every } a^* \in A \text{ and every } x^{**} \in \text{ext}(B_{X^{**}}),$$

then $n(X) = 1$. In particular, condition (1) implies $n(X) = 1$.

Proof. Fix $T \in L(X)$. Since T^* is w^* -continuous, for every $\varepsilon > 0$ we can find $a^* \in A$ such that $\|T^*(a^*)\| \geq \|T\| - \varepsilon$. Now, take $x^{**} \in \text{ext}(B_{X^{**}})$ with

$$|x^{**}(T^*(a^*))| = \|T^*(a^*)\|,$$

and observe that $|x^{**}(a^*)| = 1$, giving

$$v(T) = v(T^*) \geq |x^{**}(T^*(a^*))| \geq \|T\| - \varepsilon. \quad \square$$

By considering strongly exposed points, two necessary conditions to have numerical index 1 are established in [33, Lemma 1]. Actually, as is noted in the Remark 6 of the same paper, we can replace the hypothesis of having numerical index 1 by the hypothesis of having the alternative Daugavet property.

Lemma 1.2 *Let X be a Banach space with the alternative Daugavet property. Then:*

- (a) $|x^{**}(x^*)| = 1$ for every $x^{**} \in \text{ext}(B_{X^{**}})$ and every w^* -strongly exposed point $x^* \in B_{X^*}$.
- (b) $|x^*(x)| = 1$ for every $x^* \in \text{ext}(B_{X^*})$ and every strongly exposed point $x \in B_X$.

Recall that the dual unit ball of an Asplund space is the w^* -closed convex hull of its w^* -strongly exposed points (see [43], for instance) so, by just applying the above two lemmas, we easily get the following characterization.

Corollary 1.3 *Let X be an Asplund space. Then, the following are equivalent:*

- (i) $n(X) = 1$.
- (ii) X has the alternative Daugavet property.
- (iii) $|x^{**}(x^*)| = 1$ for every $x^{**} \in \text{ext}(B_{X^{**}})$ and every w^* -strongly exposed point $x^* \in B_{X^*}$.

For similar results concerning Banach spaces with the Radon-Nikodým property and numerical index 1 we refer the reader to [36].

2 JB^* -triples and preduals of JBW^* -triples

We recall that a JB^* -triple is a complex Banach space X endowed with a continuous triple product $\{\cdot\cdot\cdot\} : X \times X \times X \rightarrow X$ which is linear and symmetric in the outer variables, conjugate-linear in the middle variable, and satisfies:

1. For all x in X , the mapping $y \mapsto \{xxy\}$ from X to X is a hermitian operator on X and has nonnegative spectrum.

2. The main identity

$$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$$

holds for all a, b, x, y, z in X .

3. $\|\{xxx\}\| = \|x\|^3$ for every x in X .

Concerning condition 1 above, we also recall that a bounded linear operator T on a complex Banach space X is said to be *hermitian* if $\|\exp(irT)\| = 1$ for every $r \in \mathbb{R}$. The main interest of JB^* -triples relies on the fact that, up to biholomorphic equivalence, there are no bounded symmetric domains in complex Banach spaces others than the open unit balls of JB^* -triples (see [29, 31]).

Every C^* -algebra becomes a JB^* -triple under the triple product

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x).$$

Moreover, norm-closed subspaces of C^* -algebras closed under the above triple product form the class of JB^* -triples known as JC^* -triples.

By a JBW^* -triple we mean a JB^* -triple whose underlying Banach space is a dual space in metric sense. It is known (see [5]) that every JBW^* -triple has a unique predual up to isometric linear isomorphisms and its triple product is separately w^* -continuous in each variable. We will apply without notice that the bidual of every JB^* -triple X is a JBW^* -triple under a suitable triple product which extends the one of X [15].

Let X be a JB^* -triple. Given $a, b \in X$, we write $D(a, b)$ for the multiplication operator $x \mapsto \{abx\}$. The elements a and b are said to be *orthogonal* if $D(a, b) = 0$ (equivalently, $D(b, a) = 0$). For $a \in X$, the conjugate linear operator $x \mapsto \{axa\}$ is denoted by Q_a . An element $u \in X$ is said to be a *tripotent* if $\{uuu\} = u$. Associated to any tripotent u , we define the *Peirce projections*

$$P_2(u) = Q_u^2, \quad P_1(u) = 2(D(u, u) - Q_u^2), \quad P_0(u) = \text{Id} - 2D(u, u) + Q_u^2,$$

which are mutually orthogonal with sum Id , and ranges

$$X_j(u) := P_j(u)(X) = \left\{ x : \{uuu\} = \frac{j}{2}x \right\} \quad (j = 0, 1, 2),$$

giving $X = X_2(u) \oplus X_1(u) \oplus X_0(u)$ (*Peirce decomposition*). A tripotent $u \in X$ is said to be *minimal* if $u \neq 0$ and $X_2(u) = \mathbb{C}u$ (equivalently, $\{uXu\} = \mathbb{C}u$) and u is said to be *diagonalizing* if $X_1(u) = 0$. A *triple ideal* of X is a subspace M of X such that $\{XXM\} + \{XMX\} \subseteq M$. It is well-known that the triple ideals of a JB^* -triple are precisely its M -ideals [5, Theorem 3.2]. The *socle* of X , $K_0(X)$, is the closed linear span of the

minimal tripotents of X . Then, $K_0(X)$ is a triple ideal of X , the one generated by the minimal tripotents in X . The triple X is said to be *non-atomic* if $K_0(X) = 0$, that is, if it does not contain any minimal tripotent. By [38, Theorem 4.7] or [6, Corollary 3.4], a JB^* -triple is non-atomic if and only if it has the Daugavet property. On the other side, a JBW^* -triple Y is said to be *atomic* if it equals the w^* -closure of its socle. The JBW^* -triple Y is said to be a *factor* if it cannot be written as an ℓ_∞ -sum of two nonzero ideals or, equivalently [23], if 0 and Y are the only w^* -closed ideals in Y . An especially important class of JBW^* -triples are the so-called *Cartan factors*. This class falls into six subclasses as follows, where H and H' are arbitrary complex Hilbert spaces and $J : H \rightarrow H$ is a conjugation (that is, a conjugate linear isometry with $J^2 = \text{Id}$):

- (1) *rectangular*: $L(H, H')$;
- (2) *symplectic*: $\{x \in L(H) : Jx^*J = -x\}$;
- (3) *hermitian*: $\{x \in L(H) : Jx^*J = x\}$;
- (4) *spin*: H with $\dim(H) \geq 3$, with a certain (non-Hilbertian) equivalent norm.
- (5) $M_{1,2}(\mathbb{O})$: the 1×2 matrices over the complex octonions \mathbb{O} ;
- (6) $H_3(\mathbb{O})$: the hermitian 3×3 matrices over \mathbb{O} .

The rectangular, symplectic and hermitian factors have the usual JC^* -triple product. For the definition of the other triple products see [31]. The Cartan factors are appropriately “irreducible” JBW^* -triples [22] in terms of which there is a useful representation theory. For instance, every JB^* -triple is isometrically isomorphic to a subtriple of an ℓ_∞ -sum of Cartan factors [19].

A JB^* -triple M is said to be *elementary* if it is isometric to the socle $K_0(C)$ of a Cartan factor C . We have $K_0(C)^{**} = C$, and that a JB^* -triple is elementary if and only if its bidual is a Cartan factor [11, Lemma 3.2]. Let us also recall that, for every JB^* -triple X , $K_0(X)$ is equal to the c_0 -sum of all elementary triple ideal of X [12, Lemma 3.3].

Of course, no elementary JB^* -triple has the Daugavet property. The following result says that, actually, not so many elementary JB^* -triples have the alternative Daugavet property.

Proposition 2.1 *The only Cartan factor which has the alternative Daugavet property is \mathbb{C} . Actually, the only elementary JB^* -triple which has the alternative Daugavet property is \mathbb{C} .*

To prove the above proposition, we need the following easy consequence of [18, Proposition 4] and [7, Corollary 2.11] (see the proof of [6, Theorem 3.2]). For the sake of completeness, we include a proof here.

Lemma 2.2 *Let X be a JBW^* -triple. Then, every extreme point x_* of B_{X_*} is strongly exposed by a minimal tripotent of X .*

Proof. Given $x_* \in \text{ext}(B_{X_*})$, [18, Proposition 4] assures the existence of a minimal tripotent u of X such that $u(x_*) = 1$. But u is a point of Fréchet-smoothness of the norm of X [7, Corollary 2.11] so, as we commented in the introduction, this implies that x_* is strongly exposed by u (see [14, Corollary I.1.5], for instance). \square

Proof of Proposition 2.1. Let M be an elementary JB^* -triple and write $C = M^{**}$, which is a Cartan factor. Suppose that M has the alternative Daugavet property. By the above lemma, every extreme point x^* of B_{M^*} is strongly exposed by a minimal tripotent of M^{**} , which is actually in M , so x^* is a w^* -strongly exposed point of B_{M^*} . Then, Lemma 1.2 gives us that

$$|e(x_*)| = 1 \quad (e \in \text{ext}(B_C), x_* \in \text{ext}(B_{C_*})). \quad (2)$$

We claim that

$$|u(x_*)| \in \{0, 1\} \quad (3)$$

for every minimal tripotent u of C and every $x_* \in \text{ext}(B_{C_*})$. Indeed, we fix a minimal tripotent u of C and $x_* \in \text{ext}(B_{C_*})$, and we find a maximal orthogonal family of minimal tripotents $\{u_i\}_{i \in I}$ such that $u_{i_0} = u$ for

some $i_0 \in I$. Now, for each family $\{\lambda_i\}_{i \in I}$ of elements of \mathbb{T} , we consider $e = \sum_{i \in I} \lambda_i u_i$, which is a complete tripotent of C [44, Lemma 2.1] and thus, an extreme point of B_C [32], so Eq. (2) gives

$$\left| \sum_{i \in I} \lambda_i u_i(x_*) \right| = |e(x_*)| = 1.$$

In particular, choosing a suitable family $\{\lambda_i\}_{i \in I}$, we get

$$\sum_{i \in I} |u_i(x_*)| = 1.$$

If we write $\alpha = u_{i_0}(x_*)$, $\beta = \sum_{i \in I \setminus \{i_0\}} u_i(x_*)$, the above two equations give that

$$|\alpha| \leq 1, |\beta| \leq 1, \quad \text{and} \quad |\alpha + \lambda\beta| = 1 \quad \text{for every } \lambda \in \mathbb{T}.$$

This implies that $|\alpha|, |\beta| \in \{0, 1\}$, and the claim is proved.

Now, the set of minimal tripotents of a Cartan factor is connected; indeed, for Cartan factors of finite-rank this can be easily deduced from [30, (4.5) and (4.6)], and the remaining examples follow by a direct inspection (compare [30, Section 3]). Therefore, Eq. (3) gives us that $|u(x_*)| = 1$ for every minimal tripotent u of C and every $x_* \in \text{ext}(B_{C_*})$. This implies that every minimal tripotent of C is complete (there is no pair of orthogonal minimal tripotents in C) and thus, $\text{rank}(C) = 1$ and C is a Hilbert space [13]; but, obviously, if a Hilbert space satisfies (2), then it is one-dimensional. \square

If X is a JBW^* -triple, it is well known that $X_* = A \oplus_1 N$, where A is the closed linear span of the extreme points of B_{X_*} , and the unit ball of N has no extreme points [18]. Therefore, $X = \mathcal{A} \oplus_\infty \mathcal{N}$, where $\mathcal{A} = N^\perp \equiv A^*$ is atomic and $\mathcal{N} = A^\perp \equiv N^*$ is non-atomic. On the one hand, by [19, Proposition 2.2] and [22, Corollary 1.8], an atomic JBW^* -triple is the ℓ_∞ -sum of Cartan factors. On the other hand, the non-atomic part \mathcal{N} of X has the Daugavet property, hence the alternative Daugavet property. Now, since an ℓ_∞ -sum of Banach spaces has the alternative Daugavet property if and only if all the summands do [38, Proposition 3.1], the above proposition says that X has the alternative Daugavet property if and only if all the Cartan factors appearing in the atomic part are equal to \mathbb{C} . This is part of the following result, which characterizes those JBW^* -triples whose preduals have the alternative Daugavet property.

Theorem 2.3 *Let X be a JBW^* -triple. Then, the following are equivalent:*

- (i) X has the alternative Daugavet property.
- (ii) X_* has the alternative Daugavet property.
- (iii) $|x(x_*)| = 1$ for every $x \in \text{ext}(B_X)$ and every $x_* \in \text{ext}(B_{X_*})$.
- (iv) The atomic part of X is isometrically isomorphic to $\ell_\infty(\Gamma)$ for a convenient set Γ ; i.e. $X = \ell_\infty(\Gamma) \oplus_\infty \mathcal{N}$ where \mathcal{N} is non-atomic.

Proof. (i) \Rightarrow (ii) is clear.

(ii) \Rightarrow (iii). Just apply Lemmas 1.2 and 2.2.

(iii) \Rightarrow (iv). By the comments preceding this theorem, we only have to prove that every Cartan factor appearing in the atomic decomposition of X is equal to \mathbb{C} . Let us fix such a Cartan factor C , and let K be an elementary JB^* -triple with $K^{**} = C$. Since K^* is an L -summand of X_* and K^{**} is an M -summand of X , it is straightforward to show that condition (iii) implies

$$|x^{**}(x^*)| = 1 \quad (x^{**} \in \text{ext}(B_{K^{**}}), x^* \in \text{ext}(B_{K^*})).$$

Now, Lemma 1.1 implies that $n(K) = 1$, so K has the alternative Daugavet property and Proposition 2.1 gives us that K (and so does C) is equal to \mathbb{C} .

(iv) \Rightarrow (i). Since $n(\ell_\infty(\Gamma)) = 1$ and \mathcal{N} has the Daugavet property, both have the alternative Daugavet property and so does its ℓ_∞ -sum [38, Proposition 3.1]. \square

The equivalences (i) \Leftrightarrow (ii) \Leftrightarrow (iv) of the above theorem appear in [38, Theorem 4.6 and Corollary 4.8] with a different proof.

Just applying Theorem 2.3 and Lemma 1.1 to the JBW^* -triple X^{**} , we get the following result.

Corollary 2.4 *Let X be a JB^* -triple. If X^* has the alternative Daugavet property, then $|x^{**}(x^*)| = 1$ for every $x^{**} \in \text{ext}(B_{X^{**}})$ and every $x^* \in \text{ext}(B_{X^*})$ and, in particular, $n(X) = 1$.*

Remark 2.5 It is worth mentioning that, for an arbitrary Banach space Z , the condition

$$|z^*(z)| = 1 \quad \text{for every } z \in \text{ext}(B_Z) \text{ and every } z^* \in \text{ext}(B_{Z^*})$$

does not necessarily imply that Z has the alternative Daugavet property. Indeed, let H be the 2-dimensional Hilbert space and let us consider $Z = c_0(H)$. Since $\text{ext}(B_Z) = \emptyset$, it is clear that Z satisfies the above condition. But since Z is an Asplund space and $n(Z) = n(H) < 1$ by [39, Proposition 1], Corollary 1.3 gives us that Z does not have the alternative Daugavet property.

The last result of the section is a characterization of the alternative Daugavet property for JB^* -triples.

Theorem 2.6 *Let X be a JB^* -triple. Then, the following are equivalent:*

- (i) X has the alternative Daugavet property.
- (ii) $|x^{**}(x^*)| = 1$ for every $x^{**} \in \text{ext}(B_{X^{**}})$ and every w^* -strongly exposed point x^* of B_{X^*} .
- (iii) Each elementary triple ideal of X is equal to \mathbb{C} ; equivalently, $K_0(X)$ is isometric to $c_0(\Gamma)$ for some index set Γ .
- (iv) There exists a closed triple ideal Y with $c_0(\Gamma) \subseteq Y \subseteq \ell_\infty(\Gamma)$ for convenient index set Γ , such that X/Y is non-atomic.
- (v) All minimal tripotents of X are diagonalizing.

Proof. (i) \Rightarrow (ii). This is the first part of Lemma 1.2.

(ii) \Rightarrow (iii). Let K be an elementary triple ideal of X . Since K is an M -ideal of X , it is clear that condition (ii) goes down to K . Since K is an Asplund space, Corollary 1.3 gives us that K has the alternative Daugavet property, so $K = \mathbb{C}$ by Proposition 2.1.

(iii) \Rightarrow (iv). The JBW^* -triple X^{**} decomposes into its atomic and non-atomic part as $X^{**} = \mathcal{A} \oplus_\infty \mathcal{N}$. Let us consider $Y = \mathcal{A} \cap X$. By [11, Proposition 3.7], Y is an Asplund space and X/Y is non-atomic. Since $K_0(Y) = K_0(X)$ (see [12, Corollary 3.5]) and $K_0(X) = c_0(\Gamma)$, Proposition 4.4 of [12] shows that

$$c_0(\Gamma) \subset Y \subset c_0(\Gamma)^{**} = \ell_\infty(\Gamma).$$

(iv) \Rightarrow (i). On the one hand, X/Y has the alternative Daugavet property since it has the Daugavet property. On the other hand, a sight to [17, Remark 8] gives us that $n(Y) = 1$ and thus, Y has the alternative Daugavet property. Since Y is an M -ideal, Proposition 3.4 of [38] gives us that X has the alternative Daugavet property.

(iii) \Leftrightarrow (v). On the one hand, a minimal tripotent of X is diagonalizing if and only if it is diagonalizing in the unique Cartan factor of X^{**} containing it. On the other hand, a Cartan factor has a diagonalizing minimal tripotent (if and) only if it is equal to \mathbb{C} ; indeed, if a Cartan factor C has a minimal diagonalizing tripotent u , then $C = C_0(u) \oplus C_2(u) = C_0(u) \oplus \mathbb{C}u$; but the above direct sum is an ℓ_∞ -sum (see [18, Lemma 1.3]) so, by the definition of a factor, $C_0(u) = 0$. \square

The equivalence (i) \Leftrightarrow (v) of the above theorem appears in [38, Theorem 4.5] with a different proof.

Remark 2.7 It is worth mentioning that condition (ii) of Theorem 2.6 for arbitrary Banach spaces does not necessarily imply the alternative Daugavet property. For instance, let H be the two-dimensional Hilbert space and let us consider $X = \ell_1(H)$. Since X has the Radon-Nikodým property and $n(X) = n(\ell_1(H)) < 1$ by [39, Proposition 1], X does not have the alternative Daugavet property (see [33, Remark 6]). But since the norm of X is not Fréchet-smooth at any point, the unit ball of X^* does not have any w^* -strongly exposed point, thus condition (ii) of Theorem 2.6 is satisfied.

3 C^* -algebras and von Neumann preduals

The last section of the paper is devoted to specialize the results of the above section to C^* -algebras and von Neumann preduals. Let us introduce the basic definitions and some preliminary results.

If X is a C^* -algebra, we write $Z(X)$ for the center of the algebra, i.e. the subalgebra consisting of those elements of X which commute with all elements of X . A *projection* in X is an element $p \in X$ such that $p^* = p$

and $p^2 = p$. The tripotents of X are the *partial isometries*, i.e. elements $u \in X$ satisfying that $uu^*u = u$. It is clear that projections are partial isometries (and so tripotents), but there are partial isometries which are not projections. A projection p in X is said to be *atomic* if $p \neq 0$ and $pXp = \mathbb{C}p$. The C^* -algebra X is said to be *non-atomic* if it does not have any atomic projection. It is easy to show (see [38, pp. 170] or the paragraph before [6, Corollary 4.4], for example) that a partial isometry u in X is minimal if and only if $d = u^*u$ and $r = uu^*$ (the domain and range projections associated to u) are atomic. Therefore, a C^* -algebra is non-atomic (as an algebra) if and only if it does not have any minimal tripotent, i.e. it is non-atomic as a JB^* -triple.

The following two corollaries particularize the main results of the paper to the case of C^* -algebras. The first one is a direct consequence of Theorem 2.3; the second one follows from Theorem 2.6 with just a little bit of work.

Corollary 3.1 *Let X be a von Neumann algebra. Then, the following are equivalent:*

- (i) X has the alternative Daugavet property.
- (ii) X_* has the alternative Daugavet property.
- (iii) $|x(x_*)| = 1$ for every $x \in \text{ext}(B_X)$ and every $x_* \in \text{ext}(B_{X_*})$.
- (iv) $X = \mathcal{C} \oplus_\infty \mathcal{N}$ where \mathcal{C} is commutative and \mathcal{N} is non-atomic.

Corollary 3.2 *Let X be a C^* -algebra. Then, the following are equivalent:*

- (i) X has the alternative Daugavet property.
- (ii) $|x^{**}(x^*)| = 1$ for every $x^{**} \in \text{ext}(B_{X^{**}})$ and every w^* -strongly exposed point x^* of B_{X^*} .
- (iii) There exists a two-sided commutative ideal Y such that X/Y is non-atomic.
- (iv) $K_0(X)$ is isometric to $c_0(\Gamma)$.
- (v) $K_0(X)$ is commutative.
- (vi) All atomic projections in X are central.
- (vii) $K_0(X) \subseteq Z(X)$.

Proof. (i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) follows from Theorem 2.6.

(iv) \Leftrightarrow (v). We have that $K_0(X) = [\oplus_{\gamma \in \Gamma} K(H_\gamma)]_{c_0}$ for convenient index set Γ and convenient family of Hilbert spaces $\{H_\gamma\}$. Thus, $K_0(X)$ is commutative if and only if $\dim(H_\gamma) \leq 1$ for every $\gamma \in \Gamma$.

(v) \Rightarrow (vi). First, we observe that every atomic projection of X is contained in $K_0(X)$. Now, we take an atomic projection $p \in X$ and we claim that $p \in Z(X)$. Indeed, fix $x \in X$ and observe that $p, px, xp \in K_0(X)$ (since it is an ideal of X), so

$$px = p(px) = (px)p = p(xp) = (xp)p = xp.$$

(vi) \Rightarrow (vii). We are going to show that every minimal partial isometry on X is central, which implies $K_0(X) \subseteq Z(X)$. Indeed, let $u \in X$ be a minimal partial isometry. We write $d = u^*u$ and $r = uu^*$ for the domain and range projections of u , which are atomic and so, $d, r \in Z(X)$. Then

$$r = r^2 = (uu^*)(uu^*) = u((u^*u)u^*) = u(u^*(u^*u)) = u(u^*u^*)u = \lambda u,$$

where the last equality comes from the fact that u is minimal. Since $r \neq 0$ (otherwise $u = ru = 0$), $u \in \text{span}(r) \subseteq Z(X)$.

(vii) \Rightarrow (v) is immediate. □

Next, we compile some known results concerning C^* -algebras and von Neumann preduals having numerical index 1. Since we have not found explicitly some of these results in the literature, and we can now relate them to the alternative Daugavet property, we include them here for the sake of completeness.

C^* -algebras with numerical index 1 have been characterized in [24] (see also [28, pp. 202]) as the commutative ones. With not much work, we can obtain the following result.

Proposition 3.3 *Let X be a C^* -algebra. Then, the following are equivalent:*

- (i) $n(X^*) = 1$.
- (ii) X^* has the alternative Daugavet property.
- (iii) $|x^{**}(x^*)| = 1$ for every $x^{**} \in \text{ext}(B_{X^{**}})$ and every $x^* \in \text{ext}(B_{X^*})$.
- (iv) $n(X) = 1$.

(v) X is commutative.

Proof. (i) \Rightarrow (ii) is clear. (ii) \Rightarrow (iii) is Corollary 2.4. (iii) \Rightarrow (iv) is Lemma 1.1. (iv) \Rightarrow (v) is the already mentioned result of [24]. Finally, (v) \Rightarrow (i) follows from the fact that X^{**} is also commutative and so, $n(X^*) \geq n(X^{**}) = 1$. \square

Remark 3.4 In [38, Example 4.4] an example is given of a Banach space with the alternative Daugavet property whose dual does not share this property. The above proposition gives a lot of examples of this kind: every non-atomic and non-commutative C^* -algebra. Actually, the example given there fits this scheme.

For von Neumann preduals, we have the following.

Proposition 3.5 *Let X be a von Neumann algebra. Then, the following are equivalent:*

- (i) $n(X) = 1$.
- (ii) $|x^*(x)| = 1$ for every $x^* \in \text{ext}(B_{X^*})$ and every $x \in \text{ext}(B_X)$.
- (iii) $n(X_*) = 1$.

Proof. (i) \Rightarrow (ii) is immediate, since X is commutative and hence, a $C(K)$ space.

(ii) \Rightarrow (iii) follows from Lemma 1.1 applied to X_* .

(iii) \Rightarrow (i) is done in [28, Proposition 1.4]. \square

Let us finish the paper by raising the following question: *is it true that, for a Banach space Y , $n(Y^*) = 1$ whenever $n(Y) = 1$?* This is the case when Y is a C^* -algebra or Y is the predual of a von Neumann algebra (Propositions 3.3 and 3.5 above), but we do not know the answer even in the case when Y is a JB^* -triple or when Y is the predual of a JBW^* -triple.

Note added in May 2006. Very recently, K. Boyko, V. Kadets, M. Martín, and D. Werner [Numerical index and duality, *Math. Proc. Cambridge Phil. Soc.* (to appear)] have shown that there exists a Banach space Y with $n(Y) = 1$ and such that $n(Y^*) < 1$. The same space Y also shows that condition (1) in page 3 is not necessary to have numerical index 1.

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