NORM EQUALITIES FOR OPERATORS ON BANACH SPACES

Vladimir Kadets,

Faculty of Mechanics and Mathematics
Kharkov National University
pl. Svobody 4, 61077 Kharkov, Ukraine
E-mail: vova1kadets@yahoo.com

Miguel Martín, and Javier Merí

Departamento de Análisis Matemático
Facultad de Ciencias
Universidad de Granada
18071 Granada, Spain
E-mail: mmartins@ugr.es, jmeri@ugr.es

Abstract. A Banach space X has the Daugavet property if the Daugavet equation $\|\mathrm{Id}+T\| = 1+\|T\|$ holds for every rank-one operator $T : X \longrightarrow X$. We show that the most natural attempts to introduce new properties by considering other norm equalities for operators (like $\|g(T)\| = f(\|T\|)$ for some functions f and g) lead in fact to the Daugavet property of the space. On the other hand there are equations (for example $\|\mathrm{Id} + T\| = \|\mathrm{Id} - T\|$) that lead to new, strictly weaker properties of Banach spaces.

1. Introduction

The purpose of this paper is to study equalities involving the norm of bounded linear operators on Banach spaces, and to discuss the possibility of defining isometric properties for Banach spaces by requiring that all operators of a suitable class satisfy such a norm equality.

The interest in this topic goes back to 1963, when the Russian mathematician I. K. Daugavet [10] showed that each compact operator T on $C[0,1]$ satisfies the norm equality

$$\|\mathrm{Id} + T\| = 1 + \|T\|. \tag{DE}$$

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The above equation is nowadays referred to as *Daugavet equation*. Few years later, this result was extended to various classes of operators on some Banach spaces, including weakly compact operators on $C(K)$ for perfect K and on $L_1(\mu)$ for atomless μ (see [27] for an elementary approach). A new wave of interest in this topic surfaced in the eighties, when the Daugavet equation was studied by many authors in various contexts. Let us cite, for instance, that a compact operator T on a uniformly convex Banach space (in particular, on a Hilbert space) satisfies (DE) only if the norm of T is an eigenvalue [3]. We refer the reader to the books [1, 2] for a brief study of this equation from different points of view.

In the late nineties, new ideas were infused into this field and, instead of looking for new spaces and new classes of operators on them for which (DE) is valid, the geometry of Banach spaces having the so-called Daugavet property was studied. Following [18, 19], we say that a Banach space X has the *Daugavet property* if every rank-one operator $T \in L(X)$ satisfies (DE) (we write $L(X)$ for the Banach algebra of all bounded linear operators on X). In such a case, every operator on X not fixing a copy of ℓ_1 also satisfies (DE) [26]; in particular, this happens to every compact or weakly compact operator on X [19]. There are several characterizations of the Daugavet property which do not involve operators (see [19, 28]). For instance, a Banach space X has the Daugavet property if and only if for every $x \in S_X$ and every $\varepsilon > 0$ the closed convex hull of the set

$$B_X \setminus \big(x + (2-\varepsilon)\, B_X\big)$$

coincides with the whole B_X. Here and subsequently, B_X and S_X stand, respectively, for the closed unit ball and the unit sphere of a Banach space X. Let us observe that the above characterization shows that the Daugavet property is somehow extremely opposite to the Radon-Nikodým property.

Although the Daugavet property is clearly of isometric nature, it induces various isomorphic restrictions. For instance, a Banach space with the Daugavet property does not have the Radon-Nikodým property [29] (actually, every slice of the unit ball has diameter 2 [19]), it contains ℓ_1 [19], it does not have unconditional basis [15] and, moreover, it does not isomorphically embed into an unconditional sum of Banach spaces without a copy of ℓ_1 [26]. It is worthwhile to remark that the latter result continues a line of generalization ([16], [17], [19]) of the known theorem of A. Pełczyński [24] from 1961 saying that $L_1[0,1]$ (and so $C[0,1]$) does not embed into a space with unconditional basis.

The state-of-the-art on the Daugavet property can be found in [28]; for very recent results we refer the reader to [4, 6, 14] and references therein.

In view of the deep consequences that the Daugavet property has on the geometry of a Banach space, one may wonder whether it is possible to define other interesting properties by requiring all rank-one operators on a Banach space to satisfy a suitable norm equality. This is the aim of the present paper.

Let us give some remarks on the question which will also serve to present the outline of our further discussion. First, the Daugavet property clearly implies that the norm of $\mathrm{Id} + T$ only depends on the norm of T. Then, a possible generalization of the Daugavet property is to require that every rank-one operator T on a Banach space X satisfies a norm equality of the form

$$\|\mathrm{Id} + T\| = f(\|T\|)$$

for a fixed function $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$. It is easy to show (see Proposition 2.2) that the only property which can be defined in this way is the Daugavet property. Therefore, we should look for equations in which $\mathrm{Id}+T$ is replaced by another function of T, i.e. we fix functions g and f and we require that every rank-one operator T on a Banach space X satisfies the norm equality

$$\|g(T)\| = f(\|T\|).$$

We need g to carry operators to operators and to apply to arbitrary rank-one operators, so it is natural to impose g to be a power series with infinite radius of convergence, i.e. an entire function (when $\mathbb{K} = \mathbb{C}$ this is the usual definition; when $\mathbb{K} = \mathbb{R}$, g is the restriction to $\mathbb{R}$ of a complex entire function which carries the real line into itself). Again, the

only non trivial possibility is the Daugavet property, as we will show in section 3. Section 4 is devoted to the last kind of equations we would like to study. Concretely, we consider an entire function g, a continuous function f, and a Banach space X, and we require each rank-one operator $T \in L(X)$ to satisfy the norm equality

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|). \tag{1}$$

If X is a Banach space with the Daugavet property and g is an entire function, then it is easy to see that the norm equality

$$\|\mathrm{Id} + g(T)\| = |1 + g(0)| - |g(0)| + \|g(T)\|$$

holds for every rank-one $T \in L(X)$. Therefore, contrary to the previous cases, our aim here is not to show that only few functions g are possible in (1), but to prove that many functions g produce the same property. Unfortunately, we have to separate the complex case and the real case, and only in the first one we are able to give fully satisfactory results. More concretely, we consider a complex Banach space X, an entire function g and a continuous function f, such that (1) holds for every rank-one operator $T \in L(X)$. If $\mathrm{Re}\, g(0) \neq -1/2$, then X has the Daugavet property. Surprisingly, the result is not true when $\mathrm{Re}\, g(0) = -1/2$ and another family of properties strictly weaker than the Daugavet property appears: there exists a modulus one complex number ω such that the norm equality

$$\|\mathrm{Id} + \omega\, T\| = \|\mathrm{Id} + T\| \tag{2}$$

holds for every rank-one $T \in L(X)$. In the real case, the discussion above depends upon the surjectivity of g, and there are many open questions when

g is not onto. Finally, we give in section 5 some remarks about the properties defined by norm equalities of the form given in (2).

We finish the introduction by commenting that, although we have been unable to find any result like the above ones in the literature, there are several papers in which the authors work on inequalities which remind the Daugavet equation. For instance, it is proved in [5] (see also [25, §9]) that for every $1 < p < \infty$, $p \neq 2$, there exists a function $\varphi_p : (0, \infty) \longrightarrow (0, \infty)$ such that the inequality

$$\|\mathrm{Id} + T\| \geqslant 1 + \varphi_p(\|T\|) \tag{3}$$

holds for every nonzero compact operator T on $L_p[0,1]$ (see [8] for the φ_p estimates). This result has been recently carried to non-commutative $\mathcal{L}_p$-spaces and to some spaces of operators [22, 23]. Finally, inequalities as (3) in which φ_p is linear are studied in [20].

2. Preliminaries

Let us start by fixing some notation. We use the symbols $\mathbb{T}$ and $\mathbb{D}$ to denote, respectively, the unit sphere and the closed unit ball of the base field $\mathbb{K}$, and we write $\mathrm{Re}(\cdot)$ to denote the real part function, which is nothing than the identity when $\mathbb{K} = \mathbb{R}$. Given a Banach space X, the dual space of X is denoted by X^* and, when X is a complex space, $X_{\mathbb{R}}$ is the underlying real space. Given $x \in X$ and $x^* \in X^*$, we write $x^* \otimes x$ to denote the bounded linear operator

$$y \longmapsto x^*(y)\, x \qquad (y \in X),$$

whose norm is equal to $\|x\|\,\|x^*\|$.

It is straightforward to check that if an operator T on a Banach space X satisfies (DE), then all the operators of the form αT for $\alpha > 0$ also satisfy (DE) (see [3, Lemma 2.1] for instance). We will use this fact along the paper without explicit mention.

Next, let us observe that the definition of the Daugavet property for complex spaces may be a bit confusing, since the meaning of a rank-one operator can be understood in two different forms. Indeed, an operator T on a Banach space X is a rank-one operator if $\dim T(X) \leqslant 1$. Then, when X is a complex space, two different classes of rank-one operators may be considered: the complex-linear operators whose ranks have complex-dimension one (i.e. operators of the form $x^* \otimes x$ for $x^* \in X^*$ and $x \in X$) and, on the other hand, the real-linear operators whose ranks have real-dimension one (i.e. operators of the form $\mathrm{Re}\, x^* \otimes x$ for $\mathrm{Re}\, x^* \in (X_{\mathbb{R}})^*$ and $x \in X_{\mathbb{R}}$). As a matter of facts, the two possible definitions of the Daugavet property that one may state are equivalent.

Remark 2.1. Let X be a complex Banach space. Then, the following are equivalent:

(i) Every (real) rank-one operator on $X_{\mathbb{R}}$ satisfies (DE).
(ii) Every (complex) rank-one operator on X satisfies (DE).

Proof. $(i) \Rightarrow (ii)$ follows immediately from [19, Theorem 2.3], but a direct proof is very easy to state. Given a complex rank-one operator $T = x_0^* \otimes x_0$ with $\|x_0^*\| = 1$ and $\|x_0\| = 1$, we have $\|\mathrm{Id} + \operatorname{Re} x_0^* \otimes x_0\| = 2$. Therefore, given $\varepsilon > 0$, there exists $x \in S_X$ such that $\|x + \operatorname{Re} x_0^*(x)x_0\| \geqslant 2 - \varepsilon$. It follows that

$$|\operatorname{Re} x_0^*(x)| \geqslant 1 - \varepsilon, \qquad \text{and so} \qquad |\operatorname{Im} x_0^*(x)| \leqslant \varepsilon.$$

Now, it is clear that

$$\begin{aligned}\|\mathrm{Id} + T\| &\geqslant \|x + Tx\| = \|x + x_0^*(x)x_0\| \\ &\geqslant \|x + \operatorname{Re} x_0^*(x)x_0\| - \|\operatorname{Im} x_0^*(x)x_0\| \geqslant 2 - 2\varepsilon.\end{aligned}$$

$(ii) \Rightarrow (i)$. Let $T = \operatorname{Re} x_0^* \otimes x_0$ be a real rank-one operator with

$$\|\operatorname{Re} x_0^*\| = \|x_0^*\| = 1 \qquad \text{and} \qquad \|x_0\| = 1.$$

By (ii), we have that $\|\mathrm{Id} + x_0^* \otimes x_0\| = 2$ so, given $\varepsilon > 0$, there exists $x \in S_X$ satisfying

$$\|x + x_0^*(x)x_0\| \geqslant 2 - \varepsilon.$$

If we take $\omega \in \mathbb{T}$ such that $\omega x_0^*(x) = |x_0^*(x)|$, then

$$x_0^*(\omega x) = \omega x_0^*(x) = \operatorname{Re} x_0^*(\omega x).$$

Therefore,

$$\begin{aligned}\|\mathrm{Id} + T\| &\geqslant \|\omega x + \operatorname{Re} x_0^*(\omega x)x_0\| \\ &= \|\omega x + \omega x_0^*(x)x_0\| = \|x + x_0^*(x)x_0\| \geqslant 2 - \varepsilon.\end{aligned}$$

□

From now on, by a rank-one operator on a Banach space over $\mathbb{K}$, we will mean a bounded $\mathbb{K}$-linear operator whose image has $\mathbb{K}$-dimension less or equal than one.

As we commented in the introduction, the aim of this paper is to discuss whether there are other isometric properties apart from the Daugavet property which can be defined by requiring all rank-one operators on a Banach space to satisfy a norm equality. A first observation is that the Daugavet property implies that for every rank-one operator T, the norm of $\mathrm{Id} + T$ only depends on the norm of T. It is easy to check that the above fact only may happen if the Banach space involved has the Daugavet property. We state and prove a slightly more general version of this result which we will use later on.

Proposition 2.2. *Let $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ be an arbitrary function. Suppose that there exist $a, b \in \mathbb{K}$ and a non-null Banach space X over $\mathbb{K}$ such that the norm equality*

$$\|a\mathrm{Id} + b\,T\| = f(\|T\|)$$

holds for every rank-one operator $T \in L(X)$. Then, $f(t) = |a| + |b|\,t$ for every $t \in \mathbb{R}_0^+$. In particular, if $a \neq 0$ and $b \neq 0$, then X has the Daugavet property.

Proof. If $a\,b = 0$ we are trivially done, so we may assume that $a \neq 0$, $b \neq 0$ and we write $\omega_0 = \frac{\bar{b}}{|b|}\frac{a}{|a|} \in \mathbb{T}$. Now, we fix $x_0 \in S_X$, $x_0^* \in S_{X^*}$ such that $x_0^*(x_0) = \omega_0$

and, for each $t \in \mathbb{R}_0^+$, we consider the rank-one operator $T_t = t\, x_0^* \otimes x_0 \in L(X)$. Observe that $\|T_t\| = t$, so we have

$$f(t) = \|a\mathrm{Id} + b\, T_t\| \qquad (t \in \mathbb{R}_0^+).$$

Then, it follows that

$$\begin{aligned} |a| + |b|\, t \geqslant f(t) &= \|a\mathrm{Id} + b\, T_t\| \geqslant \big\|[a\mathrm{Id} + b\, T_t](x_t)\big\| \\ &= \|a\, x_t + b\, \omega_0 t\, x_t\| = |a + b\, \omega_0 t|\, \|x_t\| \\ &= \left| a + b\frac{\overline{b}}{|b|}\frac{a}{|a|}\, t\right| = |a| + |b|\, t. \end{aligned}$$

Finally, if the norm equality

$$\|a\mathrm{Id} + b\, T\| = |a| + |b|\, \|T\|$$

holds for every rank-one operator on X, then X has the Daugavet property. Indeed, we fix a rank-one operator $T \in L(X)$ and apply the above equality to $S = \frac{a}{b}\, T$ to get

$$|a|\big(1 + \|T\|\big) = |a| + |b|\, \|S\| = \|a\mathrm{Id} + b\, S\| = |a|\, \|\mathrm{Id} + T\|. \qquad \square$$

With the above property in mind, we have to look for Daugavet-type norm equalities in which $\mathrm{Id} + T$ is replaced by another function of T. If we want such a function to carry operators to operators and to be applied to arbitrary rank-one operators on arbitrary Banach spaces, it is natural to consider power series with infinite radius of convergence. Let us introduce some notation. We say that $g : \mathbb{K} \longrightarrow \mathbb{K}$ is an *entire function* if g is represented by an everywhere convergent Taylor series; in other words, when $\mathbb{K} = \mathbb{C}$ this is the usual definition of entire function, but when $\mathbb{K} = \mathbb{R}$, g is the restriction to $\mathbb{R}$ of a complex entire function which carries the real line into itself. Given an entire function g, for each operator $T \in L(X)$, we define

$$g(T) = \sum_{k=0}^{\infty} a_k\, T^k,$$

where $g(\zeta) = \sum_{k=0}^{\infty} a_k\, \zeta^k$ is the power series expansion of g. The following easy result shows how to calculate $g(T)$ when T is a rank-one operator.

Lemma 2.3. *Let $g : \mathbb{K} \longrightarrow \mathbb{K}$ be an entire function with power series expansion*

$$g(\zeta) = \sum_{k=0}^{\infty} a_k\, \zeta^k \qquad (\zeta \in \mathbb{K}),$$

and let X be a Banach space over $\mathbb{K}$. For $x^* \in X^*$ and $x \in X$, we write $T = x^* \otimes x$ and $\alpha = x^*(x)$. Then, for each $\lambda \in \mathbb{K}$,

$$g(\lambda T) = \begin{cases} a_0\mathrm{Id} + a_1 \lambda T & \text{if } \alpha = 0 \\ a_0\mathrm{Id} + \frac{\widetilde{g}(\alpha\lambda)}{\alpha} T & \text{if } \alpha \neq 0, \end{cases}$$

where

$$\widetilde{g}(\zeta) = g(\zeta) - a_0 \qquad \big(\zeta \in \mathbb{K}\big).$$

Proof. Given $\lambda \in \mathbb{K}$, it is immediate to check that

$$(\lambda T)^k = \alpha^{k-1}\lambda^k T \qquad (k \in \mathbb{N}).$$

Now, if $\alpha = 0$, then $T^2 = 0$ and the result is clear. Otherwise, we have

$$\begin{aligned} g(\lambda T) &= a_0 \mathrm{Id} + \sum_{k=1}^{\infty} a_k\, \alpha^{k-1}\, \lambda^k\, T \\ &= a_0 \mathrm{Id} + \left(\frac{1}{\alpha}\sum_{k=1}^{\infty} a_k\, \alpha^k \lambda^k\right) T = a_0 \mathrm{Id} + \frac{\widetilde{g}(\alpha\lambda)}{\alpha} T. \end{aligned}$$

□

3. Norm equalities of the form $\|g(T)\| = f(\|T\|)$

We would like to study now norm equalities for operators of the form

$$\|g(T)\| = f(\|T\|), \tag{4}$$

where $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ is an arbitrary function and $g : \mathbb{K} \longrightarrow \mathbb{K}$ is an entire function.

Our goal is to show that the Daugavet property is the only non-trivial property that it is possible to define by requiring all rank-one operators on a Banach space of dimension greater than one to satisfy a norm equality of the form (4). We start by proving that g has to be a polynomial of degree less or equal than one, and then we will deduce the result from Proposition 2.2.

Theorem 3.1. *Let $g : \mathbb{K} \longrightarrow \mathbb{K}$ be an entire function and $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ an arbitrary function. Suppose that there is a Banach space X over $\mathbb{K}$ with* $\dim(X) \geqslant 2$ *such that the norm equality*

$$\|g(T)\| = f(\|T\|)$$

holds for every rank-one operator T on X. Then, there are $a, b \in \mathbb{K}$ such that

$$g(\zeta) = a + b\zeta \qquad \big(\zeta \in \mathbb{K}\big).$$

Proof. Let $g(\zeta) = \sum_{k=0}^{\infty} a_k\, \zeta^k$ be the power series expansion of g and let $\widetilde{g} = g - a_0$. Given $\alpha \in \mathbb{D}$, we take $x_\alpha^* \in S_{X^*}$ and $x_\alpha \in S_X$ such that $x_\alpha^*(x_\alpha) = \alpha$ (we can do it since $\dim(X) \geqslant 2$), and we write $T_\alpha = x_\alpha^* \otimes x_\alpha$, which satisfies $\|T_\alpha\| = 1$. Using Lemma 2.3, for each $\lambda \in \mathbb{K}$ we obtain that

$$g(\lambda T_0) = a_0 \mathrm{Id} + a_1 \lambda\, T_0$$

and

$$g(\lambda T_\alpha) = a_0 \mathrm{Id} + \frac{1}{\alpha}\widetilde{g}(\lambda\alpha)\, T_\alpha \qquad (\alpha \neq 0).$$

Now, fixed $\lambda \in \mathbb{K}$, we have

$$f(|\lambda|) = \|g(\lambda T_0)\| = \|a_0 \mathrm{Id} + a_1 \lambda T_0\|,$$

and

$$f(|\lambda|) = \|g(\lambda T_\alpha)\| = \left\| a_0 \mathrm{Id} + \frac{1}{\alpha} \widetilde{g}(\lambda\alpha) T_\alpha \right\|.$$

Therefore, we have the equality

$$\left\| a_0 \mathrm{Id} + \frac{1}{\alpha} \widetilde{g}(\lambda\alpha) T_\alpha \right\| = \|a_0 \mathrm{Id} + a_1 \lambda T_0\| \qquad (\lambda \in \mathbb{K},\ 0 < |\alpha| \leqslant 1). \tag{5}$$

In the complex case it is enough to consider the above equality for $\alpha = 1$ and to use the triangle inequality to get that

$$|\widetilde{g}(\lambda)| \leqslant 2|a_0| + |a_1|\,|\lambda| \qquad (\lambda \in \mathbb{C}). \tag{6}$$

From this, it follows by just using Cauchy's estimates, that $\widetilde{g}$ is a polynomial of degree one (see [9, Exercise 1, p. 80] or [12, Theorem 3.4.4], for instance), and we are done.

In the real case, it is not possible to deduce from inequality (6) that $\widetilde{g}$ is a polynomial, so we have to return to (5). From this equality, we can deduce by just applying the triangle inequality that

$$\left|\frac{\widetilde{g}(\lambda\alpha)}{\alpha}\right| - |a_0| \leqslant |a_0| + |a_1|\,|\lambda| \qquad \text{and} \qquad |a_1|\,|\lambda| - |a_0| \leqslant \left|\frac{\widetilde{g}(\lambda\alpha)}{\alpha}\right| + |a_0|$$

for every $\lambda \in \mathbb{R}$ and every $\alpha \in [-1,1] \setminus \{0\}$. It follows that

$$\left| \left|\frac{\widetilde{g}(\lambda\alpha)}{\alpha}\right| - |a_1|\,|\lambda| \right| \leqslant 2|a_0| \qquad \big(\lambda \in \mathbb{R},\ \alpha \in [-1,1] \setminus \{0\}\big). \tag{7}$$

Next, for $t \in]1,+\infty[$ and $k \in \mathbb{N}$, we use (7) with $\lambda = t^k$ and $\alpha = \frac{1}{t^{k-1}}$ to obtain that

$$\big|\,|\widetilde{g}(t)| - |a_1|\,t\big| \leqslant \frac{2|a_0|}{t^{k-1}}$$

so, letting $k \longrightarrow \infty$, we get that

$$|\widetilde{g}(t)| = |a_1|\,t \qquad \big(t \in]1,+\infty[\big).$$

Finally, an obvious continuity argument allows us to deduce from the above equality that $\widetilde{g}$ coincides with a degree one polynomial in the interval $]1,+\infty[$, thus the same is true in the whole $\mathbb{R}$ by analyticity. □

We summarize the information given in Proposition 2.2 and Theorem 3.1.

Corollary 3.2. *Let $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ be an arbitrary function and $g : \mathbb{K} \longrightarrow \mathbb{K}$ an entire function. Suppose that there is a Banach space X over $\mathbb{K}$ with $\dim(X) \geqslant 2$ such that the norm equality*

$$\|g(T)\| = f(\|T\|)$$

holds for every rank-one operator T on X. Then, only three possibilities may happen:

(a) *g is a constant function (trivial case).*
(b) *There is a non-null $b \in \mathbb{K}$ such that $g(\zeta) = b\,\zeta$ for every $\zeta \in \mathbb{K}$ (trivial case).*
(c) *There are non-null $a, b \in \mathbb{K}$ such that $g(\zeta) = a + b\,\zeta$ for every $\zeta \in \mathbb{K}$, and X has the Daugavet property.*

4. Norm equalities of the form $\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$

Let X be a Banach space over $\mathbb{K}$. Our next aim is to study norm equalities of the form

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|) \tag{8}$$

where $g : \mathbb{K} \longrightarrow \mathbb{K}$ is entire and $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ is continuous.

When X has the Daugavet property, it is clear that Eq. (8) holds for every rank-one operator if we take $g(\zeta) = \zeta$ and $f(t) = 1 + t$. But, actually, every entire function g works with a suitable f.

Remark 4.1. *If X is a real or complex Banach space with the Daugavet property and $g : \mathbb{K} \longrightarrow \mathbb{K}$ is an entire function, the norm equality*

$$\|\mathrm{Id} + g(T)\| = |1 + g(0)| - |g(0)| + \|g(T)\|$$

holds for every weakly compact operator $T \in L(X)$. Indeed, write $\widetilde{g} = g - g(0)$ and observe that $\widetilde{g}(T)$ is weakly compact whenever T is. Since X satisfies the Daugavet property, we have

$$\begin{aligned}
\|\mathrm{Id} + g(T)\| &= \|\mathrm{Id} + g(0)\,\mathrm{Id} + \widetilde{g}(T)\| = |1 + g(0)| + \|\widetilde{g}(T)\| \\
&= \big(|1 + g(0)| - |g(0)|\big) + \big(|g(0)| + \|\widetilde{g}(T)\|\big) \\
&= |1 + g(0)| - |g(0)| + \|g(0)\,\mathrm{Id} + \widetilde{g}(T)\| \\
&= |1 + g(0)| - |g(0)| + \|g(T)\|.
\end{aligned}$$

With the above result in mind, it is clear that the aim of this section cannot be to show that only few g's are possible in (8), but it is to show that many g's produce only few properties. Previous to formulate our results, let us discuss the case when the Banach space we consider is one-dimensional.

Remark 4.2.

(a) Complex case: *It is not possible to find a non-constant entire function g and an arbitrary function $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ such that the equality*

$$|1 + g(\zeta)| = f(|g(\zeta)|)$$

holds for every $\zeta \in \mathbb{C} \equiv L(\mathbb{C})$. Indeed, we suppose otherwise that such functions g and f exist, and we use Picard Theorem to assure the existence of $\lambda > 0$ such that $-\lambda$, $\lambda \in g(\mathbb{C})$. We get

$$f(\lambda) = |1 + \lambda| \neq |1 - \lambda| = f(|-\lambda|) = f(\lambda),$$

a contradiction.

(b) Real case: *The equality*

$$|1+t^2| = 1+|t^2|$$

holds for every $t \in \mathbb{R} \equiv L(\mathbb{R})$.

It follows that real and complex spaces do not behave in the same way with respect to equalities of the form given in (8). Therefore, from now on we study separately the complex and the real cases. Let us also remark that when a Banach space X has dimension greater than one, it is clear that

$$\|g(T)\| \geqslant |g(0)|$$

for every entire function $g : \mathbb{K} \longrightarrow \mathbb{K}$ and every rank-one operator $T \in L(X)$. Therefore, the function f in (8) has to be defined only in the interval $[|g(0)|, +\infty[$.

• Complex case:

Our key lemma here states that the function g in (8) can be replaced by a degree one polynomial.

Lemma 4.3. *Let* $g : \mathbb{C} \longrightarrow \mathbb{C}$ *be a non-constant entire function, let* $f : [|g(0)|, +\infty[\longrightarrow \mathbb{R}$ *be a continuous function and let* X *be a Banach space with dimension greater than one. Suppose that the norm equality*

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

holds for every rank-one operator $T \in L(X)$. *Then,*

$$\big\|\big(1+g(0)\big)\,\mathrm{Id} + T\big\| = |1+g(0)| - |g(0)| + \|g(0)\,\mathrm{Id} + T\|$$

for every rank-one operator $T \in L(X)$.

Proof. We claim that the norm equality

$$\big\|\big(1+g(0)\big)\,\mathrm{Id} + T\big\| = f\big(\|g(0)\,\mathrm{Id} + T\|\big) \tag{9}$$

holds for every rank-one operator $T \in L(X)$. Indeed, we write $\widetilde{g} = g - g(0)$ and we use Picard Theorem to assure that $\widetilde{g}(\mathbb{C})$ equals $\mathbb{C}$ except, at most, one point $\alpha_0 \in \mathbb{C}$. Next, we fix a rank-one operator $T = x^* \otimes x$ with $x^* \in X^*$ and $x \in X$. If $x^*(x) \neq 0$ and $x^*(x) \neq \alpha_0$, we may find $\zeta \in \mathbb{C}$ such that $\widetilde{g}(\zeta) = x^*(x)$, and we use Lemma 2.3 to get that

$$g\left(\frac{\zeta}{x^*(x)}\,T\right) = g(0)\,\mathrm{Id} + \frac{\widetilde{g}(\zeta)}{x^*(x)}\,T = g(0)\,\mathrm{Id} + T.$$

We deduce that

$$\begin{aligned}\big\|\big(1+g(0)\big)\,\mathrm{Id} + T\big\| &= \left\|\mathrm{Id} + g\left(\frac{\zeta}{x^*(x)}\,T\right)\right\| \\ &= f\left(\left\|g\left(\frac{\zeta}{x^*(x)}\,T\right)\right\|\right) = f\big(\|g(0)\,\mathrm{Id} + T\|\big).\end{aligned}$$

The remaining cases in which $x^*(x) = 0$ or $x^*(x) = \alpha_0$ follow from the above equality thanks to the continuity of f.

To finish the proof, we have to show that

$$f(t) = |1 + g(0)| - |g(0)| + t \qquad \big(t \geqslant |g(0)|\big).$$

Suppose first that $g(0) = -1$. We take $x \in S_X$ and $x^* \in S_{X^*}$ such that $x^*(x) = 1$ and, for every $t \geqslant 1$, we define the rank-one operator

$$T_t = (1 - t)\, x^* \otimes x.$$

It is immediate to show that

$$\| -\mathrm{Id} + T_t\| = t \qquad \text{and} \qquad \|T_t\| = t - 1.$$

Then, it follows from (9) that $f(t) = t - 1$, and we are done.

Suppose otherwise that $g(0) \neq -1$. We take $x \in S_X$ and $x^* \in S_{X^*}$ such that $x^*(x) = 1$ and, for every $t \geqslant |g(0)|$, we define the rank-one operator

$$T_t = \frac{1 + g(0)}{|1 + g(0)|}\big(t - |g(0)|\big)\, x^* \otimes x.$$

It is routine to show, by just evaluating at the point x, that

$$\big\|\big(1 + g(0)\big)\,\mathrm{Id} + T_t\big\| = |1 + g(0)| + t - |g(0)|.$$

Therefore, if follows from (9) that

$$f\big(\|g(0)\,\mathrm{Id} + T_t\|\big) = |1 + g(0)| + t - |g(0)|, \tag{10}$$

and the proof finishes by just proving that

$$\|g(0)\,\mathrm{Id} + T_t\| = t.$$

On the one hand, it is clear that

$$\|g(0)\,\mathrm{Id} + T_t\| \leqslant |g(0)| + \|T_t\| = |g(0)| + t - |g(0)| = t.$$

On the other hand, the converse inequality trivially holds when $g(0) = 0$, so we may suppose $g(0) \neq 0$ and we define the rank-one operator

$$S_t = \frac{g(0)}{|g(0)|}\big(\|g(0)\,\mathrm{Id} + T_t\| - |g(0)|\big)\, x^* \otimes x.$$

It is routine to show, by using that $\|g(0)\,\mathrm{Id} + S_t\| \geqslant |g(0)|$ and evaluating at the point x, that

$$\|g(0)\,\mathrm{Id} + S_t\| = \|g(0)\,\mathrm{Id} + T_t\|.$$

From the above equality, (9), and (10), we deduce that

$$\begin{aligned}|1 + g(0)| + t - |g(0)| = f\big(\|g(0)\,\mathrm{Id} + S_t\|\big) &= \big\|\big(1 + g(0)\big)\,\mathrm{Id} + S_t\big\| \\ &\leqslant |1 + g(0)| + \|S_t\| = |1 + g(0)| + \|g(0)\,\mathrm{Id} + T_t\| - |g(0)|,\end{aligned}$$

so $t \leqslant \|g(0)\,\mathrm{Id} + T_t\|$ and we are done. $\square$

In view of the norm equality appearing in the above lemma, two different cases arise: either $|1+g(0)| \neq |g(0)|$ or $|1+g(0)| = |g(0)|$; equivalently, $\operatorname{Re} g(0) \neq -1/2$ or $\operatorname{Re} g(0) = -1/2$. In the first case, we get the Daugavet property.

Theorem 4.4. *Let X be a complex Banach space with $\dim(X) \geqslant 2$. Suppose that there exist a non-constant entire function $g : \mathbb{C} \longrightarrow \mathbb{C}$ with $\operatorname{Re} g(0) \neq -\frac{1}{2}$ and a continuous function $f : \big[|g(0)|, +\infty\big[\longrightarrow \mathbb{R}_0^+$, such that the norm equality*

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

holds for every rank-one operator $T \in L(X)$. Then, X has the Daugavet property.

Proof. Let us first suppose that $\operatorname{Re} g(0) > -1/2$ so that

$$M = |1+g(0)| - |g(0)| > 0.$$

Then, dividing by M the equation given by Lemma 4.3, we get that the norm equality

$$\left\| \tfrac{1+g(0)}{M} \mathrm{Id} + \tfrac{1}{M} T \right\| = 1 + \left\| \tfrac{g(0)}{M} \mathrm{Id} + \tfrac{1}{M} T \right\| \tag{11}$$

holds for every rank-one operator $T \in L(X)$. Now, we take ω, $\xi \in \mathbb{T}$ such that

$$\omega\,(1+g(0)) = |1+g(0)| \qquad \text{and} \qquad \xi\, g(0) = |g(0)|,$$

we fix a rank-one operator $T \in L(X)$, and we observe that

$$\left\| \tfrac{1+g(0)}{M} \mathrm{Id} + \tfrac{1}{M} T \right\| = \left\| \tfrac{|1+g(0)|}{M} \mathrm{Id} + \tfrac{\omega}{M} T \right\| = \left\| \mathrm{Id} + \tfrac{|g(0)|}{M} \mathrm{Id} + \tfrac{\omega}{M} T \right\|$$

and

$$\left\| \tfrac{g(0)}{M} \mathrm{Id} + \tfrac{1}{M} T \right\| = \left\| \tfrac{|g(0)|}{M} \mathrm{Id} + \tfrac{\xi}{M} T \right\|.$$

Therefore, from (11) we get

$$\left\| \mathrm{Id} + \tfrac{|g(0)|}{M} \mathrm{Id} + \tfrac{\omega}{M} T \right\| = 1 + \left\| \tfrac{|g(0)|}{M} \mathrm{Id} + \tfrac{\xi}{M} T \right\|.$$

It follows straightforwardly from the arbitrariness of T that the norm equality

$$\left\| \mathrm{Id} + \tfrac{|g(0)|}{M} \mathrm{Id} + T \right\| = 1 + \left\| \tfrac{|g(0)|}{M} \mathrm{Id} + \tfrac{\xi}{\omega} T \right\| \tag{12}$$

holds for every rank-one operator $T \in L(X)$. Now, we claim that

$$\left\| \tfrac{|g(0)|}{M} \mathrm{Id} + T \right\| = \left\| \tfrac{|g(0)|}{M} \mathrm{Id} + \tfrac{\xi}{\omega} T \right\| \tag{13}$$

for every rank-one operator $T \in L(X)$. Indeed, by the triangle inequality, we deduce from (12) that

$$\left\| \tfrac{|g(0)|}{M} \mathrm{Id} + T \right\| \geqslant \left\| \tfrac{|g(0)|}{M} \mathrm{Id} + \tfrac{\xi}{\omega} T \right\|$$

for every rank-one operator $T \in L(X)$ and, for every $n \in \mathbb{N}$, applying the above inequality n times, we get that

$$\left\| \tfrac{|g(0)|}{M} \mathrm{Id} + T \right\| \geqslant \left\| \tfrac{|g(0)|}{M} \mathrm{Id} + \tfrac{\xi}{\omega} T \right\| \geqslant \cdots \geqslant \left\| \tfrac{|g(0)|}{M} \mathrm{Id} + \left(\tfrac{\xi}{\omega}\right)^n T \right\|;$$

the claim follows from the easy fact that the sequence $\left\{\left(\frac{\xi}{\omega}\right)^n\right\}_{n\in\mathbb{N}}$ has a subsequence which converges to 1.

Now, given a rank-one operator $T \in L(X)$, it follows from (12) and (13) that

$$\left\|\mathrm{Id} + \tfrac{|g(0)|}{M}\,\mathrm{Id} + T\right\| = 1 + \left\|\ \tfrac{|g(0)|}{M}\,\mathrm{Id} + T\right\|$$

holds and, therefore, for every $n \in \mathbb{N}$ we get

$$\left\|\mathrm{Id} + \tfrac{1}{n}\left(\tfrac{|g(0)|}{M}\,\mathrm{Id} + T\right)\right\| = 1 + \left\|\tfrac{1}{n}\left(\tfrac{|g(0)|}{M}\,\mathrm{Id} + T\right)\right\|.$$

To finish the proof, we fix a rank-one operator $S \in L(X)$ and $n \in \mathbb{N}$, and we apply the above equality to $T = n\,S$ to get that

$$\left\|\mathrm{Id} + S + \tfrac{|g(0)|}{n\,M}\,\mathrm{Id}\right\| = 1 + \left\|\tfrac{|g(0)|}{n\,M}\,\mathrm{Id} + S\right\|.$$

We let $n \longrightarrow \infty$ to deduce that

$$\|\mathrm{Id} + S\| = 1 + \|S\|,$$

so X has the Daugavet property.

In case that $\operatorname{Re} g(0) < -\frac{1}{2}$, we write

$$M = -|1 + g(0)| + |g(0)| > 0$$

and we deduce from Lemma 4.3 that the norm equality

$$\left\|\tfrac{g(0)}{M}\,\mathrm{Id} + \tfrac{1}{M}\,T\right\| = 1 + \left\|\tfrac{1+g(0)}{M}\,\mathrm{Id} + \tfrac{1}{M}\,T\right\|$$

holds for every rank-one operator $T \in L(X)$. The rest of the proof is completely analogous, using the above equality instead of (11). $\square$

When $\operatorname{Re} g(0) = -\frac{1}{2}$, the above proof does not work. Actually, contrary to all the previous cases, another family of properties apart from the Daugavet property appears. More concretely, let X be a complex Banach space with dimension greater than one, let $g : \mathbb{C} \longrightarrow \mathbb{C}$ be a non-constant entire function with $\operatorname{Re} g(0) = -1/2$, and let $f : [|g(0)|, +\infty[\longrightarrow \mathbb{R}$ be a continuous function, such that the norm equality

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

holds for every rank-one operator $T \in L(X)$. Then, we get from Lemma 4.3 that

$$\left\|\big(1 + g(0)\big)\,\mathrm{Id} + T\right\| = \|g(0)\,\mathrm{Id} + T\|$$

for every rank-one operator $T \in L(X)$. Therefore, being $|1 + g(0)| = |g(0)|$, we deduce that there are $\omega_1, \omega_2 \in \mathbb{C}$ with $\omega_1 \neq \omega_2$ and $|\omega_1| = |\omega_2|$, such that

$$\|\mathrm{Id} + \omega_1\,T\| = \|\mathrm{Id} + \omega_2\,T\|$$

for every rank-one operator $T \in L(X)$ or, equivalently, that there is $\omega \in \mathbb{T}\backslash\{1\}$ such that

$$\|\mathrm{Id} + \omega\,T\| = \|\mathrm{Id} + T\|$$

for every rank-one operator $T \in L(X)$. It is routine to check that, fixed a Banach space X, the set of those $\omega \in \mathbb{T}$ which make true the above equality for all rank-one operators on X is a multiplicative closed subgroup of $\mathbb{T}$. Recall that such a subgroup of $\mathbb{T}$ is either the whole $\mathbb{T}$ or the set of those n^{th}-roots of unity for an integer $n \geqslant 2$. Let us state the result we have just proved.

Theorem 4.5. *Let X be a complex Banach space with $\dim(X) \geqslant 2$. Suppose that there exist a non-constant entire function $g : \mathbb{C} \longrightarrow \mathbb{C}$ with $\operatorname{Re} g(0) = -\frac{1}{2}$ and a continuous function $f : \big[|g(0)|, +\infty\big[\longrightarrow \mathbb{R}_0^+$, such that the norm equality*

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

holds for every rank-one operator $T \in L(X)$. Then, there is $\omega \in \mathbb{T} \setminus \{1\}$ such that

$$\|\mathrm{Id} + \omega T\| = \|\mathrm{Id} + T\|$$

for every rank-one operator $T \in L(X)$. Moreover, two possibilities may happen:

(a) *If $\omega^n \neq 1$ for every $n \in \mathbb{N}$, then*

$$\|\mathrm{Id} + \xi\, T\| = \|\mathrm{Id} + T\|$$

for every rank-one operator $T \in L(X)$ and every $\xi \in \mathbb{T}$.

(b) *Otherwise, if we take the minimum $n \in \mathbb{N}$ such that $\omega^n = 1$, then*

$$\|\mathrm{Id} + \xi\, T\| = \|\mathrm{Id} + T\|$$

for every rank-one operator $T \in L(X)$ and every n^{th}-root ξ of unity.

Remark 4.6. *The above theorem is actually a characterization. Namely, let us fix $\omega \in \mathbb{T} \setminus \{1\}$ and let X be a Banach space such that*

$$\|\mathrm{Id} + \omega\, T\| = \|\mathrm{Id} + T\|$$

for every rank-one operator $T \in L(X)$. Then, there are an entire function g with $\operatorname{Re} g(0) = -\frac{1}{2}$, and a continuous function f such that the norm equality

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

holds for every rank-one operator $T \in L(X)$. Indeed, if we take $\alpha \in \mathbb{R}$ such that

$$\left(\frac{1}{2} + \alpha\, i\right) \left(-\frac{1}{2} + \alpha\, i\right)^{-1} = \omega$$

and we consider

$$f(t) = t \quad (t \in \mathbb{R}_0^+), \qquad g(\zeta) = \left(-\frac{1}{2} + \alpha\, i\right) + \left(-\frac{1}{2} + \alpha\, i\right) \zeta \quad (\zeta \in \mathbb{C}),$$

it is routine to check that the norm equality

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

holds for every rank-one operator $T \in L(X)$.

The next example shows that all the properties appearing in Theorem 4.5 are strictly weaker than the Daugavet property.

Example 4.7. *The real or complex Banach space $X = C[0,1] \oplus_2 C[0,1]$ does not have the Daugavet property. However, the norm equality*

$$\|\mathrm{Id} + \omega\, T\| = \|\mathrm{Id} + T\|$$

holds for every rank-one operator $T \in L(X)$ and every $\omega \in \mathbb{T}$.

Proof. X does not have the Daugavet property since the ℓ_2 sum of two non-zero spaces never has the Daugavet property [6, Corollary 5.4]. For the second assertion, we fix $\omega \in \mathbb{T}$ and a rank-one operator $T = x^* \otimes x$ on X (we take $x \in S_X$ and $x^* \in X^*$), and it is enough to check that

$$\|\mathrm{Id} + \omega\, T\|^2 \geqslant \|\mathrm{Id} + T\|^2.$$

To do so, we write $x^* = (x_1^*, x_2^*)$ and $x = (x_1, x_2)$, with x_1^*, $x_2^* \in C[0,1]^*$ and x_1, $x_2 \in C[0,1]$, and we may and do assume that

$$x_1^* = \mu_1 + \sum_{j=1}^{n_1} \alpha_j \delta_{r_j} \qquad x_2^* = \mu_2 + \sum_{j=1}^{n_2} \beta_j \delta_{s_j},$$

where $\alpha_1,\ldots,\ \alpha_{n_1},\ \beta_1,\ldots,\ \beta_{n_2} \in \mathbb{C},\ r_1,\ldots,\ r_{n_1},\ s_1,\ldots,\ s_{n_2} \in [0,1]$, and μ_1, μ_2 are non-atomic measures on $[0,1]$ (indeed, each rank-one operator can be approximated by operators satisfying the preceding condition).

Now, we fix $0 < \varepsilon < 1$ and we consider $y = (y_1, y_2) \in S_X$ such that

$$\|y + Ty\|^2 = \|y_1 + x^*(y)x_1\|^2 + \|y_2 + x^*(y)x_2\|^2 \geqslant \|\mathrm{Id} + T\|^2 - \varepsilon. \tag{14}$$

Since x_1, x_2, y_1, and y_2 are continuous functions and $[0,1]$ is perfect, we can find open intervals Δ_1, $\Delta_2 \subset [0,1]$ so that

$$|y_i(t) + x^*(y)x_i(t)| \geqslant \|y_i + x^*(y)x_i\| - \varepsilon \qquad \big(t \in \Delta_i,\ i = 1,2\big) \tag{15}$$

and

$$\Delta_1 \cap \{r_j \ :\ j = 1,\ldots,n_1\} = \emptyset, \qquad \Delta_2 \cap \{s_j \ :\ j = 1,\ldots,n_2\} = \emptyset \tag{16}$$

Furthermore, using that μ_1, μ_2 are non-atomic and reducing Δ_1, Δ_2 if necessary, we can assume that they also satisfy

$$|\mu_1(f)| < \varepsilon\, \|f\|, \qquad |\mu_2(g)| < \varepsilon\, \|g\| \tag{17}$$

for every $f, g \in C[0,1]$ with $\mathrm{supp}(f) \subset \Delta_1$, $\mathrm{supp}(g) \subset \Delta_2$.

Now, we fix $t_1 \in \Delta_1$ and $t_2 \in \Delta_2$. For $i = 1,2$, we take piece-wise linear continuous functions $\phi_i : [0,1] \longrightarrow [0,1]$ such that

$$\phi_i(t_i) = 1 \qquad \text{and} \qquad \phi_i\big([0,1] \setminus \Delta_i\big) = \{0\},$$

and we define

$$\widetilde{y_i} = y_i(1 - \phi_i + \omega\phi_i).$$

It is easy to check that

$$|1 - \phi_i(t) + \omega\phi_i(t)| \leqslant 1 \qquad \big(t \in [0,1]\big),$$

so, $\|\widetilde{y_i}\| \leqslant \|y_i\|$. This implies that $\|\widetilde{y}\| = \|(\widetilde{y_1}, \widetilde{y_2})\| \leqslant 1$ and, therefore,

$$\|\mathrm{Id} + \omega T\|^2 \geqslant \|\widetilde{y} + \omega T(\widetilde{y})\|^2 = \|\widetilde{y_1} + \omega x^*(\widetilde{y})x_1\|^2 + \|\widetilde{y_2} + \omega x^*(\widetilde{y})x_2\|^2. \tag{18}$$

Since, clearly,

$$y_i - \widetilde{y_i} = (1-\omega) y_i \phi_i, \qquad (i = 1, 2)$$

we deduce from (16), (17), and the fact that $\mathrm{supp}(\phi_i) \subset \Delta_i$, that

$$|x^*(y - \widetilde{y})| \leqslant |x_1^*(y_1 - \widetilde{y_1})| + |x_2^*(y_2 - \widetilde{y_2})| \leqslant 4\,\varepsilon M,$$

where $M = \max\{\|x_1^*\|, \|x_2^*\|\}$. Using this, (15), and the fact that $\|x\| = 1$, we obtain the following for $i = 1, 2$:

$$\begin{aligned}
\|\widetilde{y_i} + \omega\, x^*(\widetilde{y}) x_i\| &\geqslant \|\widetilde{y_i} + \omega\, x^*(y)\, x_i\| - |x^*(y - \widetilde{y})| \\
&\geqslant |\widetilde{y_i}(t_i) + \omega\, x^*(y)\, x_i(t_i)| - 4\,\varepsilon M \\
&= |\omega\, y_i(t_i) + \omega\, x^*(y)\, x_i(t_i)| - 4\,\varepsilon M \\
&= |y_i(t_i) + x^*(y)\, x_i(t_i)| - 4\,\varepsilon M \\
&\geqslant \|y_i + x^*(y)\, x_i\| - \varepsilon - 4\,\varepsilon M.
\end{aligned}$$

Finally, using this together with (14) and (18), it is not hard to find a suitable constant $K > 0$ such that

$$\|\mathrm{Id} + \omega T\|^2 \geqslant \|\mathrm{Id} + T\|^2 - K\,\varepsilon,$$

which finishes the proof. □

• Real case:

The situation in the real case is far away from being so clear. On the one hand, the proof of Lemma 4.3 remains valid if the function g is surjective (this substitutes Picard Theorem) and then, the proofs of Theorems 4.4 and 4.5 are valid. In addition, Example 4.7 was also stated for the real case. The following result summarizes all these facts.

Theorem 4.8. *Let X be a real Banach space with dimension greater or equal than two. Suppose that there exists a* ***surjective*** *entire function $g : \mathbb{R} \longrightarrow \mathbb{R}$ and a continuous function $f : \big[|g(0)|, +\infty\big[\longrightarrow \mathbb{R}_0^+$, such that the norm equality*

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

holds for every rank-one operator $T \in L(X)$.

(a) *If $g(0) \neq -1/2$, then X has the Daugavet property.*
(b) *If $g(0) = -1/2$, then the norm equality*

$$\|\mathrm{Id} - T\| = \|\mathrm{Id} + T\|$$

holds for every rank-one operator $T \in L(X)$.

(c) *The real space* $X = C[0,1] \oplus_2 C[0,1]$ *does not have the Daugavet property but the norm equality*

$$\|\mathrm{Id} - T\| = \|\mathrm{Id} + T\|$$

holds for every rank-one operator $T \in L(X)$.

On the other hand, we do not know if a result similar to the above theorem is true when the function g is not onto. Let us give some remarks about two easy cases:

$$g(t) = t^2 \quad (t \in \mathbb{R}) \qquad \text{and} \qquad g(t) = -t^2 \quad (t \in \mathbb{R}).$$

In the first case, it is easy to see that if the norm equality

$$\|\mathrm{Id} + T^2\| = f(\|T^2\|)$$

holds for every rank-one operator, then $f(t) = 1 + t$ and, therefore, the interesting norm equality in this case is

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|. \tag{19}$$

This equation is satisfied by every rank-one operator T on a Banach space X with the Daugavet property. Let us also recall that the equality

$$|1 + t^2| = 1 + |t^2|$$

holds for every $t \in L(\mathbb{R}) \equiv \mathbb{R}$ (Remark 4.2). For the norm equality

$$\|\mathrm{Id} - T^2\| = f(\|T^2\|),$$

we are not able to get any information about the shape of the function f. Going to the one-dimensional case, we get that

$$|1 - t^2| = \max\{1 - |t^2|, |t^2| - 1\}$$

for every $t \in L(\mathbb{R}) \equiv \mathbb{R}$, but it is not possible that the corresponding norm equality holds for all rank-one operators on a Banach space with dimension greater than one (in this case, $\|\mathrm{Id} - T^2\| \geqslant 1$). On the other hand, if a Banach space X has the Daugavet property, then

$$\|\mathrm{Id} - T^2\| = 1 + \|T^2\| \tag{20}$$

for every rank-one operator $T \in L(X)$. Therefore, an interesting norm equality of this form could be the above one.

Let us characterize the properties which flow out from the norm equalities (19) and (20). We need some notation. By a *slice* of a subset A of a normed space X we mean a set of the form

$$S(A, x^*, \alpha) = \big\{x \in A \ : \ \operatorname{Re} x^*(x) > \sup \operatorname{Re} x^*(A) - \alpha\big\}$$

where $x^* \in X^*$ and $\alpha \in \mathbb{R}^+$. If X is a dual space, by a *weak*-slice* of a subset A of X we mean a slice of A defined by a weak*-continuous functional or, equivalently, a weak*-open slice of A.

Proposition 4.9. *Let* X *be a real Banach space.*

(a) *The following are equivalent:*

(i) $\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$ *for every rank-one operator* T.

(ii) $\|\mathrm{Id} + x^* \otimes x\| = 1 + \|x^* \otimes x\|$ *for* $x^* \in X^*$, $x \in X$ *with* $x^*(x) \geqslant 0$.

(iii) For every $x \in S_X$, $x^* \in S_{X^*}$ *with* $x^*(x) \geqslant 0$, *and every* $\varepsilon > 0$, *there exists* $y \in S_X$ *such that*

$$\|x + y\| > 2 - \varepsilon \qquad \text{and} \qquad x^*(y) > 1 - \varepsilon.$$

(iv) For every $x \in S_X$, $x^* \in S_{X^*}$ *with* $x^*(x) \geqslant 0$, *and every* $0 < \varepsilon < 1$, *there exist* $\delta > 0$ *and* $y^* \in S_{X^*}$ *with* $y^*(x) \geqslant 0$ *such that every* $y \in S(B_X, y^*, \delta)$ *satisfies*

$$y \in S(B_X, x^*, \varepsilon) \qquad \text{and} \qquad \|x + y\| > 2 - \varepsilon.$$

(b) *The following are equivalent:*

(i) $\|\mathrm{Id} - T^2\| = 1 + \|T^2\|$ *for every rank-one operator* T.

(ii) $\|\mathrm{Id} + x^* \otimes x\| = 1 + \|x^* \otimes x\|$ *for* $x^* \in X^*$, $x \in X$ *with* $x^*(x) \leqslant 0$.

(iii) For every $x \in S_X$, $x^* \in S_{X^*}$ *with* $x^*(x) \leqslant 0$, *and every* $\varepsilon > 0$, *there exists* $y \in S_X$ *such that*

$$\|x + y\| > 2 - \varepsilon \qquad \text{and} \qquad x^*(y) > 1 - \varepsilon.$$

(iv) For every $x \in S_X$, $x^* \in S_{X^*}$ *with* $x^*(x) < 0$, *and every* $0 < \varepsilon < 1$, *there exist* $\delta > 0$ *and* $y^* \in S_{X^*}$ *with* $y^*(x) = 0$ *such that every* $y \in S(B_X, y^*, \delta)$ *satisfies*

$$y \in S(B_X, x^*, \varepsilon) \qquad \text{and} \qquad \|x + y\| > 2 - \varepsilon.$$

Proof. We start by proving item (a).

$(i) \Rightarrow (ii)$. We consider $x \in X$ and $x^* \in X^*$ with $x^*(x) > 0$, and we write $T = x^* \otimes x$. Let us observe that the rank-one operator $S = \big(x^*(x)\big)^{-1/2} T$ satisfies $S^2 = T$, so we get

$$1 + \|T\| = 1 + \|S^2\| = \|\mathrm{Id} + S^2\| = \|\mathrm{Id} + T\|.$$

To finish the argument, it suffices to observe that any $x^* \otimes x$ with $x^*(x) = 0$ can be approximated in norm by operators of the form $y^* \otimes y$ with $y^*(y) > 0$, and that the set of rank-one operators satisfying (DE) is closed.

$(ii) \Rightarrow (i)$. Just observe that for every rank-one operator $T = x^* \otimes x$, it is clear that $T^2 = x^*(x)\, T$ and, therefore, $T^2 = y^* \otimes x$ where $y^* = x^*(x)\, x^*$ with $y^*(x) = \big(x^*(x)\big)^2 \geqslant 0$.

Finally, for the equivalence between (ii), (iii), and (iv) just follow the proof of [19, Lemma 2.1] or [1, Lemma 11.46].

For item (b), the proofs of the equivalences between (i), (ii) and (iii) are analogous to those for item (a). Thus, we only prove the equivalence between (ii) and (iv).

$(ii) \Rightarrow (iv)$. Let us mention that the following argument follows the lines of that in [19, Lemma 2.1] or [1, Lemma 11.46]. Let $x \in S_X$, $x^* \in S_{X^*}$ with

$x^*(x) < 0$. Then

$$\|\mathrm{Id}_{X^*} + x \otimes x^*\| = \|\mathrm{Id} + x^* \otimes x\| = 2$$

and, therefore,

$$\|{-x^*(x)}\,\mathrm{Id}_{X^*} + x \otimes x^*\| = 1 + |x^*(x)|.$$

Thus, there is a functional $y_0^* \in S_{X^*}$ such that

$$\| - x^*(x)\, y_0^* + y_0^*(x)\, x^*\| > 1 + |x^*(x)| - \varepsilon\, |x^*(x)| \qquad \text{and} \qquad y_0^*(x) > 0.$$

If we write

$$y^* = \frac{-x^*(x)\, y_0^* + y_0^*(x)\, x^*}{\| - x^*(x)\, y_0^* + y_0^*(x)\, x^*\|}\,, \qquad \delta = 1 - \frac{1 + |x^*(x)| - \varepsilon\, |x^*(x)|}{\| - x^*(x)\, y_0^* + y_0^*(x)\, x^*\|},$$

it is clear that $y^*(x) = 0$ and, on the other hand, given $y \in S(B_X, y^*, \delta)$, we have

$$\begin{aligned}\big(- x^*(x)\, y_0^* + y_0^*(x)\, x^*\big)(y) &> (1-\delta)\| - x^*(x)\, y_0^* + y_0^*(x)\, x^*\| \\ &= 1 + |x^*(x)| - \varepsilon\, |x^*(x)|.\end{aligned}$$

Therefore,

$$|x^*(x)|\, y_0^*(y) + y_0^*(x)\, x^*(y) > 1 + |x^*(x)| - \varepsilon\, |x^*(x)|, \tag{21}$$

which implies (using the fact that $y_0^*(x) > 0$) that

$$x^*(y) > 1 - \varepsilon\, |x^*(x)| \geqslant 1 - \varepsilon.$$

Furthermore, (21) also tells us that

$$|x^*(x)|\, y_0^*(y) + y_0^*(x) > 1 + |x^*(x)| - \varepsilon\, |x^*(x)|$$

and, therefore

$$|x^*(x)|\, \big(y_0^*(y) + y_0^*(x)\big) > (2-\varepsilon)|x^*(x)|.$$

Finally we obtain that

$$\|x + y\| \geqslant y_0^*(x) + y_0^*(y) > 2 - \varepsilon.$$

(iv) $\Rightarrow$ *(ii)*. It is clear that *(ii)* holds for $x \in X$, $x^* \in X^*$ with $x^*(x) < 0$. The argument finishes using that the set of rank-one operators satisfying (DE) is closed. $\square$

Remark 4.10. We do not know if any of the two properties appearing in Proposition 4.9 implies the Daugavet property. However, both imply the so-called alternative Daugavet property. Following [21], we say that a Banach space X has the *alternative Daugavet property* if for every rank-one operator T there is $\omega \in \mathbb{T}$ such that ωT satisfies (DE). Examples of Banach spaces having the alternative Daugavet property and failing the Daugavet property are c_0, ℓ_1 and ℓ_∞. It is easy to check that they also fail the two properties appearing in Proposition 4.9.

Let us finish the section with the following open question which was asked to us by Gilles Godefroy when discussing these topics.

Question 4.11. Is there any real Banach space X (different from $\mathbb{R}$) such that the norm equality
$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$$
holds for every $T \in L(X)$?

The following remark shows that the answer is negative for a wide class of spaces.

Remark 4.12. *Suppose that a real Banach space X decomposes in the form $X = Z \oplus E$ for some closed subspaces Z and $E \neq 0$, where E is isomorphic to $Y \oplus Y$ for some Banach space Y. If the norm equality*
$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\|$$
holds for every $T \in L(X)$, then the norm of every projection $P_Z \in L(X)$ from X onto Z is greater or equal than 2. In particular, neither finite-dimensional spaces nor spaces which are isomorphic to squares solve positively Question 4.11.

Indeed, let $P_Z \in L(X)$ be such a projection. Then its kernel is isomorphic to E. So we can represent X as $X = Z \oplus Y \oplus Y$ in such a way that $\ker P_Z = Y \oplus Y$. Consider the operator $T \in L(X)$ given by
$$T(z, y_1, y_2) = (0, -y_2, y_1) \qquad \big((z, y_1, y_2) \in Z \oplus Y \oplus Y\big).$$
Since $P_Z = \mathrm{Id} + T^2$, we have
$$\|P_Z\| = \|\mathrm{Id} + T^2\| = 1 + \|T^2\| \geqslant 2.$$

5. Additional properties

Our aim in this section is to give some remarks concerning the properties appearing at Theorems 4.5 and 4.8.b, i.e. we consider a non-trivial multiplicative subgroup A of $\mathbb{T}$ and study those Banach spaces for which all rank-one operators satisfy the norm equalities
$$\|\mathrm{Id} + \omega\, T\| = \|\mathrm{Id} + T\| \qquad (\omega \in A).$$
In the real case, only one property arises; in the complex case, there are infinitely many properties and we do not know if all of them are equivalent.

Our first (easy) observation is that all these properties pass from the dual of a Banach space to the space. We will see later that the converse result is not valid.

Remark 5.1. *Let X be a Banach space and let $\omega \in \mathbb{T}$. Suppose that the norm equality*
$$\|\mathrm{Id} + \omega\, T\| = \|\mathrm{Id} + T\|$$
holds for every rank-one operator $T \in L(X^*)$. Then, the same is true for every rank-one operator on X. Indeed, the result follows routinely by just considering the adjoint operators of the rank-one operators on X.

Our next results deal with the shape of the unit ball of the Banach spaces having any of these properties.

Proposition 5.2. *Let X be a real or complex Banach space and let A be a non-trivial closed subgroup of $\mathbb{T}$. Suppose that the norm equality*

$$\|\mathrm{Id} + \omega\, T\| = \|\mathrm{Id} + T\|$$

holds for every rank-one operator $T \in L(X)$ and every $\omega \in A$. Then, the slices of B_X and the weak-slices of B_{X^*} have diameter greater or equal than*

$$2 - \inf\bigl\{|1+\omega| \ : \ \omega \in A\bigr\}.$$

Proof. We give the arguments only for slices of B_X, being the proof for weak*-slices of B_{X^*} completely analogous. We fix $x^* \in S_{X^*}$ and $0 < \alpha < 2$. Given $0 < \varepsilon < \alpha$, we take $x \in S_X$ such that $\operatorname{Re} x^*(x) > 1 - \varepsilon$ and so, in particular, $x \in S(B_X, x^*, \alpha)$. We define the rank-one operator $T = x^* \otimes x$ and observe that

$$\|\mathrm{Id} + T\| \geqslant \|x + x^*(x)\, x\| = \|x\|\, |1 + x^*(x)| \geqslant |1 + \operatorname{Re} x^*(x)| > 2 - \varepsilon.$$

By hypothesis, for every $\omega \in A$ we may find $y \in S_X$ such that

$$\|y + \omega\, x^*(y)\, x\| > 2 - \varepsilon$$

so, in particular,

$$|x^*(y)| > 1 - \varepsilon.$$

We take $\xi \in \mathbb{T}$ such that $\xi\, x^*(y) = |x^*(y)|$ and we deduce that

$$\xi\, y \in S(B_X, x^*, \alpha).$$

From the inequalities

$$\begin{aligned}\|\xi y - x\| &= \bigl\|\xi y + \omega x^*(\xi y) x - \omega x^*(\xi y) x - x\bigr\| \\ &\geqslant \bigl\|\xi y + \omega x^*(\xi y) x\bigr\| - \bigl|1 + \omega x^*(\xi y)\bigr| \\ &> 2 - \varepsilon - \bigl|1 + \omega |x^*(y)|\bigr|\end{aligned}$$

and

$$\begin{aligned}\bigl|1 + \omega |x^*(y)|\bigr| &\leqslant \bigl|1 + \omega + \omega(|x^*(y)| - 1)\bigr| \\ &\leqslant |1+\omega| + \bigl|1 - |x^*(y)|\bigr| < |1+\omega| + \varepsilon,\end{aligned}$$

we get

$$\|\xi y - x\| \geqslant 2 - 2\,\varepsilon - |1+\omega|. \qquad \square$$

It is well-known that the unit ball of a Banach space X with the Radon-Nikodým property has many denting points and the unit ball of the dual of an Asplund space has many weak*-denting points. Recall that $x_0 \in B_X$ is said to be a *denting point* of B_X if it belongs to slices of B_X with arbitrarily small diameter. If X is a dual space and the slices can be taken to be weak*-open, then we say that x_0 is a *weak*-denting point*. We refer to [7, 11] for more information on these concepts.

Corollary 5.3. *Let X be a real or complex Banach space and let $\omega \in \mathbb{T} \setminus \{1\}$. If the norm equality*

$$\|\mathrm{Id} + \omega\, T\| = \|\mathrm{Id} + T\|$$

holds for every rank-one operator $T \in L(X)$, then B_X does not have any denting point and B_{X^*} does not have any w^*-denting point. In particular, X is not an Asplund space and it does not have the Radon-Nikodým property.

In view of this result, it is easy to show that the converse of Remark 5.1 is not true.

Example 5.4. *The real or complex space $X = C[0,1]$ satisfies that the norm equality*

$$\|\mathrm{Id} + \omega\, T\| = \|\mathrm{Id} + T\|$$

holds for every rank-one operator $T \in L(X)$ and every $\omega \in \mathbb{T}$, in spite of the fact that for every $\omega \in \mathbb{T} \setminus \{1\}$, there is a rank-one operator $S \in L(X^*)$ such that

$$\|\mathrm{Id} + \omega\, S\| < \|\mathrm{Id} + S\|.$$

Indeed, the first assertion follows from the fact that X has the Daugavet property; the second one follows from Corollary 5.3 since the unit ball of X^* is plenty of denting points.

One of the properties we are dealing with is related to a property for one-codimensional projections.

Remark 5.5. Consider a Banach space X such that the equality

$$\|\mathrm{Id} + T\| = \|\mathrm{Id} - T\|$$

holds true for every rank-one operator $T \in L(X)$. If we apply this equality to an operator P which is a rank-one projection, we get

$$\|\mathrm{Id} - P\| \geqslant 2,$$

i.e. every one-codimensional projection in $L(X)$ is at least of norm 2. Such spaces were introduced recently [13] and are called "spaces with bad projections".

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