THE DAUGAVET EQUATION FOR POLYNOMIALS

YUN SUNG CHOI, DOMINGO GARCÍA, MANUEL MAESTRE, AND MIGUEL MARTÍN

Abstract. In this paper we study when the Daugavet equation is satisfied for weakly compact polynomials on a Banach space X, i.e. when the equality

$$\|\mathrm{Id} + P\| = 1 + \|P\|$$

is satisfied for all weakly compact polynomials $P : X \longrightarrow X$. We show that this is the case when $X = C(K)$, the real or complex space of continuous functions on a compact space K without isolated points. We also study the alternative Daugavet equation

$$\max_{|\omega|=1} \|\mathrm{Id} + \omega\, P\| = 1 + \|P\|$$

for polynomials $P : X \longrightarrow X$. We show that this equation holds for every polynomial on the complex space $X = C(K)$ (K arbitrary) with values in X. The result is not true in the real case. Finally, we study the Daugavet and the alternative Daugavet equations for k-homogeneous polynomials.

In 1963, I. K. Daugavet [13] showed that every compact linear operator T on $C[0,1]$ satisfies

$$\|\mathrm{Id} + T\| = 1 + \|T\|,$$

a norm equality which has currently become known as the Daugavet equation. Over the years, the validity of the above equality has been established for many classes of operators on many Banach spaces. For instance, weakly compact linear operators on $C(K)$, K perfect, and $L_1(\mu)$, μ atomless, satisfy Daugavet equation (see [25] for an elementary approach). We refer the reader to the books [1, 2] and the papers [20, 26] for more information and background. It is also a remarkable result given in 1970 by J. Duncan et al. [16] that, for every compact Hausdorff space K and every bounded linear operator T on $C(K)$, the equality

$$\max_{\omega\in\mathbb{T}} \|\mathrm{Id} + \omega T\| = 1 + \|T\|$$

holds, where we use $\mathbb{T}$ to denote the unit sphere of the base field. The above norm equality is now known as the alternative Daugavet equation [22], and it is satisfied by all bounded linear operators on $C(K)$ and $L_1(\mu)$, K and μ arbitrary. We refer the reader to [16, 21, 22] and references there in for background. The aim of this paper is to study the Daugavet equation and the alternative Daugavet equation for polynomials in Banach spaces.

There is a concept, the numerical range of an operator (see below for the definition), intimately related to the Daugavet and alternative Daugavet equations. The definition of numerical range for bounded linear operators on Banach spaces was given in 1962 by F. Bauer [5] (see [9, 10] for background) extending the 1918 classical definition of numerical range (or field of values) of a matrix given by O. Toeplitz [24]. In 1968, the concept of numerical range was extended to arbitrary continuous functions from the unit sphere of a real or complex Banach space into the space by F. Bonsall, B. Cain, and H. Schneider [7]. In the seventies,

Date: March 24th, 2006; revised September 13th, 2006.

2000 *Mathematics Subject Classification.* Primary 46G25; Secondary 46B20, 47A12.

Key words and phrases. Polynomial, Banach space, Daugavet equation, numerical range.

The first author was supported by grant No. R01-2004-000-10055-0 from the Basic Research Program of the Korea Science and Engineering Foundation. The second and third authors were supported by MEC and FEDER Project MTM2005-08210. The fourth author was supported by MCYT and FEDER Project BFM2003-01681 and Junta de Andalucía grant FQM-185.

L. Harris [17, 18] showed that a good setting to deal with numerical ranges is the space of bounded uniformly continuous functions on the unit sphere of a Banach space with values in the space (see also Rodríguez-Palacios [23]). Once we have given up the linearity, we have realized that also the study of the Daugavet and the alternative Daugavet equations is clarified if it is done for bounded uniformly continuous functions on the unit ball of a Banach space with values in the space.

Let us introduce the necessary definitions and notations.

Throughout the article, all the Banach spaces considered will be real or complex unless the scalar field is specified and all polynomials will be continuous. Let X be a Banach space. By B_X we denote the closed unit ball and by S_X the unit sphere of X. Given $k \geqslant 0$, we denote by $\mathcal{P}\left({}^kX;X\right)$ the space of all k-homogeneous polynomials from X to X, and by $\mathcal{P}\left({}^kX\right)$ the space of all k-homogeneous scalar polynomials. If $k=0$, $\mathcal{P}\left({}^0X;X\right)$ identify with the space of constant functions, i.e. $\mathcal{P}\left({}^0X;X\right) \equiv X$; for $k=1$, $\mathcal{P}\left({}^1X;X\right)$ is equal to $L(X)$, the algebra of all bounded linear operators on X. We say that $P : X \longrightarrow X$ is a *polynomial* on X, writing $P \in \mathcal{P}(X;X)$, if P is a finite sum of homogeneous polynomials from X to X. We will use the notation $\mathcal{P}(X)$ to denote the space of all finite sums of homogeneous scalar polynomial. Let us recall that $\mathcal{P}(X;X)$ is a normed space if we endowed it with the norm

$$\|P\| := \sup\{\|P(x)\| \ : \ x \in B_X\}.$$

Therefore, $\mathcal{P}(X;X)$ embeds isometrically into $\ell_\infty(B_X, X)$, the Banach space of all bounded functions from B_X to X endowed with the supremum norm. We will write $\ell_\infty(B_X)$ when only scalar valued functions are considered. Analogous definitions will be used changing B_X to S_X.

Generalizing the linear case, we say that $\Phi \in \ell_\infty(B_X, X)$ satisfies the *Daugavet equation* if the norm equality

$$\text{(DE)} \qquad \|\mathrm{Id} + \Phi\| = 1 + \|\Phi\|$$

holds, and we say that Φ satisfies the *alternative Daugavet equation* if there exists $\omega \in \mathbb{T}$ such that $\omega\,\Phi$ satisfies (DE) or, equivalently, if

$$\text{(ADE)} \qquad \max_{|\omega|=1} \|\mathrm{Id} + \omega\,\Phi\| = 1 + \|\Phi\|.$$

It is clear that (DE) implies (ADE) but, in general, they are not the same ($-\mathrm{Id}$ always satisfies (ADE) but never (DE)). Let us mention that a function Φ satisfies (DE) (resp. (ADE)) if and only if so does $\alpha\,\Phi$ for every $\alpha \in \mathbb{R}_0^+$ (see [3, Lemma 2.2], for instance), a fact that we will use in the sequel without any explicit mention. We say that a Banach space X has the *k-order Daugavet property* (*k-DP* in short) if all rank-one k-homogeneous polynomials satisfy (DE). When $k=1$, we simply say that X has the Daugavet property. Analogously, X has the *k-order alternative Daugavet property* (*k-ADP* in short) if all rank-one k-homogeneous polynomials satisfy (ADE), and we use the name alternative Daugavet property in the linear case.

If X is a Banach space, by $C_u(B_X, X)$ we denote the Banach space of all uniformly continuous X valued functions on B_X endowed with the supremum norm. Note that since B_X is convex and bounded, then every function in $C_u(B_X, X)$ is also bounded. If X is a complex Banach space, we denote by $\mathcal{A}_\infty(B_X)$ (resp. $\mathcal{A}_\infty(B_X, X)$) the Banach space of all complex valued (resp. X valued) functions on B_X which are holomorphic in the open unit ball, and bounded and continuous in B_X. $\mathcal{A}_\infty(B_X, X)$ embeds isometrically into $\ell_\infty(S_X, X)$. $\mathcal{A}_u(B_X)$ (resp. $\mathcal{A}_u(B_X, X)$) will stand for the closed subspace of $\mathcal{A}_\infty(B_X)$ (resp. $\mathcal{A}_\infty(B_X, X)$) formed by the functions which admit (a unique) uniformly continuous extension to the closed unit ball of X.

For a Banach space X, we write $\Pi(X)$ to denote the subset of $X \times X^*$ given by

$$\Pi(X) := \{(x, x^*) \ : \ x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1\}.$$

Given a bounded function $\Phi : S_X \longrightarrow X$, its *numerical range* is

$$V(\Phi) := \{x^*(\Phi(x)) \ : \ (x, x^*) \in \Pi(X)\}$$

and the associated *numerical radius* is

$$v(\Phi) := \sup\{|\lambda| \ : \ \lambda \in V(\Phi)\}.$$

Let us comment that for a bounded function $\Phi : \Omega \longrightarrow X$, where $S_X \subset \Omega \subset X$, the above definitions apply by just considering $V(\Phi) := V(\Phi|_{S_X})$.

The outline of the paper is at follows. In the first section we give preliminary results about the (DE) and (ADE) for arbitrary bounded functions on the unit ball of a Banach space with values in the spaces, analogous to those given in [20, Lemma 2.2 and Theorem 2.3] and [22, Proposition 2.1 and Theorem 2.2] for bounded linear operators. We also relate both equations with numerical ranges in the case of bounded and uniformly continuous functions. We devote section 2 to (non-homogeneous) polynomials. We prove that every weakly compact polynomial on $C_b(\Omega, X)$ satisfies (DE), where Ω is a completely regular Hausdorff topological space without isolated points and X is an arbitrary Banach space. We also prove that every polynomial on the complex spaces c_0 and $C(K)$ (K arbitrary) satisfies (ADE). Finally, we study in section 3 the k-DP and the k-ADP. We show that both properties are equivalent in the complex case for $k \geqslant 2$; the same is true in the real case for k even, but not for k odd. Examples of spaces with the k-DP for every $k \geqslant 2$ are the spaces $C_b(\Omega, X)$ when Ω has no isolated points, and the complex spaces c_0 and $C(K)$ (K arbitrary). The real spaces c_0, and $C(K)$ when K has isolated points and more than one point do not have the k-ADP for every $k \geqslant 2$. The real or complex space ℓ_1 does not have the k-ADP for every $k \geqslant 2$.

1. Preliminary generalities

Our first preliminary result is a generalization of [20, Lemma 2.2 and Theorem 2.3], where only bounded linear operators were considered. Actually, our proof is inspired in the proof of those results. The following notation can be useful. If X is a Banach space and $\mathcal{Z}$ is a subspace of $\ell_\infty(B_X)$, we write $\mathcal{Z}^X$ to denote the space of all functions $\Phi : B_X \longrightarrow X$ such that $x^* \circ \Phi \in \mathcal{Z}$ for every $x^* \in X^*$. The simplest elements of $\mathcal{Z}^X$ are the tensors of functions of $\mathcal{Z}$ and elements of X, namely, for $\varphi \in \mathcal{Z}$ and $x_0 \in X$, we denote by $\varphi \otimes x_0$ for the element of $\mathcal{Z}^X$ given by $[\varphi \otimes x_0](x) = \varphi(x) x_0$ for every $x \in B_X$. Let us mention that in this paper we will use the result below for $\mathcal{Z} = \mathcal{P}(X)$ and $\mathcal{Z} = \mathcal{P}\left({}^k X\right)$.

Theorem 1.1. *Let X be a Banach space and let $\mathcal{Z}$ be a subspace of $\ell_\infty(B_X)$. Then, the following are equivalent:*

(i) For every $\varphi \in \mathcal{Z}$ and every $x_0 \in X$, $\varphi \otimes x_0$ satisfies (DE).

(ii) For every $\varphi \in S_{\mathcal{Z}}$, every $x_0 \in S_X$, and every $\varepsilon > 0$, there exist $\omega \in \mathbb{T}$ and $y \in B_X$ such that

$$\operatorname{Re} \omega\varphi(y) > 1 - \varepsilon \qquad \textit{and} \qquad \|x_0 + \omega y\| > 2 - \varepsilon.$$

(iii) Every $\Phi \in \mathcal{Z}^X$ whose image is relatively weakly compact satisfies (DE).

Proof. *(i)* $\Rightarrow$ *(ii)*. Let $\Phi = \varphi \otimes x_0$, which satisfies $\|\mathrm{Id} + \Phi\| = 2$ by *(i)*, so there exists $y \in B_X$ such that

$$\|y + \varphi(y)\, x_0\| > 2 - \varepsilon/2.$$

It follows that $|\varphi(y)| > 1 - \varepsilon/2$ and, calling $\omega = \dfrac{|\varphi(y)|}{\varphi(y)} \in \mathbb{T}$, we have

$$\operatorname{Re} \omega\varphi(y) = |\varphi(y)| > 1 - \varepsilon/2 > 1 - \varepsilon,$$

and

$$\begin{aligned}\|x_0 + \omega\, y\| = \|y + \overline{\omega}\, x_0\| &= \left\| y + \frac{\varphi(y)}{|\varphi(y)|}\, x_0 \right\| \\ &\geqslant \|y + \varphi(y)\, x_0\| - \big||\varphi(y)| - 1\big| > 2 - \varepsilon.\end{aligned}$$

$(ii) \Rightarrow (iii)$. Let us suppose $\|\Phi\| = 1$. Since the set $K = \overline{\mathrm{co}}\left(\mathbb{T}\Phi(B_X)\right)$ is weakly compact, then it coincides with the closed convex hull of its denting points (see [14] for instance). Therefore, given $\varepsilon > 0$, we may take a denting point $y_0 \in K$ with $\|y_0\| > 1 - \varepsilon$. Then, for some $0 < \delta < \varepsilon$, there is a slice

$$S = \{y \in K \ : \ \operatorname{Re} y_0^*(y) \geqslant 1 - \delta\}$$

of K containing y_0 and having diameter less than ε; here $y_0^* \in X^*$ and

$$\sup_{y\in K} \operatorname{Re} y_0^*(y) = \sup_{y\in K} |y_0^*(y)| = 1$$

(observe that K is balanced). In particular,

$$y \in K,\ \operatorname{Re} y_0^*(y) > 1 - \delta \quad \Longrightarrow \quad \|y - y_0\| < \varepsilon. \tag{1}$$

If we take $\varphi := y_0^* \circ \Phi$, then $\varphi \in \mathcal{Z}$ and

$$\|\varphi\| = \sup_{x\in B_X} |y_0^*(\Phi(x))| = \sup_{y\in K} |y_0^*(y)| = 1.$$

Now, we use (ii) with $x_0 = y_0/\|y_0\|$ and φ to get $y \in B_X$ and $\omega \in \mathbb{T}$ such that

$$\operatorname{Re} \omega\varphi(y) > 1 - \delta \qquad \text{and} \qquad \left\| \frac{y_0}{\|y_0\|} + \omega y \right\| > 2 - \delta > 2 - \varepsilon.$$

We observe that

$$\operatorname{Re} y_0^*(\omega\Phi(y)) = \operatorname{Re} \omega\varphi(y) > 1 - \delta,$$

so Eq. (1) and the fact that $\omega\Phi(y) \in K$ gives

$$\|\omega\Phi(y) - y_0\| < \varepsilon.$$

On the other hand,

$$\|y_0 + \omega y\| \geqslant \left\| \omega y + \frac{y_0}{\|y_0\|} \right\| - \left\| y_0 - \frac{y_0}{\|y_0\|} \right\| = \left\| \omega y + \frac{y_0}{\|y_0\|} \right\| - \big|\|y_0\| - 1\big| > 2 - 2\varepsilon.$$

Finally,

$$\|\mathrm{Id} + \Phi\| \geqslant \|y + \Phi(y)\| = \|\omega(y + \Phi(y))\| \geqslant \|\omega y + y_0\| - \|\omega\Phi(y) - y_0\| > 2 - 3\varepsilon.$$

Letting $\varepsilon \downarrow 0$, we conclude that Φ satisfies (DE).

$(iii) \Rightarrow (i)$ is clear. □

For the alternative Daugavet equation we can state an analogous result, as it is done in [22, Proposition 2.1 and Theorem 2.2] for the linear case. Part of it is a direct consequence of the above theorem, and the proof of the other part can be easily adapted.

Corollary 1.2. *Let X be a Banach space and let $\mathcal{Z}$ be a subspace of $\ell_\infty(B_X)$. Then, the following are equivalent:*

- *(i) For every $\varphi \in \mathcal{Z}$ and every $x_0 \in X$, $\varphi \otimes x_0$ satisfies* (ADE).
- *(ii) For every $\varphi \in S_{\mathcal{Z}}$, every $x_0 \in S_X$, and every $\varepsilon > 0$, there exist $\omega_1, \omega_2 \in \mathbb{T}$ and $y \in B_X$ such that*

$$\operatorname{Re} \omega_1\varphi(y) > 1 - \varepsilon \qquad \text{and} \qquad \|x_0 + \omega_2 y\| > 2 - \varepsilon.$$

(*iii*) *For every* $\varphi \in S_{\mathcal{Z}}$*, every* $x_0 \in S_X$*, and every* $\varepsilon > 0$*, there exist* $\omega \in \mathbb{T}$ *and* $y \in B_X$ *such that*
$$|\varphi(y)| > 1 - \varepsilon \qquad \text{and} \qquad \|x_0 + \omega y\| > 2 - \varepsilon.$$
(*iv*) *Every* $\Phi \in \mathcal{Z}^X$ *whose image is relatively weakly compact satisfies* (ADE).

Proof. The equivalence between (*i*) and (*ii*) follows from Theorem 1.1 and the fact that $\varphi \otimes x_0$ satisfies (ADE) if and only if there exists $\omega \in \mathbb{T}$ such that $\omega\varphi \otimes x_0$ satisfies (DE). (*ii*) and (*iii*) are trivially equivalent, and (*iv*) $\Rightarrow$ (*i*) is clear. It is only remains to proof (*ii*) $\Rightarrow$ (*iv*), which is an straightforward adaptation of the one given in the proof of Theorem 1.1. □

For uniformly continuous functions we can state other characterizations of the equations (DE) and (ADE). The proof can be deduced from the general theory of numerical ranges [18], by just proving that the norm of a uniformly continuous function satisfying (DE) can be calculated using only elements of the unit sphere of the space. By the same price, we will give a direct proof.

Proposition 1.3. *Let* X *be a Banach space and let* Φ *be an element of* $C_u(B_X, X)$*. Then, we have*

(a) Φ *satisfies* (DE) *if and only if* $\|\Phi\| = \sup \operatorname{Re} V(\Phi)$.
(b) Φ *satisfies* (ADE) *if and only if* $\|\Phi\| = v(\Phi)$.

Proof. (a). Let us suppose that Φ satisfies (DE). Since Φ is uniformly continuous, for every $\varepsilon > 0$, there exists $0 < \delta < \varepsilon$ such that
$$y, z \in B_X, \ \|y - z\| < \delta \quad \Longrightarrow \quad \|\Phi(y) - \Phi(z)\| < \varepsilon.$$
For every $0 < \varepsilon < 1$ fixed, we may find $y \in B_X$ such that $\|y + \Phi(y)\| > 1 + \|\Phi\| - \delta^2/4$, and then, we may also find $y^* \in S_{X^*}$ such that
$$\operatorname{Re} y^*(y) + \operatorname{Re} y^*(\Phi(y)) > 1 + \|\Phi\| - \delta^2/4.$$
It clearly follows that
$$\operatorname{Re} y^*(y) > 1 - \delta^2/4, \tag{2}$$
$$\operatorname{Re} y^*(\Phi(y)) > \|\Phi\| - \delta^2/4 > \|\Phi\| - \varepsilon. \tag{3}$$
By the Bishop-Phelps-Bollobás' Theorem [10, §16], we deduce from Eq. (2) the existence of a pair $(z, z^*) \in \Pi(X)$ such that
$$\|y - z\| < \delta \qquad \text{and} \qquad \|y^* - z^*\| < \delta < \varepsilon.$$
Now, we have $\|\Phi(y) - \Phi(z)\| < \varepsilon$ by the uniform continuity of Φ and then,
$$\begin{aligned} &|\operatorname{Re} z^*(\Phi(z)) - \operatorname{Re} y^*(\Phi(y))| \\ &\quad \leqslant |\operatorname{Re} z^*(\Phi(z) - \Phi(y))| + |\operatorname{Re} [y^* - z^*](\Phi(z))| + |\operatorname{Re} y^*(\Phi(y) - \Phi(z))| \\ &\quad\quad \leqslant \|\Phi(z) - \Phi(y)\| + \|y^* - z^*\| + \|\Phi(y) - \Phi(z)\| < 3\varepsilon. \end{aligned}$$
With this in mind and Eq. (3) we deduce that
$$\operatorname{Re} z^*(\Phi(z)) > \|\Phi\| - 4\varepsilon.$$
Let us prove the reverse implication. For every $\varepsilon > 0$, we may find a pair $(z, z^*) \in \Pi(X)$ such that
$$\operatorname{Re} z^*(\Phi(z)) > \|\Phi\| - \varepsilon.$$
Therefore,
$$\|\mathrm{Id} + \Phi\| \geqslant \operatorname{Re} z^*(z + \Phi(z)) = 1 + \operatorname{Re} z^*(\Phi(z)) > 1 + \|\Phi\| - \varepsilon,$$
and the result follows by letting $\varepsilon \downarrow 0$.

(b). If Φ satisfies (ADE), then there exists $\omega \in \mathbb{T}$ such that $\omega\,\Phi$ satisfies (DE) and, by (a), we have

$$\|\Phi\| = \|\omega\,\Phi\| = \sup \operatorname{Re} V(\omega\Phi) \leqslant v(\Phi) \leqslant \|\Phi\|.$$

Conversely, given $(x, x^*) \in \Pi(X)$, we have

$$\max_{\omega\in\mathbb{T}} \|\mathrm{Id} + \omega\,\Phi\| \geqslant \max_{\omega\in\mathbb{T}} |x^*(x) + \omega\, x^*(\Phi(x))| = 1 + |x^*(\Phi(x))|.$$

Therefore,

$$\max_{\omega\in\mathbb{T}} \|\mathrm{Id} + \omega\,\Phi\| \geqslant 1 + \sup\{|x^*(\Phi(x))| \ : \ (x, x^*) \in \Pi(X)\} = 1 + \|\Phi\|,$$

and the result follows since the other inequality is always true. □

Remark 1.4. Let X be a Banach space and $\Phi \in B_u(B_X, X)$. As a consequence of the above proposition, if Φ satisfies (ADE), then the norm of Φ can be calculated using only elements in S_X, that is, $\|\Phi\| = \sup\{\|\Phi(y)\| \ : \ y \in S_X\}$.

2. The Daugavet equation for polynomials

We start the section studying the simplest examples: $\mathbb{R}$ and $\mathbb{C}$.

Examples 2.1.

(a) Since the linear operators on a Banach space are polynomials, *neither* $\mathbb{R}$ *nor* $\mathbb{C}$ *have the property of that every (weakly compact) polynomial on them satisfies* (DE).

(b) *Every polynomial on* $\mathbb{C}$ *satisfies* (ADE). Indeed, let us fix $P \in \mathcal{P}(\mathbb{C};\mathbb{C})$ with $\|P\| = 1$. Then, by the Maximum Modulus Theorem, there exists $y \in S_{\mathbb{C}} \equiv \mathbb{T}$ such that $|P(y)| = \|P\| = 1$, and we may find $\omega_1 \in \mathbb{T}$ such that

$$\operatorname{Re}\, \omega_1 P(y) = |P(y)| = 1.$$

On the other hand, we may find $\omega_2 \in \mathbb{T}$ such that

$$\operatorname{Re}\, \omega_2 y = |y| = 1.$$

Then, writing $\omega = \overline{\omega_2}\,\omega_1 \in \mathbb{T}$, we have

$$\|\mathrm{Id} + \omega\, P\| \geqslant |y + \omega\, P(y)| = |\omega_2 y + \omega_1\, P(y)| = 2.$$

(c) The above result is not valid in the real case, i.e. *there exists a (weakly compact) polynomial* $P \in \mathcal{P}(\mathbb{R};\mathbb{R})$ *such that* $\|\mathrm{Id} \pm P\| < 1 + \|P\|$. Indeed, if we define

$$P(t) = 1 - t^2 \qquad (t \in \mathbb{R}),$$

we have $\|P\| = 1$ and

$$\|\mathrm{Id} \pm P\| = \max_{t\in[-1,1]} |t \pm (1 - t^2)| = \frac{5}{4} < 2.$$

Our next aim is to present a wide family of Banach spaces in which all weakly compact polynomials satisfy (DE). We will use Theorem 1.1, which in terms of polynomials read as follows.

Corollary 2.2. *Let X be a Banach space. Then, the following are equivalent:*

(i) For every $p \in \mathcal{P}(X)$ and every $x_0 \in X$, the polynomial $p \otimes x_0$ satisfies (DE).

(ii) For every $p \in \mathcal{P}(X)$ with $\|p\| = 1$, every $x_0 \in S_X$, and every $\varepsilon > 0$, there exist $\omega \in \mathbb{T}$ and $y \in B_X$ such that

$$\operatorname{Re}\, \omega p(y) > 1 - \varepsilon \qquad \textit{and} \qquad \|x_0 + \omega y\| > 2 - \varepsilon.$$

(iii) Every weakly compact $P \in \mathcal{P}(X;X)$ satisfies (DE).

Definition 2.3. Let Ω be a completely regular Hausdorff topological space and let X be a Banach space. We say that a subspace $\mathcal{F}$ of $C_b(\Omega, X)$, the Banach space of all bounded X valued continuous functions on Ω endowed with the supremum norm, is *C_b-rich* if for every open subset U of Ω, every $x \in X$ and every $\varepsilon > 0$ there exists a continuous function $\varphi : \Omega \longrightarrow [0,1]$ of norm one with support included in U such that the distance of $\varphi \otimes x$ to $\mathcal{F}$ is less than ε.

We can assume in the definition that there is $t_0 \in U$ such that $\varphi(t_0) = 1$. Indeed, given $0 < \varepsilon < 1$, consider φ as above. Since φ has norm one and support in U, we can find $t_0 \in U$ and an open neighborhood V of t_0 such that $\varphi(t) > 1 - \varepsilon$ for all $t \in V$. As Ω is completely regular, there exists $\eta : \Omega \longrightarrow [0,1]$ such that $\eta(t_0) = 1$ and η vanishes in $\Omega \setminus V$. If we take $\phi(t) = \max\{\varphi(t), \eta(t)\}$ for $t \in \Omega$ then $\phi : \Omega \longrightarrow [0,1]$ is continuous, it has support included in U, $\phi(t_0) = 1$ and $\|\varphi - \phi\| \leqslant \varepsilon$, thus $\operatorname{dist}(\phi \otimes x, \mathcal{F}) < 2\varepsilon$.

This definition is an straightforward extension of the definition of C-rich subspace of a C(K) given in [8, Definition 2.3].

Now, we can state the main result of the section.

Theorem 2.4. *Let Ω be a completely regular Hausdorff topological space without isolated points, let X be a Banach space and let $\mathcal{F}$ be a C_b-rich subspace of $C_b(\Omega, X)$. Then, every weakly compact polynomial from $\mathcal{F}$ into itself satisfies the Daugavet equation.*

Proof. Let $p \in \mathcal{P}(\mathcal{F})$ such that $\|p\| = 1$ and fix $f_0 \in S_{\mathcal{F}}$. Since p is uniformly continuous when restricted to any bounded subset of $\mathcal{F}$, given $\varepsilon > 0$, there exists $0 < \delta < \min\{\frac{\varepsilon}{10}, \frac{1}{6}\}$ such that

$$|p(f) - p(g)| < \frac{\varepsilon}{2}, \tag{4}$$

for all $f, g \in \mathcal{F}$ satisfying $\|f - g\| < 2\delta$, $\|f\| \leqslant 2$, $\|g\| \leqslant 2$. We take $h \in \mathcal{F}$ with $\|h\| \leqslant 1$ such that $|p(h)| > 1 - \frac{\varepsilon}{4}$, and we also take $\omega \in \mathbb{T}$ such that

$$\operatorname{Re} \omega p(h) = |p(h)| > 1 - \frac{\varepsilon}{4}. \tag{5}$$

Fix a point $\tau \in \Omega$ with $\|f_0(\tau)\| > 1 - \frac{\delta}{4}$ and find an open neighborhood U of τ such that

$$\|f_0(t) - f_0(\tau)\| < \frac{\delta}{4}$$

for all $t \in U$. Decreasing U if it is necessary, we may assume that

$$\|h(s) - h(t)\| < \frac{\delta}{2}$$

for all $s, t \in U$. Select a sequence $U_1, U_2, \dots$ of non-empty pairwise disjoint open subsets of U. We fix $s_j \in U_j$ for all $j = 1, 2, \dots$ and denote $x_j = \omega^{-1} f_0(s_j) - h(s_j)$. As $\mathcal{F}$ is C_b-rich, we can find continuous non-negative functions $\eta_j : \Omega \longrightarrow [0,1]$ of norm one and $z_j \in \mathcal{F}$ such that

$$\operatorname{supp} \eta_j \subset U_j \qquad \text{and} \qquad \|\eta_j \otimes x_j - z_j\| < \frac{\delta}{2^{j+2}} \|\eta_j \otimes x_j\| \tag{6}$$

for all $j \in \mathbb{N}$. By disjointness of their supports the functions $\eta_j \otimes x_j / \|\eta_j \otimes x_j\|$ form a sequence 1-equivalent to the canonical basis of c_0. Thanks to (6), the sequence z_j spans a subspace isomorphic to c_0 and tends weakly to zero. So by the Bogdanowicz Theorem on weak continuity of polynomials on c_0 [6] (see also [15, Proposition 1.59]), $|p(h + z_j) - p(h)|$ tend to 0 as $j \to \infty$. So there is such a j_0 that

$$|p(h + z_{j_0}) - p(h)| < \frac{\varepsilon}{4}.$$

Combining with Eq. (5), we have

$$\text{Re } \omega p(h - z_{j_0}) > 1 - \frac{\varepsilon}{2}. \tag{7}$$

We define

$$A = h + z_{j_0}, \qquad B = h + \eta_j \otimes x_j, \qquad g = \frac{A}{\|A\|}.$$

Clearly, $B \in C_b(\Omega, X)$, $A, g \in \mathcal{F}$, $\|g\| = 1$, and $\|A - B\| < \frac{\delta}{2}$. We have that

$$1 + 2\delta > \|A\| > 1 - 2\delta. \tag{8}$$

Indeed, for the lower bound, select $t_0 \in U_{j_0}$ in which $\eta_{j_0}(t_{j_0}) = 1$; we have

$$\begin{aligned}\|A\| &\geqslant \|B(t_{j_0})\| - \|A - B\| \\ &= \|h(t_{j_0}) - h(s_{j_0}) + \omega^{-1} f_0(s_{j_0})\| - \|A - B\| \\ &\geqslant \|f_0(s_{j_0})\| - \|h(t_{j_0}) - h(s_{j_0})\| - \frac{\delta}{2} > 1 - 2\delta.\end{aligned}$$

On the other hand, for the upper bound,

$$\|A\| \leqslant \|B\| + \|A - B\| < \|B\| + \frac{\delta}{2},$$

and $\|B(t)\| = \|h(t)\| \leqslant 1$ if $t \notin U_{j_0}$; if $t \in U_{j_0}$ then

$$\|B(t)\| \leqslant \|h(t) - h(s_{j_0})\| + (1 - \eta_{j_0}(t))\|h(s_{j_0})\| + \eta_{j_0}(t)\|f_0(s_{j_0})\| < 1 + \delta.$$

Now, by Eq. (8),

$$\|g - A\| = |1 - \|A\|| < 2\delta.$$

Hence, by equations (4) and (7),

$$\text{Re } \omega p(g) > 1 - \varepsilon,$$

and

$$\begin{aligned}\|f_0 + \omega g\| &\geqslant \left\| f_0 + \omega \frac{B}{\|A\|} \right\| - \frac{\|A - B\|}{\|A\|} \\ &\geqslant \left\| f_0(t_{j_0}) + \omega \frac{B(t_{j_0})}{\|A\|} \right\| - \frac{\delta}{2\|A\|} \\ &\geqslant \left\| f_0(t_{j_0}) + \frac{\omega(h(t_{j_0}) - h(s_{j_0})) + f_0(s_{j_0})}{\|A\|} \right\| - \frac{\delta}{2\|A\|} \\ &\geqslant \left\| f_0(s_{j_0}) + \frac{f_0(s_{j_0})}{\|A\|} \right\| - \|f_0(t_{j_0}) - f_0(s_{j_0})\| - \frac{\|h(t_{j_0}) - h(s_{j_0})\|}{\|A\|} - \frac{\delta}{2\|A\|} \\ &> \|f_0(s_{j_0})\| \left(1 + \frac{1}{\|A\|}\right) - \frac{\delta}{2} - \frac{\delta}{\|A\|} - \frac{\delta}{2\|A\|} > 2 - \varepsilon.\end{aligned}$$

The last inequality is a consequence of the fact that $\|f_0(s_{j_0})\| > 1 - \frac{\delta}{2}$, Eq. (8), and the way in which δ has been chosen. By Corollary 2.2, we have that every weakly compact polynomial from $\mathcal{F}$ into itself satisfies the Daugavet equation. □

If Ω is a completely regular Hausdorff topological space and X is a Banach space, then $C_b(\Omega, X)$ is C_b-rich of itself. Hence we have the following straightforward corollary.

Corollary 2.5. *Let Ω be a completely regular Hausdorff topological space without isolated points and let X be a Banach space. Then, every weakly compact polynomial from $C_b(\Omega, X)$ into itself satisfies the Daugavet equation.*

The density of the space of polynomials on a complex Banach space Z in $\mathcal{A}_u(B_Z, Z)$, gives us the following consequence of the above corollary.

Corollary 2.6. *Let Ω be a completely regular Hausdorff topological space without isolated points and let X be a complex Banach space. Then, every weakly compact Φ in $\mathcal{A}_u\big(B_{C_b(\Omega,X)}, C_b(\Omega, X)\big)$ satisfies the Daugavet equation.*

Proof. We denote by P_k the k-homogeneous polynomial of the Taylor series expansion of Φ at 0. By the Cauchy Integral Formula, we have $P_k(B_X)$ is contained in the closed and absolutely convex hull of $\Phi(B_X)$. As a consequence, P_k is weakly compact for all k. Hence, the Taylor polynomials of Φ are weakly compact too. For each n, $\Phi_n(f) := \Phi(\frac{n-1}{n} f)$ belongs to $\mathcal{A}_u\big(B_{C_b(\Omega,X)}, C_b(\Omega, X)\big)$ and the sequence (Φ_n) converges uniformly to Φ on the closed unit ball of $C_b(\Omega, X)$. Moreover, for each Φ_n its Taylor series expansion converges again uniformly to Φ_n on the closed unit ball of $C_b(\Omega, X)$. Thus, given $\varepsilon > 0$ there exists a weakly compact polynomial P such that

$$\|\Phi - P\| < \varepsilon.$$

Hence, by Theorem 2.4, we have

$$\begin{aligned}\|\mathrm{Id} + \Phi\| &\geqslant \|\mathrm{Id} + P\| - \|\Phi - P\| = 1 + \|P\| - \|\Phi - P\| \\ &\geqslant 1 + \|\Phi\| - 2\|\Phi - P\| > 1 + \|\Phi\| - 2\varepsilon,\end{aligned}$$

and the conclusion follows. □

More examples of C_b-rich subspaces of $C_b(\Omega, X)$ spaces appear in [19] (see [8, Proposition 2.5]): if K is a compact space without isolated points, any finite-codimensional subspace of $C(K)$ is C-rich in $C(K)$, and Theorem 2.4 applies. Therefore, we get the following.

Corollary 2.7. *Let K be a compact Hausdorff topological space without isolated points, and let Y be a finite-codimensional subspace of $C(K)$. Then, every weakly compact polynomial in $\mathcal{P}(Y; Y)$ satisfies* (DE).

If K is a compact Hausdorff topological space with isolated points, the above result is not valid, since there exist weakly compact polynomials from $C(K)$ into itself which do not satisfy (DE) (actually, the examples can be operators). One may wonder if at least every weakly compact polynomial from $C(K)$ into itself satisfies (ADE). In the real case, this is not the case, since we have seen at the beginning of this section (Examples 2.1.c) that even in the simplest case in which K has only one element (i.e. $C(K) = \mathbb{R}$), there are (weakly compact) polynomials which do not satisfy (ADE). In the complex case, the situation is completely different, as we see in the next result.

Theorem 2.8. *Let K be a compact Hausdorff space and let X be the complex space $C(K)$. Then, for every $\Phi \in \mathcal{A}_\infty(B_X, X)$, we have $v(\Phi) = \|\Phi\|$.*

To prove the above theorem, we need a preliminary result.

Lemma 2.9. *Let Ω be a set, let $\mathcal{F}$ be a subspace of $\ell_\infty(\Omega)$, and let $\Lambda \subseteq \Omega$ be a norming set for $\mathcal{F}$ (i.e. for every $f \in \mathcal{F}$, $\|f\| = \sup\{|f(\lambda)| \ : \ \lambda \in \Lambda\}$). Then, given a Banach space Y and a function $\Phi \in \ell_\infty(\Omega, Y)$ such that $y^* \circ \Phi \in \mathcal{F}$ for every $y^* \in Y^*$, we have*

$$\|\Phi\| = \sup\left\{\|\Phi(\lambda)\| \ : \ \lambda \in \Lambda\right\}.$$

Proof. We clearly have

$$\begin{aligned}\|\Phi\| &= \sup_{t\in\Omega} \|\Phi(t)\| = \sup_{t\in\Omega} \sup_{y^*\in B_{Y^*}} |y^*(\Phi(t))| \\ &= \sup_{y^*\in B_{Y^*}} \sup_{t\in\Omega} |y^*(\Phi(t))| = \sup_{y^*\in B_{Y^*}} \|y^* \circ \Phi\| \\ &= \sup_{y^*\in B_{Y^*}} \sup_{t\in\Lambda} |[y^* \circ \Phi](t)| = \sup_{t\in\Lambda} \sup_{y^*\in B_{Y^*}} |[y^* \circ \Phi](t)| = \sup_{t\in\Lambda} \|\Phi(t)\|.\end{aligned}$$

□

Proof of Theorem 2.8. By [4, Theorem 4.3], the set $\mathrm{Ext}(B_{C(K)})$ of all extreme points of $B_{C(K)}$ is a norming set for $A_\infty(B_{C(K)})$ and, by the above lemma, given $\varepsilon > 0$, there exists $e \in \mathrm{Ext}(B_{C(K)})$ such that $\|\Phi(e)\| > \|\Phi\| - \varepsilon$. On one hand, since $\Phi(e) \in C(K)$, there exists $t \in K$ such that $[\Phi(e)](t) = \|\Phi(e)\|$. On the other hand, since $e \in \mathrm{Ext}(B_{C(K)})$, we have $|e(t)| = 1$. Now, we observe that

$$|\delta_t(\Phi(e))| = \big|[\Phi(e)](t)\big| = \|\Phi(e)\| > \|\Phi\| - \varepsilon,$$

and $|\delta_t(e)| = |e(t)| = 1$. Therefore, $v(\Phi) \geqslant \|\Phi\| - \varepsilon$. □

In particular,

Corollary 2.10. *Let K be a compact Hausdorff space. Then, every polynomial from the complex space $C(K)$ into itself satisfies* (ADE).

Example 2.11. *Let X be the complex space c_0. Then, every polynomial from X into X satisfies* (ADE). Indeed, given $\Phi \in \mathcal{A}_u(B_X, X)$ and $\varepsilon > 0$, there exists $x_0 \in S_X$ such that $\|\Phi(x_0)\| > \|\Phi\| - \varepsilon$. Thus, we can find j such that

$$|e_j^*(\Phi(x_0))| > \|\Phi\| - \varepsilon,$$

where e_j^* is the associated functional to the jth-element e_j of the canonical basis of X. We define $f : \mathbb{C} \longrightarrow \mathbb{C}$ by

$$f(z) = e_j^*(\Phi(x_0 + (z - e_j^*(x_0))e_j)) \qquad (z \in \mathbb{C}).$$

It is an element of $\mathcal{A}_u(B_{\mathbb{C}})$. By the Maximum Modulus Theorem, there exists $z_0 \in \mathbb{C}$, $|z_0| = 1$ such that $|f(z)| \leqslant |f(z_0)|$ for all z in the closed unit disk of $\mathbb{C}$. In particular,

$$\|\Phi\| - \varepsilon < |e_j^*(\Phi(x_0))| = |f(e_j^*(x_0))| \leqslant |f(z_0)| = |e_j^*(\Phi(x_0 + (z_0 - e_j^*(x_0))e_j))|.$$

But $x_1 := x_0 + (z_0 - e_j^*(x_0))e_j \in S_X$, and clearly $|e_j^*(x_1)| = 1$. Hence,

$$\|\Phi\| - \varepsilon < |e_j^*(\Phi(x_1))| \leqslant v(\Phi).$$

Finally, Proposition 1.3 would apply to get the conclusion.

Let us comment that in [12, Theorem 3.1] it is proved that every continuous k-linear mapping $A : X^k \longrightarrow X$ satisfies that $v(A) = \|A\|$ for X being c_0, c and ℓ_∞, and it is claimed that every k-homogeneous polynomial P on X satisfies $v(P) = \|P\|$. Unfortunately the last claim is false for the cases of real c_0, c and ℓ_∞ as we will show in Example 3.14. Nevertheless, the claim is true for the complex c_0, c and ℓ_∞, not only for k-homogeneous polynomials but for any element of $\mathcal{A}_u(B_X, X)$, as we have shown in Theorem 2.8 and in the proof of Example 2.11.

3. The Daugavet equation for k-homogeneous polynomials

The aim of this section is to study the k-order Daugavet property and the k-order alternative Daugavet property. This study makes sense since the Examples 3.4.(b) below shows that there exist spaces with weakly compact polynomials which do not satisfy (ADE) and nevertheless every weakly compact k-homogeneous polynomial satisfies (ADE).

We start the section presenting the first examples of spaces with the k-DP and the k-ADP. The results are obvious consequences of Theorem 2.4, Corollary 2.10, and Example 2.11.

Examples 3.1.

(a) *Let Ω be a completely regular Hausdorff topological space without isolated points, and let X be a Banach space. Then, the space $C_b(\Omega, X)$ has the k-order Daugavet property for every $k \in \mathbb{N}$.*
(b) *Let K be a compact Hausdorff space. Then, the complex space $C(K)$ has the k-order alternative Daugavet property for every $k \in \mathbb{N}$.*

(c) *The complex space c_0 has the k-order alternative Daugavet property for every $k \in \mathbb{N}$.*

The following result gives us a surprising behavior of the Daugavet equation and the alternative Daugavet equation for k-homogeneous polynomials with k greater than 1.

Proposition 3.2. *Let X be a Banach space over $\mathbb{K}$ and let k be an integer, $k \geqslant 2$.*

(a) *If $\mathbb{K} = \mathbb{C}$, then the* (DE) *and the* (ADE) *are equivalent in $\mathcal{P}\left({}^k X; X\right)$.*
(b) *If $\mathbb{K} = \mathbb{R}$ and k is even, then the* (DE) *and the* (ADE) *are equivalent in $\mathcal{P}\left({}^k X; X\right)$.*

Proof. Only (ADE) $\Rightarrow$ (DE) has to be proved.

(a) If $P \in \mathcal{P}\left({}^k X; X\right)$ satisfies (ADE), then there is $\omega \in \mathbb{T}$ such that

$$\|\mathrm{Id} + \omega P\| = 1 + \|P\|.$$

We take $\beta \in \mathbb{T}$ such that $\beta^{k-1} = \omega$ (observe that we need $k \geqslant 2$ for this!). For every $\varepsilon > 0$, we may find $x \in B_X$ such that

$$\|x + \omega\, P(x)\| > 1 + \|P\| - \varepsilon.$$

If we write $y = \beta x \in B_X$, we have

$$\begin{aligned}\|y + P(y)\| &= \|\beta x + P(\beta x)\| = \|\beta x + \beta^k P(x)\| \\ &= |\beta|\, \|x + \beta^{k-1} P(x)\| = \|x + \omega\, P(x)\| > 1 + \|P\| - \varepsilon.\end{aligned}$$

Letting $\varepsilon \downarrow 0$, we get $\|\mathrm{Id} + P\| \geqslant 1 + \|P\|$ and P satisfies (DE).

(b) The above argument can be stated in the real case if $k - 1$ is odd. $\square$

As an immediate consequence, we obtain the following.

Corollary 3.3. *Let X be a Banach space and let k be an integer, $k \geqslant 2$.*

(a) *If X is complex, then the k-DP and the k-ADP are equivalent.*
(b) *If X is real and k is even, then the k-DP and the k-ADP are equivalent.*

Let us give more examples.

Examples 3.4.

(a) By Examples 2.1.(b), the complex space $\mathbb{C}$ has the k-ADP for every $k \in \mathbb{N}$. Then, by the above corollary, *$\mathbb{C}$ has the k-DP when $k \geqslant 2$, but $\mathbb{C}$ does not have the 1-DP.*
(b) Although there are non-homogeneous polynomials on $\mathbb{R}$ which do not satisfy (ADE) (see Examples 2.1.(c)), it is easy to check that *the real space $\mathbb{R}$ has the k-ADP for every $k \in \mathbb{N}$.*
(c) Thus, *$\mathbb{R}$ has the k-DP when k is even* by the above corollary. On the other hand, *if k is odd, $\mathbb{R}$ does not have the k-DP* as shown by the polynomial $P \in \mathcal{P}\left({}^k\mathbb{R}; \mathbb{R}\right)$ given by

$$P(t) = -t^k \qquad (t \in \mathbb{R}).$$

(d) By Corollary 2.10 and Example 2.11 we know that every polynomial from one of the complex spaces c_0, and $C(K)$, K arbitrary, into itself satisfies (ADE), but if we restrict ourself to the case of k-homogeneous polynomials, Corollary 3.3 gives us more: *The complex spaces c_0, and $C(K)$, K arbitrary, have the k-order Daugavet property for every $k \geqslant 2$.*

Remark 3.5. The item (c) in the above examples shows that, in the real case, the equations (DE) and (ADE) are not equivalent in $\mathcal{P}\left({}^k X; X\right)$ for odd k's. It also shows that the k-DP and the k-ADP are not equivalent in the real case for odd k's.

The next results will allow us to show that the real spaces c_0, c and ℓ_∞ do not have the k-ADP for any $k \geqslant 2$. We need some definitions. A closed subspace Y of a Banach space X is said to be an *absolute summand* of X if there exists another closed subspace Z such that $X = Y \oplus Z$ and, for every $y \in Y$ and $z \in Z$, the norm of $y+z$ only depends on $\|y\|$ and $\|z\|$. We also say that X is an *absolute sum* of Y and Z. This implies that there exists an absolute norm on $\mathbb{R}^2$ such that

$$\|x+z\| = |(\|x\|, \|z\|)|_a \qquad (x \in X,\ z \in Z),$$

where by an *absolute norm* we mean a norm $|\cdot|_a$ on $\mathbb{R}^2$ such that $|(1,0)|_a = |(0,1)|_a = 1$ and $|(a,b)|_a = |(|a|,|b|)|_a$ for every $a, b \in \mathbb{R}$. We refer the reader to [10, § 21] for background. Examples of absolute summand are the ℓ_p-sums of Banach spaces for $1 \leqslant p \leqslant \infty$.

The proof of the following result is straightforward. The reader may check [11, Proposition 2.8] to see the main idea.

Proposition 3.6. *Let X be a Banach space and let us suppose that X is an absolute sum of two closed subspaces Y and Z. If X has the k-DP (resp. the k-ADP) for some $k \in \mathbb{N}$, then so do Y and Z.*

Now, we will study the relationship between the k-DP and the k-ADP for different values of the integer k.

Proposition 3.7. *Let X be a Banach space and let k be a positive integer. If X has the $(k+1)$-ADP, then X has the k-ADP.*

Proof. Let us fix $P \in \mathcal{P}\left({}^k X; X\right)$, $P \neq 0$. For every $0 < \varepsilon < \|P\|$, we may find $y \in S_X$ and $y^* \in S_{X^*}$ such that

$$\|P(y)\| > \|P\| - \varepsilon \qquad \text{and} \qquad |y^*(y)| = 1.$$

If we define $Q \in \mathcal{P}\left({}^{k+1} X; X\right)$ by

$$Q(x) = y^*(x)\, P(x) \qquad (x \in X),$$

we have $\|Q\| \geqslant \|Q(y)\| = |y^*(y)| \|P(y)\| > \|P\| - \varepsilon$. Since X has the $(k+1)$-ADP, we may find $\omega_1 \in \mathbb{T}$ and $z \in S_X$ such that

$$1 + \|P\| - \varepsilon < \|z + \omega_1 Q(z)\| = \|z + y^*(z)\omega_1\, P(z)\|.$$

It follows that $|y^*(z)| > 1 - \frac{\varepsilon}{\|P\|}$ and, taking $\omega = \dfrac{y^*(z)}{|y^*(z)|}\,\omega_1 \in \mathbb{T}$, we have

$$\begin{aligned}\|\mathrm{Id} + \omega P\| &\geqslant \|z + \omega P(z)\| = \left\| z + \frac{y^*(z)\,\omega_1\, P(z)}{|y^*(z)|} \right\| \\ &\geqslant \|z + y^*(z)\omega_1 P(z)\| - \left| 1 - \frac{1}{|y^*(z)|} \right| \|P(z)\| > 1 + \|P\| - \varepsilon - \frac{\varepsilon \|P\|}{\|P\| - \varepsilon}.\end{aligned}$$

Letting $\varepsilon \downarrow 0$, we get $\|\mathrm{Id} + \omega P\| = 1 + \|P\|$ and X has the k-ADP. ☐

In the complex case, Proposition 3.7 can be read in terms of the k-Daugavet property for $k \geqslant 2$ since, in this case, the k-ADP and the k-DP are equivalent.

Corollary 3.8. *Let X be a complex Banach space and let k be a positive integer, $k \geqslant 2$. If X has the $(k+1)$-DP, then X has the k-DP.*

Remarks 3.9.

(a) The above result is not valid for $k = 1$. Indeed, the complex space c_0 has the 2-DP, but it does not have the 1-DP.

(b) The above result is false in the real case, since the real space $\mathbb{R}$ has the $2m$-DP for every $m \in \mathbb{N}$, but it does not have the $(2m-1)$-DP for any $m \in \mathbb{N}$.

In the real case it can be proved a result similar to the above corollary if we allow a two steps jump.

Proposition 3.10. *Let X be a real Banach space and let k be a positive integer. If X has the $(k+2)$-DP, then X has the k-DP.*

Proof. Let us fix $P \in \mathcal{P}\left({}^k X; X\right)$. For every $0 < \varepsilon < 1$, we may find $y \in S_X$ and $y^* \in S_{X^*}$ such that

$$\|P(y)\| > \|P\| - \varepsilon \qquad \text{and} \qquad y^*(y) = 1.$$

If we define $Q \in \mathcal{P}\left({}^{k+2} X; X\right)$ by

$$Q(x) = \left(y^*(x)\right)^2 P(x) \qquad (x \in X),$$

we have

$$\|Q\| \geqslant \|Q(y)\| = \left(y^*(y)\right)^2 \|P(y)\| > \|P\| - \varepsilon.$$

Since X has the $(k+2)$-DP, we may find $z \in S_X$ such that

$$1 + \|P(z)\| - \varepsilon \leqslant 1 + \|P\| - \varepsilon < \|z + Q(z)\| = \left\| z + \left(y^*(z)\right)^2 P(z) \right\|.$$

It follows that $\left(y^*(z)\right)^2 \|P(z)\| > \|P(z)\| - \varepsilon$, and so

$$\|\mathrm{Id} + P\| \geqslant \|z + P(z)\| = \left\| z + \left(y^*(z)\right)^2 P(z) \right\| - \left(1 - \left(y^*(z)\right)^2\right) \|P(z)\| > 1 + \|P\| - 2\varepsilon. \square$$

Remark 3.11. The above result is not valid in the complex case for $k = 1$. Indeed, the complex space c_0 has the 3-DP but it does not have the 1-DP.

In [12, pp. 141] it is shown a weakly compact 2-homogeneous polynomial $P \in \mathcal{P}\left({}^2\ell_1; \ell_1\right)$ (either in the real case and in the complex case) which satisfies $v(P) \leqslant \frac{1}{2}\|P\|$. This fact, together with Propositions 1.3 and 3.7, gives us the following example.

Example 3.12. *The space ℓ_1 does not have the k-ADP for any $k \geqslant 2$.*

Remark 3.13. The converse of Proposition 3.7 is not true. For instance, ℓ_1 has the 1-ADP but it does not have the 2-ADP.

It is actually proved in [12, pp. 141] that the polynomial $P : \ell_1^2 \longrightarrow \ell_1^2$ defined by

$$P(x_1, x_2) = \left(\frac{1}{2}x_1^2 + 2x_1x_2, -\frac{1}{2}x_2^2 - x_1x_2\right)$$

satisfies $\|P\| = 1$ and $v(P) = \frac{1}{2}$. Therefore, ℓ_1^2 does not have the 2-ADP. Since, in the real case, ℓ_1^2 and ℓ_∞^2 are isometric, we get that this latter space does not have the 2-ADP but, actually, much more examples can be deduced from this fact.

Example 3.14. *The real spaces c_0, c, ℓ_∞, ℓ_∞^n for every $n \geqslant 2$ do not have the k-ADP for any $k \geqslant 2$.* Indeed, since the real space ℓ_∞^2 does not have the 2-ADP, and all the above cited spaces contain ℓ_∞^2 as an ℓ_∞-summand, Proposition 3.6 give us the case $k = 2$ and the rest follows from Proposition 3.7.

The above argument also shows that real $C(K)$ when K is a compact set with at least two isolated points do not have the k-ADP for any $k \geqslant 2$. But this fact can be improved, as the next example shows.

Example 3.15. *Let K be a non-perfect compact Hausdorff space with at least two points. Then, the real space $C(K)$ does not satisfy the k-ADP for any $k \geqslant 2$.* Indeed, we consider an isolated point x_1 of K and $x_2 \in K \setminus \{x_1\}$, and we define $p : C(K) \longrightarrow \mathbb{R}$ by

$$p(f) = f(x_2)^2 - \frac{1}{2}f(x_1)^2,$$

for all $f \in C(K)$. Clearly $\|p\| = 1 = p(\chi_{K \setminus \{x_1\}})$. Suppose that $C(K)$ has the 2-ADP. Then, by Corollary 1.2, we can find a sequence (f_n) of norm-one elements of $C(K)$ such that

$$\left| f_n(x_2)^2 - \frac{1}{2} f_n(x_1)^2 \right| \longrightarrow 1 \qquad \text{and} \qquad \left\| \chi_{\{x_1\}} + f_n \right\| \longrightarrow 2. \tag{9}$$

On one hand, the first part of the above equation implies that

$$(|f_n(x_2)|) \longrightarrow 1 \qquad \text{and} \qquad (|f_n(x_1)|) \longrightarrow 0.$$

On the other hand, the only way in which $\|\chi_{\{x_1\}} + f_n\| > 1$ is that

$$\|\chi_{\{x_1\}} + f_n\| = |1 + f_n(x_1)|$$

and, therefore, the second part of Eq. (9) implies that

$$|1 + f_n(x_1)| \longrightarrow 2.$$

A contradiction.

Taking a look at all the examples that we have given, one may wonder if the k-DP or even the k-ADP for a Banach space X implies that every k-homogeneous polynomial satisfies (ADE). This is not the case, as the following example shows.

Example 3.16. Let us consider the Banach space $C([0,1], \ell_2)$ and let k be a positive integer. On the one hand, every weakly compact polynomial on $C([0,1], \ell_2)$ satisfies the (DE) by Corollary 2.5. On the other hand, [11, Propositions 2.8 and 2.9] (see also the remark after Proposition 2.9 in [11]) shows that there exists a (non-weakly compact) k-homogeneous polynomial on $C([0,1], \ell_2)$ which does not even satisfy the (ADE).

Acknowledgements: The authors would like to thank the referee, whose comments have improved the final form of the paper. In particular, for providing us with a shorter proof of Theorem 2.4.

References

[1] Y. Abramovich and C. Aliprantis, *An invitation to Operator Theory*, Graduate Texts in Math. **50**, AMS, Providence, RI, 2002.

[2] Y. Abramovich and C. Aliprantis, *Problems in Operator Theory*, Graduate Texts in Math. **51**, AMS, Providence, RI, 2002.

[3] Y. A. Abramovich, C. D. Aliprantis, and O. Burkinshaw, The Daugavet equation in uniformly convex Banach spaces, *J. Funct. Anal.* **97** (1991), 215–230.

[4] M. D. Acosta, Boundaries for spaces of holomorphic functions on $C(K)$, *Publ. Res. I. Math. Sci.* **42** (2006), 27–44.

[5] F. L. Bauer, On the field of values subordinate to a norm, *Numer. Math.*, **4** (1962), 103–111.

[6] W. Bogdanowicz, On the weak continuity of the polynomial functionals defined on the space c_0 (Russian), *Bull. Acad. Polon. Sci. Cl. III.* **5** (1957), 243–246.

[7] F. F. Bonsall, B. E. Cain, and H. Schneider, The numerical range of a continuos mapping of a normed space, *Aequationes Math.* **2** (1968), 86–93.

[8] K. Boyko, V. Kadets, M. Martín and D. Werner, Numerical index of Banach spaces and duality, *Math. Proc. Cambridge Philos. Soc.*, to appear.

[9] F. F. Bonsall and J. Duncan, *Numerical Ranges of operators on normed spaces and of elements of normed algebras*, London Math. Soc. Lecture Note Ser. **2**, Cambridge, 1971.

[10] F. F. Bonsall and J. Duncan, *Numerical Ranges II*, London Math. Soc. Lecture Note Ser. **10**, Cambridge, 1973.

[11] Y. S. Choi, D. García, S. G. Kim, and M. Maestre, The polynomial numerical index of a Banach space, *Proc. Edinburgh Math. Soc.* **49** (2006), 39û-52.

[12] Y. S. Choi and S. G. Kim, Norm or numerical radius attaining multilinear mappings and polynomials, *J. London Math. Soc.* **54** (1996), 135–147.

[13] I. K. Daugavet, On a property of completely continuous operators in the space C, *Uspekhi Mat. Nauk* **18** (1963), 157–158 (Russian).

[14] J. Diestel and J. J. Uhl, *Vector Measures*, Math. Surveys **15**, AMS, Providence 1977.

[15] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, Springer Monographs in Mathematics, Springer-Verlag, London, 1999.
[16] J. Duncan, C. M. McGregor, J. D. Pryce, and A. J. White, The numerical index of a normed space, *J. London Math. Soc.* **2** (1970), 481–488.
[17] L. A. Harris, The numerical range of holomorphic functions in Banach spaces, *American J. Math.* **93** (1971), 1005–1019.
[18] L. A. Harris, The numerical range of functions and best approximation, *Proc. Camb. Phil. Soc.* **76** (1974), 133–141.
[19] V. M. Kadets and M. M. Popov, The Daugavet property for narrow operators in rich subspaces of $C[0,1]$ and $L_1[0,1]$, *St. Petersburg Math. J.* **8** (1997), 571–584.
[20] V. M. Kadets, R. V. Shvidkoy, G. G. Sirotkin, and D. Werner, Banach spaces with the Daugavet property, *Trans. Amer. Math. Soc.* **352** (2000), 855–873.
[21] M. Martín, The alternative Daugavet property of C^*-algebras and JB^*-triples, *Math. Nachr.*, to appear.
[22] M. Martín and T. Oikhberg, The alternative Daugavet property, *J. Math. Anal. Appl.* **294** (2004), 158–180.
[23] A. Rodríguez Palacios, Numerical ranges of uniformly continuous functions on the unit sphere of a Banach space, *J. Math. Anal. Appl.* **297** (2004), 472–476.
[24] O. Toeplitz, Das algebraische Analogon zu einem Satze von Fejer, *Math. Z.* **2** (1918), 187–197.
[25] D. Werner, An elementary approach to the Daugavet equation, in: *Interaction between Functional Analysis, Harmonic Analysis and Probability* (N. Kalton, E. Saab and S. Montgomery-Smith editors), Lecture Notes in Pure and Appl. Math. **175** (1994), 449–454.
[26] D. Werner, Recent progress on the Daugavet property, *Irish Math. Soc. Bull.* **46** (2001), 77-97.

Department of Mathematics, POSTECH, Pohang (790-784), Korea

E-mail address: mathchoi@postech.ac.kr

Departamento de Análisis Matemático, Universidad de Valencia, Doctor Moliner 50, 46100 Burjasot (Valencia), Spain

E-mail address: domingo.garcia@uv.es

Departamento de Análisis Matemático, Universidad de Valencia, Doctor Moliner 50, 46100 Burjasot (Valencia), Spain

E-mail address: manuel.maestre@uv.es

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain

E-mail address: mmartins@ugr.es