ON THE INTRINSIC AND THE SPATIAL NUMERICAL RANGE

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ABSTRACT. For a bounded function f from the unit sphere of a closed subspace X of a Banach space Y, we study when the closed convex hull of its spatial numerical range $W(f)$ is equal to its intrinsic numerical range $V(f)$. We show that for every infinite-dimensional Banach space X there is a superspace Y and a bounded linear operator $T : X \longrightarrow Y$ such that $\overline{\text{co}}\, W(T) \neq V(T)$. We also show that, up to renormig, for every non-reflexive Banach space Y, one can find a closed subspace X and a bounded linear operator $T \in L(X, Y)$ such that $\overline{\text{co}}\, W(T) \neq V(T)$.

Finally, we introduce a sufficient condition for the closed convex hull of the spatial numerical range to be equal to the intrinsic numerical range, which we call the Bishop-Phelps-Bollobás property, and which is weaker than the uniform smoothness and the finite-dimensionality. We characterize strong subdifferentiability and uniform smoothness in terms of this property.

1. INTRODUCTION

Given a Banach space Y over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$), we write B_Y for the closed unit ball and S_Y for the unit sphere of Y. The dual space of Y will be denoted by Y^*. If Z is another Banach space, we write $L(Z, Y)$ for the Banach space of all bounded linear operators from Z into Y; if $Z = Y$ we simply write $L(Y) := L(Y, Y)$ to denote the the Banach algebra of all bounded linear operators on Y. For an element $u \in S_Y$, we write

$$D(Y, u) := \{y^* \in Y^* \ : \ \|y^*\| = y^*(u) = 1\},$$

the w^*-closed and convex set of all *states* of Y relative to u. Let us mention two facts, both consequence of the Hahn-Banach Theorem, which will be relevant to our discussion. On one hand, we have

$$(1) \qquad \lim_{\alpha \downarrow 0} \frac{\|u + \alpha y\| - 1}{\alpha} = \max\{\text{Re } z^*(y) \ : \ z^* \in D(Z, u)\} \qquad (y \in Z),$$

(see [10, Theorem V.9.5] for a proof). On the other hand, if X is a subspace of Y and $u \in X$, then $D(X, u)$ coincides with the restriction to X of the elements of $D(Y, u)$.

If Y is a Banach space, by a closed subspace of Y we mean a Banach space X and an inclusion operator $J : X \longrightarrow Y$ (i.e., J is a linear isometry), and we also say that Y is a *superspace* of X. When no confusion is possible, we omit J, but all the definitions below depend on the way that X is

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a subspace of Y. Let us fix X and Y as above. We write $\Pi(X,Y)$ to denote the subset of $S_X \times S_{Y^*}$ given by

$$\Pi(X,Y) := \{(x,y^*) \in S_X \times S_{Y^*} \ : \ y^* \in D(Y,Jx)\}.$$

If $X = Y$, we just write $\Pi(Y) := \Pi(Y,Y)$. We denote by $B(S_X,Y)$ the Banach space of all bounded functions from S_X to Y, endowed with the natural supremum norm, and we write $C_{\mathrm{u}}(S_X,Y)$ for its closed subspace consisting of all bounded and uniformly continuous functions. For $f \in B(S_X,Y)$ we can define two different numerical ranges, namely, the *spatial numerical range* defined as

$$W(f) := \{y^*(f(x)) \ : \ (x,y^*) \in \Pi(X,Y)\},$$

and the *intrinsic numerical range* given by

$$V(f) := \{\Phi(f) \ : \ \Phi \in D\left(B(S_X,Y), J|_{S_X}\right)\}.$$

The name of intrinsic numerical range comes from the fact that if f belongs to any closed subspace Z of $B(S_X,Y)$, we can calculate $V(f)$ using only elements in Z^*. These numerical ranges appeared in a paper by L. Harris [16] for continuous functions. In the particular case when $X = Y$ and f is (the restriction to S_Y of) a bounded linear operator, the spatial numerical range was introduced by F. Bauer (field of values subordinate to a norm [1]), extending Toeplitz's numerical range of matrices [25] and, concerning applications, it is equivalent to Lumer's numerical range [18]. Also in this case, the intrinsic numerical range appears as the *algebra numerical range* in the monographs by F. Bonsall and J. Duncan [7, 8]; we refer the reader to these books for general information and background. When f is (the restriction to S_Y of) a uniformly continuous function from B_Y to Y which is holomorphic on the interior of B_Y, both ranges appeared for the first time in [15], where some applications are given.

Let us fix a Banach space Y and a closed subspace X. For every $f \in B(S_X,Y)$, $V(f)$ is closed and convex, and we have

$$\overline{\mathrm{co}}\, W(f) \subseteq V(f), \tag{2}$$

where $\overline{\mathrm{co}}$ means closed convex hull. (Indeed, for $x \in S_X$ and $y^* \in S_{Y^*}$, the mapping $x \otimes y^*$ from $B(S_X,Y)$ to $\mathbb{K}$ defined by

$$[x \otimes y^*](g) := y^*(g(x)) \qquad \big(g \in B(S_X,Y)\big)$$

is an element of $D(B(S_X,Y),J)$.) In the case when $X = Y$, the inclusion above is known to be an equality whenever f is a uniformly continuous function [16, Theorem 1] (see also [7, §9] for bounded linear operators, [15] for holomorphic functions, and [23] for a slightly more general result). On the other hand, the equality $\overline{\mathrm{co}}\, W(f) = V(f)$ for arbitrary bounded functions cannot be expected in general. Indeed, this equality holds for every $f \in B(S_Y,Y)$ if and only if Y is uniformly smooth [22]. In the general case when X is a proper subspace, two sufficient conditions are given in [16, Theorems 2 and 3] for the equality in Eq. (2), namely, such a equality holds for all $f \in C_{\mathrm{u}}(S_X,Y)$ if either X is finite-dimensional or Y is uniformly smooth (see definition below). Let us mention that if $\overline{\mathrm{co}}\, W(f) = V(f)$ for a bounded function $f \in B(S_X,Y)$, then

$$\max \mathrm{Re}\ V(f) = \sup \mathrm{Re}\ W(f).$$

Therefore, the following formulae, consequence of Eq. (1), will be useful:

$$\max \mathrm{Re}\ V(f) = \lim_{\alpha \downarrow 0} \frac{\|J + \alpha f\| - 1}{\alpha} = \lim_{\alpha \downarrow 0} \sup_{x \in S_X} \frac{\|Jx + \alpha f(x)\| - 1}{\alpha}, \tag{3}$$

$$\sup \mathrm{Re}\ W(f) = \sup_{x \in S_X} \lim_{\alpha \downarrow 0} \frac{\|Jx + \alpha f(x)\| - 1}{\alpha}. \tag{4}$$

To state the main results of the paper, let us recall some definitions and notations.

The norm of a Banach space Y is said to be *smooth* at $u \in S_Y$ if $D(Y,u)$ reduces to a singleton, and it is said to be *Fréchet-smooth* or *Fréchet differentiable* at u whenever there exists

$$\lim_{\alpha\to 0} \frac{\|u+\alpha y\|-1}{\alpha} \tag{5}$$

uniformly for $y \in B_Y$. If this happens for all $u \in S_Y$ we say that the norm of Y is Fréchet differentiable. If, in addition, the limit in (5) is also uniform in $u \in S_X$, we say that the norm of Y is *uniformly Fréchet differentiable* at S_Y or that Y is *uniformly smooth*. A natural succedanea of Fréchet differentiability of the norm when smoothness is not required is the following notion introduced by D. Gregory [13]. The norm of Y is *strongly subdifferentiable* (*ssd* in short) at u whenever there exists

$$\lim_{\alpha\downarrow 0} \frac{\|u+\alpha y\|-1}{\alpha}$$

uniformly for $y \in B_Y$. If this happens for all $u \in S_Y$, we simply say that the norm of Y is ssd. Thus, the norm of Y is Fréchet differentiable at u if and only if it is strongly subdifferentiable at u, and Y is smooth at u. This property has been fully investigated in [11], where we refer the reader for background. It is shown in [11, Theorem 1.2] that the norm of Y is ssd at u if and only if $D(Y,u)$ *is strongly exposed by* u, i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$y^* \in B_{Y^*}, \quad \operatorname{Re} y^*(u) > 1-\delta \qquad \Longrightarrow \qquad d(y^*, D(Y,u)) < \varepsilon.$$

In this paper we study when the equality in Eq. (2) holds. The results of the paper can be divided in two categories.

The first category consists of negative results: we present examples of pairs of Banach spaces Y and closed subspaces X in which the equality in Eq. (2) fails, even for elements of $L(X,Y)$. In section 2 we show that for every infinite-dimensional Banach space X, there is a superspace Y and an element $T \in L(X,Y)$ such that $\overline{\mathrm{co}}\, W(T) \neq V(T)$. In section 3, we give concrete examples of Banach spaces Y for which there is a closed subspace X and an element $T \in L(X,Y)$ such that $\overline{\mathrm{co}}\, W(T) \neq V(T)$. Such examples are c_0, $\ell_2 \oplus_\infty (\ell_2 \oplus_1 \ell_2)$, and, up to renorming, every non-reflexive Banach space. We will use the following notation: a Banach space Y is said to have the *FR-property* if for every closed subspace X and every $T \in L(X,Y)$, the equality $\overline{\mathrm{co}}\, W(T) = V(T)$ holds.

The second category is that consisting of positive results. We introduce in section 4 a sufficient condition for the FR-property which covers all the previously known examples and may be interesting by itself. We use the name "Bishop-Phelps-Bollobás property" for it since it is related to the quantitative version of the Bishop-Phelps theorem [4] given by B. Bollobás [6]. We relate this property to the strong subdifferentiability of the norm and to the uniform smoothness.

2. When we fix the subspace

We recall that, when X is finite-dimensional, for every superspace Y and every (uniformly) continuous function $f : S_X \longrightarrow Y$, the equality $\overline{\mathrm{co}}\, W(f) = V(f)$ holds [16, Theorem 2]. The aim of this section is to show that this fact characterizes the finite-dimensionality, even if we restrict ourselves to bounded linear operators.

Theorem 2.1. *Let X be an infinite-dimensional Banach space. Then, there are a superspace Y and an operator $T \in L(X,Y)$ such that $\overline{\mathrm{co}}\, W(T) \neq V(T)$*

We need the following easy lemma.

Lemma 2.2. *If X is an infinite-dimensional Banach space, then there exists a norm-one operator $S \in L(X, c_0)$ which does not attain its norm.*

Proof. Since X is infinite-dimensional, the Josefson-Nissenzweig theorem (see [9, §XII]) assures the existence of a sequence $\{x_n^*\}$ in S_{X^*} w^*-converging to 0. Now, the operator $S : X \longrightarrow c_0$ defined by

$$[Sx](n) = \frac{n}{n+1} x_n^*(x) \qquad \big(x \in X,\ n \in \mathbb{N}\big),$$

does not attain its norm. □

Proof of Theorem 2.1. Let $Y = X \oplus c_0$ endowed with the norm

$$\|(x,t)\| = \max\left\{\|x\|,\ \|Sx\|_\infty + \|t\|_\infty\right\},$$

where $S \in L(X, c_0)$ is a norm-one operator which does not attain its norm, and let $J : X \longrightarrow Y$ be the natural inclusion $Jx = (x, 0)$ for every $x \in X$. If we define $T \in L(X, Y)$ by $Tx = (0, Sx)$ for every $x \in X$, it is straightforward to check that

$$\lim_{\alpha\downarrow 0} \sup_{x \in S_X} \frac{\|x + \alpha Tx\| - 1}{\alpha} = 1 \qquad \text{and} \qquad \sup_{x \in S_X} \lim_{\alpha\downarrow 0} \frac{\|x + \alpha Tx\| - 1}{\alpha} = 0.$$

Thus, Eq. (3) and (4) give $V(T) \neq \overline{\text{co}}\, W(T)$, as desired. □

Remark 2.3. With a bit more of work, one can show that *the superspace Y in the above theorem can be found in such a way that Y/X has dimension* 1. We divide the proof in two cases, depending on whether X is reflexive or not.

CASE 1: Suppose X is not reflexive. Then by the James theorem, there exists $x^* \in S_{X^*}$ which does not attain its norm. Thus, we can define $Y = X \oplus \mathbb{K}$ endowed with the norm

$$\|(x,t)\| = \max\{\|x\|,\ |x^*(x)| + |t|\} \qquad \big(x \in X,\ t \in \mathbb{K}\big),$$

which contains X as the subspace $\{(x, 0)\ :\ x \in X\}$. If we take $T \in L(X, Y)$ defined by $Tx = (0, x^*(x))$ for every $x \in X$, it is straightforward to show, by using Eq. (3) and (4), that $\max \operatorname{Re} V(T) = 1$ and $\sup \operatorname{Re} W(T) = 0$.

CASE 2: Suppose X is reflexive. By the Elton-Odell $(1+\varepsilon)$-separation theorem, there are $\varepsilon_0 > 0$ and a sequence $\{x_n^*\}_{n\geqslant 0}$ of elements of S_{X^*}, satisfying

$$\|x_n^* - x_m^*\| \geqslant 1 + \varepsilon_0 \qquad \big(n \neq m\big)$$

(see [9, §XIV]). Since X is reflexive, for each $n \in \mathbb{N}$ there exists $x_n \in S_X$ such that

$$|(x_n^* - x_0^*)(x_n)| = \|x_n^* - x_0^*\| \geqslant 1 + \varepsilon_0.$$

Therefore,

$$|x_0^*(x_n)| \geqslant |(x_n^* - x_0^*)(x_n)| - |x_n^*(x_n)| \geqslant 1 + \varepsilon_0 - 1 = \varepsilon_0. \tag{6}$$

On the other hand, for each $n \in \mathbb{N}$, we take

$$y_n^* = \frac{x_n^* - x_0^*}{\|x_n^* - x_0^*\|} \in S_{X^*}$$

and we observe that $y_n^*(x_n) = 1$ for every $n \in \mathbb{N}$. Since $X \not\supseteq c_0$, it can be deduced from the proof of the Elton-Odell theorem that $\{x_n^*\}$ is a basic sequence and so, it converges to zero in the $weak$ topology by the reflexivity of X^* (see [24, Theorem II.7.2]). Using this, and the fact that

$$\|x_n^* - x_0^*\| \geqslant 1 + \varepsilon_0 \qquad \text{and} \qquad \|x_0^*\| \leqslant 1,$$

we obtain

$$\overline{\lim}\, y_n^*(x) < 1 \qquad \big(x \in B_X\big).$$

This clearly implies that the operator $S \in L(X, \ell_\infty)$ given by

$$[Sx](n) = \frac{n}{n+1} y_n^*(x) \qquad \big(x \in X,\ n \in \mathbb{N}\big)$$

does not attain its norm. Now, we take $Y = X \oplus \mathbb{K}$ with the norm given by

$$\|(x,t)\| = \max\{\|x\|, \|Sx\|_\infty + |t|\} \qquad \big(x \in X,\ t \in \mathbb{K}\big),$$

we write $J \in L(X,Y)$ for the natural inclusion and, we consider the operator $T \in L(X,Y)$ defined by $Tx = (0, x_0^*(x))$ for all $x \in X$. Using Eq. (4) and the fact that S does not attain its norm, we obtain $\sup \operatorname{Re} W(T) = 0$. To compute $\max \operatorname{Re} V(T)$, we observe that

$$\|J + \alpha T\| \geqslant \|x_n + \alpha T x_n\| = \big\|\big(x_n, \alpha\, x_0^*(x_n)\big)\big\| \geqslant \|Sx_n\|_\infty + \alpha |x_0^*(x_n)|$$

so, by using Eq. (6) and the fact that $\|Sx_n\|_\infty \longrightarrow 1$, we get

$$\|J + \alpha T\| \geqslant 1 + \alpha\varepsilon_0 \qquad \text{for every } \alpha > 0.$$

By just using Eq. (3), we get $\max \operatorname{Re} V(T) \geqslant \varepsilon_0$, which finishes the proof. □

3. When we fix the superspace

As we commented in the introduction, the following result is a particular case of [16, Theorems 2 and 3].

Proposition 3.1. *Finite-dimensional spaces and uniformly smooth spaces have the FR-property.*

In the preceding section we have constructed examples ad hoc of Banach spaces Y which do not have the FR-property. The aim of this section is to present some concrete examples of this phenomenon which will also show that some natural extensions of Proposition 3.1 are not possible.

Let us give the first example.

Example 3.2. *c_0 does not have the FR-property.* Indeed, let $Y = c_0 \oplus \mathbb{K}^2$ endowed with the norm

$$\|(x, \lambda, \mu)\| = \max\{\|x\|_\infty, |\lambda| + |\mu|\} \qquad \big(x \in c_0,\ \lambda, \mu \in \mathbb{K}\big),$$

which is isometrically isomorphic to c_0. We take a norm-one functional x_0^* on c_0 not attaining its norm, we consider the closed subspace $X = \{(x, x_0^*(x), 0) \ :\ x \in c_0\}$ of Y, and we write J for the natural inclusion of X into Y. If we consider the operator $T : X \longrightarrow Y$ given by

$$T(x, x_0^*(x), 0) = (0, 0, x_0^*(x)) \qquad \big(x \in c_0\big),$$

by using Eq. (3), Eq. (4), and the fact that x_0^* does not attain its norm, it is easy to verify that

$$\max \operatorname{Re} V(T) = 1 \qquad \text{and} \qquad \sup \operatorname{Re} W(T) = 0,$$

which finish the proof.

Since the norm of c_0 is ssd (see [11, corollary 2.6], for instance), the above example shows that Proposition 3.1 cannot be extended to the class of Banach spaces with ssd norm.

On the other hand, using the ideas appearing in the above example, it is easy to prove the following.

Proposition 3.3. *Every non-reflexive Banach space admits an equivalent norm failing the FR-property.*

Proof. Let Z be a non-reflexive Banach space. Then, Z is isomorphic to $Y = V \oplus_\infty (\mathbb{K} \oplus_1 \mathbb{K})$, where V is a 2-codimensional closed subspace of Z and, therefore, it is also non-reflexive. Then, we choose $v_0^* \in S_{V^*}$ which does not attain its norm, we define the closed subspace

$$X = \{(v, v_0^*(v), 0) \ :\ v \in V\},$$

and we consider J the natural inclusion of X in Y. As in the preceding example, the operator $T : X \longrightarrow Y$ given by

$$T(v, v_0^*(v), 0) = (0, 0, v_0^*(v)) \qquad \big(x \in X\big)$$

satisfies

$$\max \operatorname{Re} V(T) = 1 \qquad \text{and} \qquad \sup \operatorname{Re} W(T) = 0.$$

□

In view of Propositions 3.1 and 3.3, one may wonder if reflexivity implies the FR-property. This is not the case, as the following example shows.

Example 3.4. *The superreflexive space $Y = \ell_2 \oplus_\infty (\ell_2 \oplus_1 \ell_2)$ does not have the FR property. Proof.* First of all, it is straightforward to show that the norm-one operator $S : \ell_2 \longrightarrow \ell_2$ defined by

$$[Sx](n) = \frac{n}{n+1} x(n) \qquad \big(x \in \ell_2,\ n \in \mathbb{N}\big)$$

does not attain its norm. Now, we consider the closed subspace

$$X = \{(x, Sx, 0)\ :\ x \in \ell_2\}$$

with its natural inclusion in Y, and we define the operator $T : X \longrightarrow Y$ by

$$T(x, Sx, 0) = (0, 0, Sx) \qquad \big(x \in X\big).$$

The proof will be finished if we show that $\sup \operatorname{Re} W(T) = 0$ and $\max \operatorname{Re} V(T) \geqslant 1$. For the first equality, given $x \in S_{\ell_2}$ we may find $\alpha_x > 0$ such that $(1+\alpha_x)\|Sx\| < 1$. Then, for each $0 < \alpha < \alpha_x$ we have

$$\|(x, Sx, 0) + \alpha T(x, Sx, 0)\| = \|(x, Sx, \alpha Sx)\| = \max\{1, (1+\alpha)\|Sx\|\} = 1,$$

and therefore

$$\lim_{\alpha \downarrow 0} \frac{\|(x, Sx, 0) + \alpha T(x, Sx, 0)\| - 1}{\alpha} = 0.$$

The arbitrariness of $x \in S_{\ell_2}$ gives $\sup \operatorname{Re} W(T) = 0$. On the other hand, for each $\alpha > 0$, we observe that

$$\|J + \alpha T\| \geqslant (1+\alpha)\frac{n}{n+1} \qquad (n \in \mathbb{N}),$$

so $\|J + \alpha T\| \geqslant 1 + \alpha$, and

$$\max \operatorname{Re} V(T) = \lim_{\alpha \downarrow 0} \frac{\|J + \alpha T\| - 1}{\alpha} \geqslant 1.$$

4. A sufficient condition: The Bishop-Phelps-Bollobás property

The aim of this section is to study a sufficient condition for the FR-property which, actually, covers all the examples given previously. The motivation for this property is the quantitative version of the classical Bishop-Phelps' Theorem [4, 5] established by B. Bollobás [6] (see [8, §16] for the below version).

Theorem 4.1 (Bishop-Phelps-Bollobás). *Let Y be a Banach space and $\varepsilon > 0$. Whenever $y_0 \in S_Y$ and $y_0^* \in S_{Y^*}$ satisfy that $\operatorname{Re} y_0^*(y_0) > 1 - \frac{\varepsilon^2}{4}$, there exists $(y, y^*) \in \Pi(Y)$ such that*

$$\|y - y_0\| < \varepsilon \qquad \textit{and} \qquad \|y^* - y_0^*\| < \varepsilon.$$

This theorem has played an outstanding role in some topics of geometry of Banach spaces (see [12, 20, 21], for instance), specially in the study of ssd norms [11] or in the study of spatial numerical range of operators [8, §16 and §17]. Also, the proof of the fact that $\overline{\mathrm{co}}\, W(f) = V(f)$ for every $f \in C_{\mathrm{u}}(S_Y, Y)$ given in [16, Theorem 1] uses the above result. For bounded linear operators, this equality can be also deduced from [17, Theorem 8], a result whose proof also uses the Bishop-Phelps-Bollobás theorem. Motivated by these facts, we introduce a property which will be sufficient for the FR-property and it may be of independent interest.

Definition 4.2. Let Y be a Banach space and let X be a closed subspace of Y. We say that (X, Y) is a *Bishop-Phelps-Bollobás pair* (*BPB-pair* in short) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $x_0 \in S_X$, $y_0^* \in S_{Y^*}$ satisfy $\operatorname{Re} y_0^*(x_0) > 1 - \delta$, there exists $(x, y^*) \in \Pi(X, Y)$ so that

$$\|x_0 - x\| < \varepsilon \qquad \text{and} \qquad \|y_0^* - y^*\| < \varepsilon.$$

We say that a Banach space Y has the *BPB property* if for every closed subspace X of Y, (X, Y) is a BPB-pair.

The next result shows that the BPB property is sufficient for the FR-property. Actually, it can be proved that the equality in Eq. (2) holds for uniformly continuous functions.

Theorem 4.3. *Let Y be a Banach space and X a closed subspace such that (X, Y) is a BPB-pair. Then, for every $f \in C_u(S_X, Y)$, the equality $\overline{\mathrm{co}}\, W(f) = V(f)$ holds.*

Proof. Let $J \in L(X, Y)$ be the inclusion map. Let $f \in C_{\mathrm{u}}(S_X, Y)$ and $\Phi \in D(C_{\mathrm{u}}(S_X, Y), J)$. By [16, Proposition 1], it suffices to show that

$$\mathrm{Re}\ \Phi(f) \leqslant \sup \mathrm{Re}\ W(f). \tag{7}$$

For each $n \in \mathbb{N}$, by using [16, Lemma 1] we may find $x_n \in S_X$ and $y_n^* \in S_{Y^*}$ such that

$$\mathrm{Re}\ \Phi(f) \leqslant \mathrm{Re}\ y_n^*(f(x_n)) + 1/n \tag{8}$$

and $y_n^*(x_n) \longrightarrow 1$. Since (X, Y) is a BPB-pair, it follows that there exists a sequence $\big\{(\widetilde{x}_n, \widetilde{y}_n^*)\big\}_{n\in\mathbb{N}} \subseteq \Pi(X, Y)$ such that

$$\{x_n - \widetilde{x}_n\}_{n\in\mathbb{N}} \longrightarrow 0 \qquad \text{and} \qquad \{y_n^* - \widetilde{y}_n^*\}_{n\in\mathbb{N}} \longrightarrow 0.$$

By Eq. (8),

$$\begin{aligned}\mathrm{Re}\ \Phi(f) &\leqslant \mathrm{Re}\ \widetilde{y}_n^*(f(\widetilde{x}_n)) + \mathrm{Re}\,[y_n^* - \widetilde{y}_n^*](f(\widetilde{x}_n)) + \mathrm{Re}\ y_n^*(f(x_n) - f(\widetilde{x}_n)) + 1/n \\ &\leqslant \sup \mathrm{Re}\ W(f) + \|y_n^* - \widetilde{y}_n^*\|\, \|f\|_\infty + \|f(x_n) - f(\widetilde{x}_n)\| + 1/n\end{aligned}$$

for all $n \in \mathbb{N}$. Thus, Eq. (7) follows from the above and the uniform continuity of f. ☐

It is worth mentioning that the above proof follows the lines of [16, Theorem 1].

Corollary 4.4. *Let Y be a Banach space with the BPB property. Then, Y has the FR-property.*

As a consequence of the above corollary and Theorem 2.1, we get the following.

Corollary 4.5. *Let X be an infinite-dimensional Banach space. Then, there is a superspace Y of X such that (X, Y) is not a BPB-pair.*

The above results imply that not every Banach space Y has the BPB property. For instance, the examples given in section 3 of Banach spaces which do not have the FR-property also fail the BPB property.

Example 4.6. *The spaces c_0 and $\ell_2 \oplus_\infty (\ell_2 \oplus_1 \ell_2)$ fail the BPB property in their canonical norms. Every non-reflexive Banach space admits an equivalent norm failing the BPB property.*

On the other hand, if we restrict ourselves to finite-dimensional subspaces, we get a characterization of the ssd norms.

Proposition 4.7. *Let Y be a Banach space. Then, the norm of Y is ssd if, and only if, for every finite-dimensional subspace $X \subseteq Y$, the pair (X, Y) is BPB.*

Proof. We suppose first that the norm of Y is ssd. Let X be a finite-dimensional subspace of Y and let $\varepsilon > 0$ be given. Since the norm of Y is ssd, [11, Theorem 1.2] gives us that for each $x \in S_X$ there exists $\delta_x > 0$ so that

$$y^* \in S_{Y^*},\ \ \mathrm{Re}\ y^*(x) > 1 - \delta_x \qquad \Longrightarrow \qquad d\,(y^*, D(Y, x)) < \varepsilon.$$

Therefore, if for each $x \in S_X$ we define

$$A_x = \left\{ u \in S_X \ :\ \|u - x\| < \min\left\{\varepsilon,\ \frac{\delta_x}{2}\right\}\right\},$$

the compactness of S_X assures the existence of $x_1, \ldots, x_n \in S_X$ such that

$$S_X = \bigcup_{i=1}^{n} A_{x_i}.$$

Then, $\delta = \min\left\{\frac{\delta_{x_i}}{2} \ : \ i = 1, \ldots, n\right\}$ satisfies the BPB condition. Indeed, let $x_0 \in S_X$ and $y_0^* \in S_{Y^*}$ be such that

$$\mathrm{Re}\ y_0^*(x_0) > 1 - \delta.$$

Since $x_0 \in S_X$, there exists $j \in \{1, \ldots, n\}$ so that $x_0 \in A_{x_j}$, that is

$$\|x_0 - x_j\| < \min\left\{\varepsilon,\ \frac{\delta_{x_j}}{2}\right\}.$$

Therefore, $\mathrm{Re}\ y_0^*(x_j) > 1 - \delta_{x_j}$ which implies the existence of $y^* \in D(Y, x_j)$ such that $\|y^* - y_0^*\| < \varepsilon$.

To prove the converse, it is enough to fix $x_0 \in S_Y$ and to show that x_0 strongly exposes $D(Y, x_0)$ [11, Theorem 1.2]. To do so, let X be the subspace of Y generated by x_0 and, fixed $\varepsilon > 0$, let $\delta > 0$ be given by the definition of the BPB for the pair (X, Y) and $\varepsilon/2$. Suppose now that $y_0^* \in S_{Y^*}$ is such that $\mathrm{Re}\ y_0^*(x_0) > 1 - \delta$, then there exists $(x, y^*) \in \Pi(X, Y)$ so that

$$\|x - x_0\| < \varepsilon/2 \qquad \text{and} \qquad \|y^* - y_0^*\| < \varepsilon/2.$$

Since $x \in \mathrm{span}(x_0)$, there exists a modulus-one $\lambda \in \mathbb{K}$ such that $x = \lambda x_0$. Therefore,

$$|\lambda - 1| = \|\lambda x_0 - x_0\| = \|x - x_0\| < \varepsilon/2,$$

and then,

$$\lambda y^* \in D(Y, x_0) \qquad \text{and} \qquad \|\lambda y^* - y_0^*\| \leqslant \|\lambda y^* - y^*\| + \|y^* - y_0^*\| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

which finishes the proof. □

Since the norm of any finite-dimensional Banach space is ssd (see [11, pp. 48]), we have the following corollary, which also implies the first part of Proposition 3.1.

Corollary 4.8. *Every finite-dimensional Banach space has the BPB property.*

The other class of spaces with the FR-property given in Proposition 3.1 is the one of uniformly smooth spaces. This result can be also deduced from Corollary 4.4, as the following proposition shows.

Proposition 4.9. *Every uniformly smooth space has the BPB property.*

Proof. Let Y be a uniformly smooth space. Then, Y^* is uniformly convex, so, for every $\varepsilon > 0$, we may find $\delta > 0$ (the modulus of convexity of Y^*) such that

$$x^*,\ y^* \in S_{Y^*}, \quad \|x^* + y^*\| > 2 - \delta \qquad \Longrightarrow \qquad \|x^* - y^*\| < \varepsilon$$

(see [2, Chapter II] for instance). Let X be a subspace of Y, and let $x_0 \in S_X$ and $y_0^* \in S_{Y^*}$ be so that $\mathrm{Re}\ y_0^*(x_0) > 1 - \delta$. If we consider $y^* \in S_{Y^*}$ such that $\mathrm{Re}\ y^*(x_0) = 1$, we have

$$\|y^* + y_0^*\| \geqslant \mathrm{Re}\,(y^* + y_0^*)(x_0) > 2 - \delta$$

and, therefore,

$$\|y^* - y_0^*\| < \varepsilon,$$

which finishes the proof. □

Observe that, in the above proof, the relation $\varepsilon - \delta$ does not depend on the subspace. The next result shows that this fact actually characterizes the uniform smoothness.

Proposition 4.10. *Let Y be a Banach space with the BPB property in such a way that the relationship between ε and δ in Definition 4.2 does not depend on the subspace X. Then, Y is uniformly smooth.*

Proof. In view of [11, Proposition 4.1], it is enough to show that the limit

$$\lim_{t\downarrow 0} \frac{\|u+ty\|-1}{t} =: \tau(u,y)$$

exists uniformly for $y \in B_Y$ and $u \in S_Y$. Given $\varepsilon > 0$, let $0 < \delta < 2$ be given by the "uniform" BPB property. Now, for $y \in B_Y$, $u \in S_Y$ and $0 < t < \frac{\delta}{2}$, we consider

$$y_t = \frac{u+ty}{\|u+ty\|} \in S_Y \qquad \text{and} \qquad y_t^* \in D(Y, y_t).$$

It is immediate to check that $\mathrm{Re}\ y_t^*(u) > 1-\delta$ so, if we take $X = \mathrm{span}(u)$, the BPB property assures the existence of $(x, z_t^*) \in \Pi(X,Y)$ such that $\|x-u\| < \varepsilon$ and $\|z_t^* - y_t^*\| < \varepsilon$. Since $x \in \mathrm{span}(u)$, there exists a modulus-one $\lambda \in \mathbb{K}$ such that $x = \lambda u$. Therefore,

$$|\lambda - 1| = \|\lambda u - u\| = \|x-u\| < \varepsilon,$$

and then,

$$\lambda z_t^* \in D(Y,u) \qquad \text{and} \qquad \|\lambda z_t^* - y_t^*\| \leqslant \|\lambda z_t^* - z_t^*\| + \|z_t^* - y_t^*\| < \varepsilon + \varepsilon = 2\varepsilon,$$

Now, the facts

$$\frac{\|u+ty\|-1}{t} = \frac{\mathrm{Re}\ y_t^*(u+ty)-1}{t} \leqslant \mathrm{Re}\ y_t^*(y)$$

and $\tau(u,y) \geqslant \mathrm{Re}\ \lambda z_t^*(y)$ (by Eq. (1)), give

$$0 \leqslant \frac{\|u+ty\|-1}{t} - \tau(u,y) \leqslant \mathrm{Re}\ y_t^*(y) - \mathrm{Re}\ \lambda z_t^*(y) \leqslant \|\lambda z_t^* - y_t^*\| < 2\varepsilon,$$

and the arbitrariness of $\varepsilon > 0$ finishes the proof. □

We conclude the paper proving that a pair (X,Y) is a BPB-pair provided that X is an absolute ideal of Y. Let us introduce the necessary definitions. We refer the reader to [8, § 21], [19], and references therein for background. A closed subspace X of a Banach space Y is said to be an *absolute summand* of Y if there exists another closed subspace Z such that $Y = X \oplus Z$ and, for every $x \in X$ and $z \in Z$, the norm of $x + z$ only depends on $\|x\|$ and $\|z\|$. We also say that Y is an *absolute sum* of X and Z. This implies that there exists an absolute norm on $\mathbb{R}^2$ such that

$$\|x+z\| = |(\|x\|, \|z\|)|_a \qquad \big(x \in X,\ z \in Z\big).$$

By an *absolute norm* we mean a norm $|\cdot|_a$ on $\mathbb{R}^2$ such that $|(1,0)|_a = |(0,1)|_a = 1$ and $|(a,b)|_a = |(|a|,|b|)|_a$ for every $a, b \in \mathbb{R}$. Useful results about absolute norms are the following inequality

$$\max\{|a|,|b|\} \leqslant |(a,b)|_a \leqslant |a| + |b| \qquad a, b \in \mathbb{R},$$

and the fact that absolute norms are nondecreasing and continuous in each variable. We say that X is an *absolute ideal* of Y if $X^\perp$ is an absolute summand of Y^*, in which case, Y^* can be identified with $X^* \oplus X^\perp$ with a convenient absolute sum. It is clear that absolute summands are absolute ideals, but the converse is not true.

Absolute summands and absolute ideals are generalizations of the well-known *M-summands*, *L-summands*, *M-ideals*, and the more general class of *L_p-summands* [3, 14].

Proposition 4.11. *Let Y be a Banach space and let X be an absolute ideal of Y. Then the pair (X,Y) is BPB.*

We need the following easy result, which we separate from the proof of the proposition for the sake of clearness.

Lemma 4.12. *Let E be $(\mathbb{R}^2, |\cdot|_a)$ where $|\cdot|_a$ is an absolute norm. We write*

$$b_0 = \max\{b \geqslant 0 \ :\ |(1,b)|_a = 1\},$$

and we define

$$A(\delta) = \{(a,b) \in B_E \ :\ a > 1-\delta,\ b \geqslant b_0\} \qquad (\delta > 0).$$

Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that diam$(A(\delta)) < \varepsilon$.

Proof. Suppose, for the sake of contradiction, that the result does not hold. Then, there exists $\varepsilon_0 > 0$ such that for every $n \in \mathbb{N}$, $\operatorname{diam}(A(\frac{1}{n})) \geqslant \varepsilon_0$. So, we may find $(a_n, b_n) \in A(\frac{1}{n})$ such that $|(a_n, b_n) - (1, b_0)|_a \geqslant \frac{\varepsilon_0}{2}$, and thus

$$\frac{\varepsilon_0}{2} \leqslant |a_n - 1| + |b_n - b_0| \qquad (n \in \mathbb{N}). \tag{9}$$

Let $\{(a_{\sigma_n}, b_{\sigma_n})\}$ be a convergent subsequence of $\{(a_n, b_n)\}$, and let $(1, b) \in S_E$ be its limit. By Eq. (9) and the fact that $(a_{\sigma_n}, b_{\sigma_n}) \in A(\frac{1}{\sigma_n})$, it is immediate to check that

$$\frac{\varepsilon_0}{2} \leqslant |b - b_0| \qquad \text{and} \qquad b \geqslant b_0.$$

So, b is strictly bigger than b_0, a contradiction. $\square$

Proof of Proposition 4.11. There exist an absolute norm $|\cdot|_a$ on $\mathbb{R}^2$ so that $Y^* = X^* \oplus X^\perp$ and

$$\|(x^*, z^*)\| = \big|(\|x^*\|, \|z^*\|)\big|_a \qquad \big(x^* \in X^*,\ z^* \in X^\perp\big).$$

For $\varepsilon > 0$ fixed, we take $\delta_1 > 0$ given by the preceding lemma applied for $\varepsilon/3$, and we define

$$\delta := \min\left\{\delta_1, \frac{\varepsilon^2}{36}\right\}.$$

To finish the proof, for $x_0 \in S_X$ and $y_0^* = (x_0^*, z_0^*) \in S_{Y^*}$ satisfying

$$\operatorname{Re}\, y_0^*(x_0) = \operatorname{Re}\, x_0^*(x_0) > 1 - \delta,$$

we have to find $(x, y^*) \in \Pi(X, Y)$ so that

$$\|y^* - y_0^*\| < \varepsilon \qquad \text{and} \qquad \|x - x_0\| < \varepsilon.$$

To this end, since

$$\|x_0\| = 1 = \left\|\frac{x_0^*}{\|x_0^*\|}\right\| \qquad \text{and} \qquad \operatorname{Re}\, \frac{x_0^*}{\|x_0^*\|}(x_0) \geqslant \operatorname{Re}\, x_0^*(x_0) > 1 - \frac{\varepsilon^2}{36},$$

we can apply the classical Bishop-Phelps-Bollobás Theorem (4.1) to $\left(x_0, \frac{x_0^*}{\|x_0^*\|}\right) \in X \times X^*$ to get $(x, x^*) \in \Pi(X)$ such that

$$\left\|x^* - \frac{x_0^*}{\|x_0^*\|}\right\| < \frac{\varepsilon}{3} \quad \text{and} \quad \|x - x_0\| < \frac{\varepsilon}{3}. \tag{10}$$

Now, we distinguish two cases. Suppose first that $\|z_0^*\| \leqslant b_0$. Then, we take $y^* := (x^*, z_0^*)$, which satisfies $\operatorname{Re}\, y^*(x) = 1$ and $\|y^*\| = \big|(1, \|z_0^*\|)\big|_a = 1$. Using Eq. (10) and the definition of δ, we get

$$\|y^* - y_0^*\| = \|x^* - x_0^*\| < \frac{\varepsilon}{3} + \delta < \varepsilon.$$

So, the pair (x, y^*) satisfies the desired condition.

Suppose otherwise that $\|z_0^*\| > b_0$. In this case, we take $y^* := \big(x^*, b_0\, \|z_0^*\|^{-1}\, z_0^*\big)$, which clearly satisfies $\operatorname{Re}\, y^*(x) = 1 = \|y^*\|$. Now, $(1, b_0)$ and $(\|x_0^*\|, \|z_0^*\|)$ belong to $A(\delta)$ and the diameter of this set is less than $\varepsilon/3$ by Lemma 4.12, so we have

$$\big|\|z_0^*\| - b_0\big| \leqslant \big|(1, b_0) - (\|x_0^*\|, \|z_0^*\|)\big|_a < \frac{\varepsilon}{3}$$

and

$$\|y^* - y_0^*\| = \big|(\|x^* - x_0^*\|, \|z_0^*\| - b_0)\big|_a \leqslant \|x^* - x_0^*\| + \big|\|z_0^*\| - b_0\big| < \varepsilon. \qquad \square$$

By just applying the above proposition and Theorem 4.3, we get the following.

Corollary 4.13. *Let Y be a Banach space and let X be an absolute ideal of Y. Then,*

$$\overline{\operatorname{co}}\, W(f) = V(f)$$

for every $f \in C_u(S_X, Y)$.

An interesting particular case is the case of M-embedded and L-embedded spaces. A Banach space X is said to be M-embedded if it is an M-ideal of X^{**}, and it is L-embedded if $X^{**} = X \oplus_1 Z$ for some closed subspace Z of X^{**}.

Corollary 4.14. *If X is an M-embedded or an L-embedded space, then (X, X^{**}) is a BPB-pair.*

We do not know if the assumption of being M-embedded or L-embedded in the above result is superabundant.

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