THE DAUGAVET PROPERTY FOR LINDENSTRAUSS SPACES

Abstract. A Banach space X is said to have the Daugavet property if every rank-one operator $T : X \longrightarrow X$ satisfies $\|\mathrm{Id} + T\| = 1 + \|T\|$. We give geometric characterizations of this property for Lindenstrauss spaces.

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1. Introduction

The study of the Daugavet property was inaugurated in 1961 when I. Daugavet [7] proved that every compact linear operator T on $C[0,1]$ satisfies the norm equality

$$\|\mathrm{Id} + T\| = 1 + \|T\|, \tag{DE}$$

now known as the *Daugavet equation*. Over the years, the validity of this equation was proved for compact linear operators on various spaces, including $C(K)$ and $L_1(\mu)$ provided that K is perfect and μ does not have any atoms (see [18] for an elementary approach), and certain function algebras such as the disk algebra $A(\mathbb{D})$ or the algebra of bounded analytic functions H^∞ [19, 21]. In the nineties, new ideas were infused into the field, and the geometry of Banach spaces having the so-called Daugavet property was initiated. Let us recall that a Banach space X is said to have the *Daugavet property* [15] if every rank-one operator $T : X \longrightarrow X$ satisfies (DE), in which case, all weakly compact operators on X also satisfy (DE) (see [15, Theorem 2.3]). Therefore, this definition of Daugavet property coincides with those that gave a briefly appearance in [6, 1]. A good introduction to the the Daugavet equation is given in the books [2, 3] and the state-of-the-art on the subject can be found in the papers [15, 20]. For very recent results we refer the reader to [4, 5, 14, 16] and references therein.

Let us mention here several facts concerning the Daugavet property which are relevant to our discussion. It is clear that X has the Daugavet property whenever its topological dual X^* does, but the converse result is false (for instance, $X = C[0,1]$). It is known that a space with the Daugavet property cannot have the Radon-Nikodým property [21]; even more, every weakly open

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subset of its unit ball has diameter 2 [17]. A space with the Daugavet property contains a copy of ℓ_1 [15], it does not have an unconditional basis [13] and it does not even embed into a space with an unconditional basis [15].

The Daugavet property is not always inherited by ultraproducts. Actually, given a Banach space X, every ultrapower $X_{\mathcal{U}}$, $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$, has the Daugavet property if and only if X has the so-called *uniform Daugavet property*, a quantitative version of the Daugavet property introduced in [5], and which is strictly stronger than the usual Daugavet property [16, Theorem 3.3]. Even though, the basic examples of spaces with the Daugavet property ($C(K)$ with K perfect and $L_1[0,1]$) are in fact spaces with the uniform Daugavet property [5, Lemmas 6.6 and 6.7]. We refer to [11, 12] for definitions and basic results about ultraproducts of Banach spaces.

The aim of this note is to give geometric characterizations of the Daugavet property valid for the so-called *Lindenstrauss spaces* (i.e. Banach spaces whose dual is isometric to an $L_1(\mu)$ space) which remind those given in [4, Corollaries 4.1 and 4.4] for C^*-algebras. We apply them to prove that the Daugavet property passes to ultraproducts of Lindenstrauss spaces.

Let us fix notation and recall some common definitions.

Let X be a Banach space. The symbols B_X and S_X denote, respectively, the closed unit ball and the unit sphere of X, and we write $\operatorname{ext}(C)$ to denote the set of extreme points of the convex set C. Let us fix u in S_X. We define the set $D(X,u)$ of all *states* of X relative to u by

$$D(X,u) := \{f \in B_{X^*} \ : \ f(u) = 1\},$$

which is a non-empty w^*-closed face of B_{X^*}. The norm of X is *smooth* at u if $D(X,u)$ reduces to a singleton, and it is *Fréchet-smooth* at $u \in S_X$ whenever there exists $\lim_{\alpha\to 0} \frac{\|u+\alpha x\|-1}{\alpha}$ uniformly for $x \in B_X$. We define the *roughness of X at u* by the equality

$$\eta(X,u) \ := \ \limsup_{\|h\|\to 0} \frac{\|u+h\| + \|u-h\| - 2}{\|h\|}.$$

We remark that the absence of roughness of X at u (i.e., $\eta(X,u) = 0$) is nothing other than the Fréchet-smoothness of the norm of X at u [8, Lemma I.1.3]. Given $\delta > 0$, the Banach space X is said to be *δ-rough* if, for every u in S_X, we have $\eta(X,u) \geqslant \delta$. We say that X is *extremely rough* whenever it is 2-rough. A *slice* of B_X is a subset of the form

$$S(B_X, f, \alpha) = \big\{x \in B_X \ : \ \mathrm{Re}\ f(x) > 1 - \alpha\big\},$$

where $f \in S_{X^*}$ and $0 < \alpha < 1$. If X is a dual space and f is actually taken from the predual, we say that $S(B_X, f, \alpha)$ is a w^*-slice. By [8, Proposition I.1.11], the norm of X is δ-rough if and only if every nonempty w^*-slice of B_{X^*} has diameter greater or equal than δ. A point $x \in S_X$ is said to be an *strongly exposed point* if there exists $f \in D(X,x)$ such that $\lim \|x_n - x\| = 0$ for every sequence (x_n) of elements of B_X such that $\lim \mathrm{Re}\ f(x_n) = 1$ or, equivalently, if there is a point of Fréchet-smoothness in $D(X,x)$ (see [8, Corollary I.1.5]).

2. The results

Our aim is to characterize those real or complex Lindenstrauss spaces with the Daugavet property. Since the Daugavet property passes from the dual of a Banach space to the space itself, one may wonder if it is enough to characterizes $L_1(\mu)$ spaces with the Daugavet property and then the mentioned result applies. Characterizations of the Daugavet property for $L_1(\mu)$ spaces can be obtained as a particular case of [4, Corollaries 4.2 and 4.4], where the work was done for von Neumann preduals. Let us state here this result for $L_1(\mu)$ spaces.

Let μ be a positive measure. Then, the following are equivalent:

(i) $L_1(\mu)$ has the Daugavet property.
(ii) Every weak-open subset of $B_{L_1(\mu)}$ has diameter 2.
(iii) $B_{L_1(\mu)}$ has no strongly exposed points.
(iv) $B_{L_1(\mu)}$ has no extreme points.
(v) The measure μ does not have any atom.

Let us observe that, as an immediate consequence of (iv) above and the Krein-Milman Theorem, we have that *an $L_1(\mu)$ space which is a dual space never has the Daugavet property.* Therefore, the Daugavet property for a Lindenstrauss space never comes from its dual space, and we need to look for other kind of characterizations which depends not only on the measure μ but also on the particular form in which a given Lindenstrauss space is the predual of $L_1(\mu)$. For $C(K)$ spaces (which are very particular examples of Lindenstrauss spaces) this was done in [4, Corollaries 4.1 and 4.4], where the Daugavet property was characterized for C^*-algebras. Let us write here the result for $C(K)$ spaces.

Let K be a Hausdorff compact topological space. Then, the following are equivalent:

(i) $C(K)$ has the Daugavet property.
(ii) The norm of $C(K)$ is extremely rough.
(iii) The norm of $C(K)$ is not Fréchet-smooth at any point.
(iv) The space K does not have any isolated point.

Just remembering that the space K is isometric to a quotient of the topological space $\operatorname{ext}\left(B_{C(K)^*}\right)$ endowed with the weak* topology, one realizes that the above result characterizes the Daugavet property of $C(K)$ either in terms of the geometry of the space or in terms of the way in which $C(K)$ is a predual of $C(K)^*$.

The aim of this paper is to give characterizations of the Daugavet property for Lindenstrauss spaces analogous to the ones given above for $C(K)$ spaces. Of course, we have to translate the meaning of (iv) to an arbitrary Lindenstrauss space, and the above paragraph gives us the idea to do so for an arbitrary Banach space.

Definition 2.1. Given a Banach space X, we define the equivalence relation $f \sim g$ if and only if f and g are linearly dependent elements of $\operatorname{ext}(B_{X^*})$, and we endowed the quotient space $\operatorname{ext}(B_{X^*})/\sim$ with the quotient topology of the weak* topology.

In [19, Theorem 3.5], D. Werner proved that a Lindenstrauss space X for which $\operatorname{ext}(B_{X^*})/\sim$ does not have any isolated point has the Daugavet property. We will show that this condition is actually a characterization. To do so, we need the following geometrical result, which may be of independent interest.

Proposition 2.2. *Let X be a Banach space and let f be an extreme point of B_{X^*} such that its equivalence class is an isolated point in $\operatorname{ext}(B_{X^*})/\sim$. Then, f is a w^*-strongly exposed point of B_{X^*}.*

Proof. We may find a w^*-neighborhood U of f in B_{X^*} such that whenever $g \in \operatorname{ext}(B_{X^*})$ belongs to U, then $f \sim g$. By Choquet's Lemma (see [10, Lemma 3.40], for instance), we may certainly suppose that U is a w^*-open slice of B_{X^*}; i.e., there are $x \in S_X$ and $0 < \alpha_0 < 1$ such that

$$g \in \operatorname{ext}(B_{X^*}),\ g \in S(B_{X^*}, x, \alpha_0) \quad \Longrightarrow \quad f \sim g. \tag{1}$$

We claim that, for $0 < \alpha \leqslant \alpha_0$ and $y \in S_X$ satisfying $\|y-x\| < \alpha$, there exists a modulus-one scalar ω_y such that $D(X,y)$ reduces to the singleton $\{\omega_y f\}$ and

$$\|\omega_y f - \omega_x f\| < \sqrt{2\,\alpha}\,.$$

Let us observe that this claim finish the proof, since it implies that every selector of the duality mapping is norm to norm continuous at x, which gives that the norm of X is Fréchet-smooth at x (see [9, Theorem II.2.1]) and then, $\omega_x f$ (and hence f) is w^*-strongly exposed (see [8, Corollary I.1.5]).

Let us prove the claim. If $\|y - x\| < \alpha$, every $g \in D(X,y)$ satisfies

$$\operatorname{Re} g(x) = \operatorname{Re} g(y) - \big(\operatorname{Re} g(y) - \operatorname{Re} g(x)\big) \geqslant 1 - \|x-y\| > 1 - \alpha \tag{2}$$

and so, $D(X,y)$ is contained in $S(B_{X^*}, x, \alpha) \subset S(B_{X^*}, x, \alpha_0)$. Then, every extreme point of the w^*-closed face $D(X,y)$ (remaining extreme in B_{X^*}) is a multiple of f by Eq. (1). Since only one multiple of f can be in the face $D(X,y)$ and, being w^*-compact, $D(X,y)$ is the w^*-closed convex hull of its extreme points, we get $D(X,y) = \{\omega_y f\}$ for a suitable modulus-one scalar ω_y. Finally, on one hand, since $|f(x)| = 1$, we have that

$$\|\omega_x f - \omega_y f\| = |\omega_x - \omega_y| = |\omega_x f(x) - \omega_y f(x)| = |1 - \omega_y f(x)|.$$

On the other hand, Eq. (2) says that $\operatorname{Re} \omega_y f(x) > 1 - \alpha$ and so, an straightforward computation gives that

$$|1 - \omega_y f(x)| < \sqrt{2\,\alpha}\,. \qquad \square$$

Remarks 2.3.

(a) In the real case, the proof of Proposition 2.2 actually gives a stronger result. Namely, *let X be a real Banach space and let f be a w^*-isolated point of $\operatorname{ext}(B_{X^*})$. Then, the face of the unit ball*

$$\{x \in B_X \ :\ f(x) = 1\}$$

has non-empty interior (relative to S_X), which implies that f is w^*-strongly exposed.

(b) Let us comment that no proper face of the unit ball of a complex Banach space has interior points relative to S_X, so the above result does not hold for complex spaces. Anyhow, a sight to the proof of Proposition 2.2 allows us to state the following improvement. *Let X be a complex Banach space and let f be an extreme point of B_{X^*} such that its equivalent class is isolated in* $\operatorname{ext}(B_{X^*})/\sim$. *Then, there exists an open subset U of S_X such that the norm of X is Fréchet-smooth at any point of U and each derivative is a multiple of f.*

We are now ready to state the main result of the paper.

Theorem 2.4. *Let X be a Lindenstrauss space. Then, the following are equivalent:*
- *(i) X has the Daugavet property.*
- *(ii) The norm of X is extremely rough.*
- *(iii) The norm of X is not Fréchet-smooth at any point.*
- *(iv)* $\operatorname{ext}(B_{X^*})/\sim$ *does not have any isolated point.*

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are clear and valid for general Banach spaces, and Proposition 2.2 gives $(iii) \Rightarrow (iv)$. Finally, $(iv) \Rightarrow (i)$ follows from [19, Theorem 3.5]. □

Remark 2.5. It is worth mentioning that the above geometric characterizations are not valid for arbitrary Banach spaces. On one hand, *the norm of ℓ_1 is extremely rough (and so ℓ_1 has no points of Fréchet-smoothness), but ℓ_1 does not have the Daugavet property.* On the other hand, *for $X = \ell_2$ the set* $\operatorname{ext}(B_{X^*})/\sim$ *does not have any w^*-isolated point, but ℓ_2 is reflexive and, therefore, it does not have the Daugavet property.*

The above theorem and the fact that the class of Lindenstrauss spaces is closed under ultraproducts, gives us the following result.

Corollary 2.6. *The ultraproduct of every family of Lindenstrauss spaces with the Daugavet property also has the Daugavet property. In particular, the Daugavet and the uniform Daugavet properties are equivalent for Lindenstrauss spaces.*

Proof. Since the class of Lindenstrauss spaces is closed under arbitrary ultraproducts [12, Proposition 2.1], the result follows from Theorem 2.4 and the fact that the roughness of the norm is inherited under arbitrary ultraproducts [4, Lemma 5.1]. □

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