THE DAUGAVET PROPERTY OF C*-ALGEBRAS, JB*-TRIPLES, AND OF THEIR ISOMETRIC PREDUALS

JULIO BECERRA GUERRERO [1]

Departamento de Matemática Aplicada
Facultad de Ciencias
Universidad de Granada
18071 Granada, SPAIN
Email address: juliobg@ugr.es

MIGUEL MARTÍN [2] [3]

Departamento de Análisis Matemático
Facultad de Ciencias
Universidad de Granada
18071 Granada, SPAIN
Email address: mmartins@ugr.es

Abstract:

A Banach space X is said to have the Daugavet property if every rank-one operator $T : X \longrightarrow X$ satisfies $\|Id + T\| = 1 + \|T\|$. We give geometric characterizations of this property in the settings of C^*-algebras, JB^*-triples and their isometric preduals. We also show that, in these settings, the Daugavet property passes to ultrapowers, and thus, it is equivalent to an stronger property called the uniform Daugavet property.

Keywords:

C^*-algebra; von Neumann predual; JB^*-triple; predual of a JBW^*-triple; Daugavet equation; Daugavet property; rough norm; Fréchet-differentiability.

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[2]Partially supported by Spanish MCYT project no. BFM2003-01681

[3]Corresponding author. *Email:* mmartins@ugr.es, *Fax:* +34958243272

1. Introduction

A Banach space X is said to have the *Daugavet property* [28] if every rank-one operator $T : X \longrightarrow X$ satisfies the norm identity

$$\text{(DE)} \qquad \|Id + T\| = 1 + \|T\|,$$

known as *Daugavet equation.* In such a case, all weakly compact operators on X also satisfy (DE) (see [28, Theorem 2.3]). Therefore, this definition of Daugavet property coincides with those that appeared in [10] and [1].

The study of the Daugavet equation was inaugurated by I. Daugavet [11] in 1961 by proving that every compact operator on $C[0,1]$ satisfies (DE). Over the years, the validity of the Daugavet equation was proved for compact operators on various spaces, including $C(K)$ and $L_1(\mu)$ provided that K is perfect and μ does not have any atoms (see [39] for an elementary approach), and certain function algebras such as the disk algebra $A(\mathbb{D})$ or the algebra of bounded analytic functions H^∞ [40, 42]. In the nineties, new ideas were infused into the field and the geometry of Banach spaces having Daugavet property was studied. The state-of-the-art on the subject can be found in [28, 41]. For very recent results we refer the reader to [8, 27, 29] and references therein.

Let us mention here several facts concerning the Daugavet property which are relevant to our discussion. It is clear that X has the Daugavet property whenever its topological dual X^* does, but the converse result is false ($X = C[0,1]$, for instance). It is known that a space with the Daugavet property cannot have the Radon-Nikodým property (RNP in short) [42]; even more, every weakly open subset of its unit ball has diameter 2 [38]. A space with the Daugavet property contains a copy of ℓ_1 [28], it does not have an unconditional basis [26] and it does not even embed into a space with an unconditional basis [28].

In 2002, T. Oikhberg [36] carried the classical results on the Daugavet property for $C(K)$ and $L_1(\mu)$ to the non-commutative case, characterizing when (complex) C^*-algebras and preduals of von Neumann algebras have the Daugavet property. A C^*-algebra has the Daugavet property if and only if it does not have atomic projections; if the algebra is a von Neumann algebra (i.e., it is a dual space), its (unique) isometric predual has the Daugavet property if and only if the algebra does. In 2004, T. Oikhberg and the second named author [35], translated these results to the non-associative case, characterizing (complex) JB^*-triples and predual of (complex) JBW^*-triples having the Daugavet property in an analogous way, replacing atomic projections by minimal tripotents. The necessary definitions and basic results on JB^*-triples are presented in section 3.

In the present paper we give geometric characterizations of the Daugavet property in the setting of real and complex JB^*-triples and their isometric preduals. In particular, our results contain the already mentioned ones of [35, 36] for complex C^*-algebras and complex JB^*-triples, but our proofs are independent.

To state the main results of the paper we need to fix notation and recall some definitions.

Let X be a Banach space. The symbols B_X and S_X denote, respectively, the closed unit ball and the unit sphere of X. Let us fix u in S_X. We define the set $D(X,u)$ of all *states* of X relative to u by

$$D(X,u) := \{f \in B_{X^*} \ : \ f(u) = 1\},$$

which is a non-empty w^*-closed face of B_{X^*}. The norm of X is said to be *smooth* at u if $D(X,u)$ reduces to a singleton, and it is said to be *Fréchet-smooth* or *Fréchet* *differentiable* at $u \in S_X$ whenever there exists $\lim\limits_{\alpha \to 0} \dfrac{\|u + \alpha x\| - 1}{\alpha}$ uniformly for $x \in B_X$. We define the *roughness of* X *at* u by the equality

$$\eta(X, u) \ := \ \limsup_{\|h\| \to 0} \frac{\|u + h\| + \|u - h\| - 2}{\|h\|}.$$

We remark that the absence of roughness of X at u (i.e., $\eta(X, u) = 0$) is nothing other than the Fréchet-smoothness of the norm of X at u [12, Lemma I.1.3]. Given $\delta > 0$, the Banach space X is said to be δ*-rough* if, for every u in S_X, we have $\eta(X, u) \geqslant \delta$. We say that X is *rough* whenever it is δ-rough for some $\delta > 0$, and *extremely rough* whenever it is 2-rough. Roughly speaking, the space X is rough if its norm is "uniformly" non-differentiable at any point. A *slice* of B_X is a subset of the form

$$S(B_X, f, \alpha) = \big\{x \in B_X \ : \ \mathrm{Re}\ f(x) > 1 - \alpha\big\},$$

where $f \in S_{X^*}$ and $0 < \alpha < 1$. If X is a dual space and f is actually taken from the predual, we say that $S(B_X, f, \alpha)$ is a w^*-slice. By [12, Proposition I.1.11], the norm of X is δ-rough if and only if every nonempty w^*-slice of B_{X^*} has diameter greater or equal than δ.

Finally, a point $x \in S_X$ is said to be an *strongly exposed point* if there exists $f \in D(X, x)$ such that $\lim \|x_n - x\| = 0$ for every sequence (x_n) of elements of B_X such that $\lim \mathrm{Re}\ f(x_n) = 1$ (equivalently, there are slices defined by f with arbitrary small diameter). It is known that x is strongly exposed if and only if there is a point of Fréchet-smoothness in $D(X, x)$ (see [12, Corollary I.1.5]).

The main results of the paper are the characterizations of the Daugavet property for JB^*-triples and preduals of JBW^*-triples given in Theorems 3.10 and 3.2 respectively. For a real or complex JB^*-triple X, the following are equivalent:

- (i) X has the Daugavet property,
- (ii) the norm of X is extremely rough,
- (iii) the norm of X is not Fréchet-smooth at any point.

For the predual X_* of a real or complex JBW^*-triple X, the following are equivalent:

- (i) X has the Daugavet property,
- (ii) X_* has the Daugavet property,
- (iii) every relative weak-open subset of B_{X_*} has diameter 2,
- (iv) B_{X_*} has no strongly exposed points,
- (v) B_{X_*} has no extreme points.

This characterizations allow us to prove that, for JB^*-triples and for preduals of JBW^*-triples, the Daugavet property passes to ultrapowers. As a consequence, a stronger version of the Daugavet property introduced in [8], called the uniform Daugavet property, is equivalent to the usual Daugavet property in the setting of JB^*-triples and their isometric preduals.

The outline of the paper is as follows. In section 2 we give sufficient conditions for a Banach space to have the Daugavet property, which will be the keys to state the rest of the paper.

Section 3 is devoted to the above cited characterizations of the Daugavet property for real or complex JB^*-triples and their isometric preduals, and we dedicate section 4 to particularize these result to the setting of real or complex C^*-algebras and von Neumann preduals.

Finally, in section 5 we study the behaviour of the Daugavet property for ultraproducts of JB^*-triples and of preduals of JBW^*-triples. As a consequence, we show that the already mentioned uniform Daugavet property and the Daugavet property coincide in real or complex JB^*-triples and their isometric preduals.

Throughout the paper, for a subset A of a Banach space, we write $\overline{\text{co}}\,(A)$ for the closed convex hull of A, we use $\text{ex}(B)$ to denote the set of extreme points of the convex set B and, finally, if X and Y are Banach spaces, we write $X \oplus_1 Y$ and $X \oplus_\infty Y$ to denote, respectively, the ℓ_1-sum and the ℓ_∞-sum of X and Y.

2. Sufficient conditions for the Daugavet property

For a better comprehension of the geometry underlying the Daugavet property, we present the following characterization from [28, Lemma 2.1] and [41, Corollary 2.3]. We shall have occasion to use it throughout the paper.

Lemma 2.1. *The following assertions are equivalent:*

(i) X *has the Daugavet property.*
(ii) *For all* $x \in S_X$, $x^* \in S_{X^*}$, *and* $\varepsilon > 0$, *there exists some* $y \in S_X$ *such that* $\text{Re}\; x^*(y) > 1 - \varepsilon$ *and* $\|x + y\| > 2 - \varepsilon$.
(iii) *For all* $x \in S_X$, $x^* \in S_{X^*}$, *and* $\varepsilon > 0$, *there exists some* $y^* \in S_X$ *such that* $\text{Re}\; y^*(x) > 1 - \varepsilon$ *and* $\|x^* + y^*\| > 2 - \varepsilon$.
(iv) *For all* $x \in S_X$ *and* $\varepsilon > 0$,
$$B_X \subset \overline{\text{co}}\,\big(\{y \in X \;:\; \|y\| \leqslant 1 + \varepsilon,\ \|x + y\| > 2 - \varepsilon\}\big).$$

Observe that condition *(ii)* in the above lemma implies that every weak slice of the unit ball of a Banach space X with the Daugavet property has diameter 2. Also, condition *(iii)* implies that every w^*-slice of the unit ball of X^* has diameter 2, thus the norm of the space is extremely rough.

The next result is a sufficient condition for a Banach space to have the Daugavet property which will be crucial in the rest of the paper. Recall that a closed subspace Z of the dual of a Banach space X is called *norming* whenever
$$\|x\| = \sup\{|z^*(x)| \;:\; z^* \in Z,\ \|z^*\| = 1\}$$
for every $x \in X$. This condition is clearly equivalent to B_Z be w^*-dense in B_{X^*}.

Theorem 2.2. *Let* X *be a Banach space such that there are two norming subspaces* Y *and* Z *of* X^* *such that* $X^* = Y \oplus_1 Z$. *Then,* X *has the Daugavet property.*

Proof. We fix $x_0 \in S_X$, $f_0 \in S_{X^*}$ and $\varepsilon > 0$. We write $f_0 = y_0 + z_0$ such that $y_0 \in Y$, $z_0 \in Z$, $\|f_0\| = \|y_0\| + \|z_0\|$, and
$$U = \{x^* \in B_{X^*} \;:\; \text{Re}\; x^*(x_0) > 1 - \varepsilon\},$$
a w^*-open slice of B_{X^*}. Since B_Z is w^*-dense in B_{X^*}, we may find $z \in Z \cap U$. Observe that, trivially, $\|z\| > 1 - \varepsilon$. Now, since B_Y is w^*-dense in B_{X^*}, we may find a net (y_λ) in B_Y which is w^*-convergent to z. Since $z \in U$, we may suppose that $y_\lambda \in U$ for every λ. On the other hand, since $(y_\lambda + y_0) \longrightarrow z + y_0$ and the norm is w^*-lower semi-continuous, we have
$$\liminf \|y_\lambda + y_0\| \geqslant \|z + y_0\| = \|z\| + \|y_0\| > 1 + \|y_0\| - \varepsilon,$$
and we may find μ such that
$$\|y_\mu + y_0\| \geqslant 1 + \|y_0\| - \varepsilon/2.$$

To finish the proof, we just observe that

$$\begin{aligned}\|f_0 + y_\mu\| &= \|(y_0 + y_\mu) + z_0\| \\ &= \|y_0 + y_\mu\| + \|z_0\| > 1 + \|y_0\| - \varepsilon + \|z_0\| = 2 - \varepsilon,\end{aligned}$$

and that Re $y_\mu(x_0) > 1 - \varepsilon$ since $y_\mu \in U$, and we use Lemma 2.1.*iii*. □

Just remembering Goldstine and Krein-Milman Theorems, we obtain the following useful particular case. Recall that a Banach space X is said to be L-embedded if $X^{**} = X \oplus_1 Z$ for some closed subspace Z of X^{**}.

Corollary 2.3. *Let X be a non-null L-embedded Banach space without extreme points. Then, X^* (and hence X) has the Daugavet property.*

Proof. We have $X^{**} = X \oplus_1 Z$ for some subspace Z. On one hand, since B_X has no extreme points and $\operatorname{ex}(B_{X^{**}}) = \operatorname{ex}(B_X) \bigcup \operatorname{ex}(B_Z)$, we have $\operatorname{ex}(B_{X^{**}}) = \operatorname{ex}(B_Z)$ and the Krein-Milman Theorem gives us that B_Z is w^*-dense in $B_{X^{**}}$. On the other hand, Goldstine Theorem gives us that B_X is w^*-dense in $B_{X^{**}}$. □

It is worth mentioning that it is proved in [33] that a Banach space X such that $X^{**} = X \oplus_1 Z$ with B_Z w^*-dense in $B_{X^{**}}$ satisfies that every weak open subset of B_X has diameter two. Actually, the proof or Theorem 2.2 has been inspired by the one given there.

Let us finish the section by showing some immediate consequences of the above result.

Corollary 2.4. *If X is an L-embedded space with $\operatorname{ex}(B_X) = \emptyset$ and $Y \subsetneq X$ is also an L-embedded space, then $(X/Y)^*$ (and hence X/Y) has the Daugavet property.*

Proof. On one hand, X/Y is a non-null L-embedded space by [20, Corollary IV.1.3]. On the other hand, [20, Propositions IV.1.12 and IV.1.14] gives us that $\operatorname{ex}\left(B_{X/Y}\right) = \emptyset$. Therefore, Corollary 2.3 applies. □

As a particular case of the above corollary we have the following result.

Corollary 2.5. *If Y is an L-embedded space which is a subspace of $L_1 \equiv L_1[0,1]$, then $(L_1/Y)^*$ has the Daugavet property. In particular, $(L_1/Y)^*$ has the Daugavet property for every reflexive subspace Y of L_1 and so do H^∞ and its predual L_1/H_0^1.*

Proof. The space L_1 is an L-embedded space with $\operatorname{ex}(B_{L_1}) = \emptyset$ and the space $H_0^1 \subset L_1$ is also an L-embedded space (see [20, Example IV.1.1] for instance). Then, the result follows immediately from Corollary 2.4. □

It is shown in [20, Proposition IV.2.11] that X/Y fails the RNP when X is an L-embedded space with $\operatorname{ex}(B_X) = \emptyset$ and $Y \subsetneq X$ is also an L-embedded space. On the other hand, it is proved in [28, Proposition 3.2] that L_1/X has the Daugavet property whenever X is a reflexive subspace of L_1. The result for H^∞ appeared in [40] and [42].

3. JB^*-triples and preduals of JBW^*-triples

We recall that a *complex JB^*-triple* is a complex Banach space X with a continuous triple product $\{\cdots\} : X \times X \times X \longrightarrow X$ which is linear and symmetric in the outer variables, and conjugate-linear in the middle variable, and satisfies:

(1) For all x in X, the mapping $y \longmapsto \{xxy\}$ from X to X is a hermitian operator on X and has nonnegative spectrum.
(2) The *main identity*

$$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$$

holds for all a, b, x, y, z in X.
(3) $\|\{xxx\}\| = \|x\|^3$ for every x in X.

Concerning Condition (1) above, we also recall that a bounded linear operator T on a complex Banach space X is said to be *hermitian* if $\|\exp(irT)\| = 1$ for every r in $\mathbb{R}$. By a *complex JBW^*-triple* we mean a complex JB^*-triple whose underlying Banach space is a dual space in metric sense. It is known (see [3]) that every complex JBW^*-triple has a unique predual up to isometric linear isomorphisms and its triple product is separately w^*-continuous in each variable.

Following [23], we define *real JB^*-triples* as norm-closed real subtriples of complex JB^*-triples. Here, by a *subtriple* we mean a subspace which is closed under triple products of its elements. In particular, complex JB^*-triples are real JB^*-triples. A *triple ideal* of a real or complex JB^*-triple X is a subspace M of X such that $\{XXM\} + \{XMX\} \subseteq M$; if merely $\{MXM\} \subseteq M$, then M is called an *inner ideal*.

Real JBW^*-triples where first introduced as those real JB^*-triples which are dual Banach spaces in such a way that the triple product becomes separately w^*-continuous (see [23, Definition 4.1 and Theorem 4.4]). Later, it has been shown in [34] that the requirement of separate w^*-continuity of the triple product is superabundant. We will apply without notice that the bidual of every real or complex JB^*-triple X is a JBW^*-triple under a suitable triple product which extends the one of X ([13] for the complex case and [23] for the real case).

Examples of real JB^*-triples are the spaces $\mathcal{L}(H, K)$, for arbitrary real, complex, or quaternionic Hilbert spaces H and K, under the triple product

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x).$$

The above examples become particular cases of those arising by considering either the so-called complex Cartan factors (regarded as real JB^*-triples) or real forms of complex Cartan factors [32]. We recall that *real forms* of a complex Banach space X are defined as the real closed subspaces of X of the form $X^\tau := \{x \in X \ : \ \tau(x) = x\}$, for some conjugation (i.e., conjugate-linear isometry of period two) on X. We note that, if X is a complex JB^*-triple, then every real form of X is a real JB^*-triple (since conjugations on X preserve triple products [30]). Conversely, if X is a real JB^*-triple, there exists [23, Proposition 2.8] a unique complex JB^*-triple structure on the algebraic complexification $X \oplus iX$ (denoted $\widehat{X}$) and a conjugation τ on $X \oplus iX$ such that $X = \widehat{X}^\tau$, i.e., every real JB^*-triple is a real form of its complexification, which is a complex JB^*-triple.

Let X be a real or complex JB^*-triple. An element $u \in X$ is said to be a *tripotent* if $\{uuu\} = u$, and it said to be a *minimal tripotent* if $u \neq 0$ and

$$\big\{x \in X \ : \ \{uxu\} = x\big\} = \mathbb{R}u.$$

In the complex setting, this is equivalent to $u \neq 0$ and $\{uXu\} = \mathbb{C}u$.

If x is a norm-one element of a real or complex JB^*-triple X, then the set $D(X, x) = D(X^{**}, x) \cap X^*$ is a proper closed face of B_{X^*}, and therefore, by [15, Lemma 2.1 and Theorem 3.7], there is a unique tripotent u in X^{**} such that

$D(X^{**},x)\cap X^* = D(X^{**},u)\cap X^*$. Such a tripotent u is called *the support of x in X^{**}*, and will be denoted by $u(X^{**},x)$.

The complex case of the following result is stated in [7, Corollary 2.11]; the real case follows from results on [6] in an analogous way than the complex version. We include the proof for the sake of completeness.

Lemma 3.1. *Let X be a real or complex JB^*-triple and let x be in S_X. Then, X is Fréchet-smooth at x if and only if $u(X^{**},x)$ lies in X and it is a minimal tripotent of X.*

Proof. Recall that the norm of a Banach space is Fréchet-smooth at x if and only if it is smooth and strongly subdifferentiable at the point (see [17]). Now, the proof follows from the following facts: the norm of X is strongly subdifferentiable at x if and only if $u(X^{**},x)$ belongs to X [6, Corollary 2.5]; X is smooth at x if and only if $D(X^{**},x)\cap X^* = \{x^*\}$ for some extreme point x^* of S_{X^*}, and this is equivalent to the fact that $u(X^{**},x)$ is a minimal tripotent of X^{**} [37, Lemma 2.7 and Corollary 2.1]; and, finally, a tripotent $u\in X$ is a minimal tripotent of X (if and) only if it is a minimal tripotent of X^{**} (by the w^*-density of X in X^{**} and the separate w^*-continuity of the triple product of X^{**}). ☐

It is known [4] that the predual of every real or complex JBW^*-triple is L-embedded. Therefore, Corollary 2.3 gives us that such a space has the Daugavet property whenever its unit ball does not have any extreme point. Actually, more can be proved:

Theorem 3.2. *Let X be a real or complex JBW^*-triple and let X_* be its predual. Then, the following are equivalent:*

(i) X has the Daugavet property.
(ii) X_* has the Daugavet property.
(iii) Every relative weak-open subset of B_{X_*} has diameter 2.
(iv) B_{X_*} has no strongly exposed points.
(v) B_{X_*} has no extreme points.

Proof. $(i)\Rightarrow(ii)$ is clear. $(ii)\Rightarrow(iii)$ is consequence of [38, Lemma 3]. $(iii)\Rightarrow(iv)$ is clear.

$(iv)\Rightarrow(v)$. Of course, it is enough to show that every extreme point of B_{X_*} is actually an strongly exposed point. Indeed, given $f\in \mathrm{ex}\,(B_{X_*})$, [37, Corollary 2.1] assures the existence of a minimal tripotent u of X such that $u(f)=1$, and u is a point of Fréchet-smoothness of the norm of X by Lemma 3.1. Therefore, there is a point of Fréchet-smoothness, u, in $D(X_*,f)$ and, as we commented in the introduction, this implies that f is strongly exposed by u (see [12, Corollary I.1.5], for instance).

$(v)\Rightarrow(i)$. X_* is an L-embedded by [4, Proposition 2.2] and B_{X_*} has no extreme points, so Corollary 2.3 applies. ☐

As an straightforward consequence of the above theorem we obtain the following result, which states the "extreme" behaviour of the diameters of the weak-open subset of the unit ball of the predual of a JBW^*-triple.

Corollary 3.3. *Let Y be the predual of some real or complex JBW^*-triple. Then, either every weak-open subset of B_Y has diameter 2 or B_Y has slices of arbitrary small diameter.*

By Corollary 2.1 of [37], a real or complex JBW^*-triple has minimal tripotents if and only if the unit ball of its predual has extreme points. Therefore, the following result follows immediately from Theorem 3.2.

Corollary 3.4. *Let X be a real or complex JBW^*-triple. Then, X has the Daugavet property if and only if it does not have any minimal tripotents.*

The complex case of the above corollary and the equivalence $(i) \Leftrightarrow (ii)$ of Theorem 3.2 appear in [35, Theorem 4.7].

As a consequence of Theorem 3.2 we obtain:

Corollary 3.5. *Neither the dual of a real or complex JB^*-triple nor a real or complex JB^*-triple which is the bidual of some space, has the Daugavet property.*

Proof. On one hand, the dual X^* of a JB^*-triple X is also the predual of the JBW^*-triple X^{**} and, as every dual space, B_{X^*} has extreme points. On the other hand, if $Y = Z^{**}$ is a JB^*-triple, then it is actually a JBW^*-triple whose predual $Y_* = Z^*$ has extreme points in its unit ball. □

Remark 3.6. It is worth mentioning that, for an arbitrary Banach space Z, the absence of extreme points in B_Z or the fact that all weak-open subsets of B_Z have diameter two, does not necessarily imply that Z has the Daugavet property. For instance, c_0 satisfies both assumptions (see [5, Lemma 2.2] for instance), but it does not have the Daugavet property.

On the other hand, the assertions (iii), (iv), and (v) of Theorem 3.2 are not equivalent for general Banach spaces. On one hand, *there exists a Banach space Z whose unit ball has slices of arbitrary small diameter, but it does not have any extreme point (so, it does not have any strongly exposed point)* [14, Proposition 1]. On the other hand, *every slice of the unit ball of ℓ_∞ has diameter 2 (and so, it does not have any strongly exposed point), but it is plenty of extreme points (it is a dual space).*

If X is a real or complex JBW^*-triple, it is well known that $X_* = A \oplus_1 N$, where A is the closed linear span of the extreme points of B_{X_*}, and the unit ball of N has no extreme points (see [18] for the complex case and [37] for the real case). Therefore, $X = \mathcal{A} \oplus_\infty \mathcal{N}$, where $\mathcal{A} = N^\perp \equiv A^*$ is an *atomic* JBW^*-triple (i.e. it is the weak*-closed span of its minimal tripotents) and $\mathcal{N} = A^\perp \equiv N^*$ is a JBW^*-triple without minimal tripotents. With this in mind, the following result is a consequence of Theorem 3.2 and a characterization of the RNP in preduals of JBW^*-triples given in [2].

Corollary 3.7. *Let X be a real or complex JBW^*-triple. Then, in the natural decomposition $X_* = A \oplus_1 N$, A has the RNP and N has the Daugavet property. Therefore, in the decomposition $X = \mathcal{A} \oplus_\infty \mathcal{N}$, $\mathcal{A}$ is a w^*-Asplund space (i.e., the dual of a space having the RNP) and $\mathcal{N}$ has the Daugavet property.*

Proof. In the complex case, since A is the predual of the atomic JBW^*-triple $\mathcal{A}$, it has the RNP by [2, Theorem 1] and, therefore, $\mathcal{A}$ is a w^*-Asplund space. In the real case, we consider $\widehat{\mathcal{A}}$, the complexification of $\mathcal{A}$. On one hand, $\widehat{\mathcal{A}}$ is a w^*-Asplund space by the above. On the other hand, $A \equiv \mathcal{A}_*$ is a (real) subspace of $\left(\widehat{\mathcal{A}}\right)_*$, and the RNP passes to subspaces.

Since $N^* = \mathcal{N}$ is a JBW^*-triple without minimal tripotents, Corollary 3.4 gives us that $\mathcal{N}$, and hence its predual N, have the Daugavet property. □

Our next aim is to prove a characterization of the Daugavet property for general JB^*-triples. We first prove that the algebraic characterization given in Corollary 3.4 for JBW^*-triples is also valid in the general case, and then we will deduce more characterizations in terms of the geometry of the norm of the triple.

We need a result about real or complex JB^*-triples which can be of independent interest. Previously, we have to recall some known facts about JB^*-triples.

If X is a real or complex JB^*-triple, X^{**} is a JBW^*-triple. Therefore, we can decompose $X^* = (X^{**})_*$ into its atomic and not atomic parts, as we have commented above, i.e., $X^* = A \oplus_1 N$ where A is the closed linear span of the extreme points of B_{X^*}, and the unit ball of N has no extreme points. Then, $X^{**} = \mathcal{A} \oplus_\infty \mathcal{N}$, where $\mathcal{A} = N^\perp \equiv A^*$ is an atomic JBW^*-triple, and $\mathcal{N} = A^\perp \equiv N^*$ is a JBW^*-triple without minimal tripotents. Let us call $\pi_{\mathcal{A}}$ (resp. $\pi_{\mathcal{N}}$) the projection from X^{**} to $\mathcal{A}$ with kernel $\mathcal{N}$ (resp. to $\mathcal{N}$ with kernel $\mathcal{A}$), and let $J_X : X \longrightarrow X^{**}$ be the natural inclusion. It is well known that $\pi_{\mathcal{A}} \circ J_X : X \longrightarrow \mathcal{A}$ is an isometric embedding (Gelfand-Naimark Theorem [19]). The next result gives the same for $\pi_{\mathcal{N}} \circ J_X$, provided X has no minimal tripotents.

Theorem 3.8. *Let X be a real or complex JB^*-triple without minimal tripotents. Then, the mapping $\pi_{\mathcal{N}} \circ J_X : X \longrightarrow \mathcal{N}$ is an isometric embedding. Therefore, N is a norming subspace of X^*.*

Proof. We start by proving the result in the complex case. Let X be a complex JB^*-triple and let us consider $Y = X \cap \mathcal{A}$, which is clearly a closed ideal of X. On one hand, Y has no minimal tripotents (indeed, if $0 \neq u \in Y$ is a minimal tripotent of Y, then $\{uYu\} = \mathbb{C}u$; since Y is a triple ideal (and hence an inner ideal), we have $\{uXu\} \subset Y$, so we obtain $\{uXu\} = \mathbb{C}u$ and u is a minimal tripotent of X, which is impossible). On the other hand, by [9, Proposition 3.7] Y^* has the RNP (i.e. Y is an Asplund space) and, if $Y \neq 0$, the norm of Y has points of Fréchet-smoothness. But the existence of points of Fréchet-smoothness in Y implies the existence of minimal tripotents in Y (Lemma 3.1), a contradiction. We deduce that Y is null and, therefore, $\pi_{\mathcal{N}} \circ J_X$ is injective. Being a triple-homomorphism, it is routine (using axiom (3)) to show that it is an isometric embedding as desired (actually, in the complex case, the converse result is also true, see [30]). Since $\mathcal{N} = A^\perp \equiv N^*$, it is clear that N is norming.

The proof for the real case is very similar. If X is a real JB^*-triple, we will show that $Y = X \cap \mathcal{A}$ has no minimal tripotents and that it is an Asplund space, and then the rest of the above proof works. First, if $0 \neq u \in Y$ is a minimal tripotent, then $\{y \in Y \,:\, \{uyu\} = y\} = \mathbb{R}u$; since Y is a inner ideal, $\{uXu\} \subseteq Y$, so if $x \in X$ is such that $\{uxu\} = x$, we obtain that $x \in Y$, which implies $x \in \mathbb{R}u$, i.e., u is a minimal tripotent of X, a contradiction. Second, we consider the complexification $\widehat{Y}$ of Y, and we observe that $\widehat{Y} = \widehat{\mathcal{A}} \cap \widehat{X}$, where $\widehat{X^{**}} = \widehat{\mathcal{A}} \oplus_\infty \widehat{\mathcal{N}}$ is the decomposition into the atomic and non-atomic part [37, Theorem 3.6]. Therefore, $\widehat{Y}$ is an Asplund space [9, Proposition 3.7] and so does its real subspace Y. □

As a consequence of the above result and Theorem 2.2, we obtain that JB^*-triples without minimal tripotents have the Daugavet property. The complex case of this result appear in [35, Theorem 4.7] with a different proof.

Proposition 3.9. *Let X be a real or complex JB^*-triple. Then, X has the Daugavet property if and only if it has no minimal tripotents.*

Proof. Suppose X has no minimal tripotents and write $X^* = A \oplus_1 N$. On one hand, since $\mathrm{ex}\,(B_{X^*}) \subseteq B_A$, the Krein-Milman Theorem gives us that A is a norming subspace of X^*. On the other hand, if X has no minimal tripotents, Theorem 3.8 gives us that N is also norming. Now, Theorem 2.2 gives us that X has the Daugavet property. Conversely, if X has a minimal tripotents, then it has a point of Fréchet-smoothness by Lemma 3.1; but the norm of a Banach space with the Daugavet property is extremely rough (use Lemma 2.1.*iii*), a contradiction. □

Actually, we can state a characterization of the Daugavet property for JB^*-triples in terms of the geometry of the norm of the triple.

Theorem 3.10. *Let X be a real or complex JB^*-triple. Then, the following are equivalent:*

(i) X has the Daugavet property.
(ii) The norm of X is extremely rough.
(iii) The norm of X is not Fréchet-smooth at any point.

Proof. $(i) \Rightarrow (ii)$. As we commented in the introduction, the norm of X is extremely rough if and only if every w^*-slice of B_{X^*} has diameter 2, and the latest fact is consequence of Lemma 2.1.*iii*.

$(ii) \Rightarrow (iii)$ is clear.

$(iii) \Rightarrow (i)$. By Lemma 3.1, the norm of X is Fréchet-smooth at the minimal tripotents, so we deduce that X has no minimal tripotents and Proposition 3.9 applies. □

Remark 3.11. It is worth mentioning that the above geometric characterizations are not valid for arbitrary Banach spaces. For instance, *the norm of ℓ_1 is extremely rough (and so ℓ_1 has no points of Fréchet-smoothness) but ℓ_1 does not have the Daugavet property.*

Also, the implication $(iii) \Rightarrow (ii)$ of the above theorem is not valid in general. Indeed, *there exists a Banach space whose norm does not have any point of Fréchet differentiability but it is not rough* (see [25, Remark 4, pp. 341]).

To finish the section, let us comment some results from [4] which are related to our development.

Remark 3.12. Let us consider the following conditions for a Banach space X:

(a) every relative weak-open subset of B_X has diameter 2,
(b) the norm of X is extremely rough.

It is proved in [4, Theorem 2.3] that condition (a) is satisfied when X is a non-reflexive real or complex JB^*-triple, while our Theorem 3.2 says that condition (a) characterizes the Daugavet property in the class of preduals of real or complex JBW^*-triples.

With respect to condition (b), it is shown in [4, Corollary 2.5] that the predual of every non-reflexive real or complex JBW^*-triple satisfies it, while condition (b) characterizes the Daugavet property for real or complex JB^*-triples (Theorem 3.10).

Since a reflexive Banach space never satisfies neither (a) nor (b), the above paragraphs contains the answer to every question about this conditions in the setting of real or complex JB^*-triples and their isometric preduals.

4. C^*-algebras and von Neumann preduals

Despite *real C^*-algebras* can be defined by different systems of intrinsic axioms (see [24] for a summary), we prefer to introduce them as the norm-closed self-adjoint real subalgebras of complex C^*-algebras. Since complex C^*-algebras are complex JB^*-triples under the triple product

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x),$$

certainly real C^*-algebras are real JB^*-triples. The concept of a *real W^*-algebra* (*real von Neumann algebra*) was first defined as a real C^*-algebra A having a complete predual A_* such that the product of A is separately w^*-continuous, but the latest condition was shown to be redundant in [24]. Real W^*-algebras are real JBW^*-triples.

Therefore, the geometric characterizations given in Theorems 3.2 and 3.10 can be stated for real or complex C^*-algebras and preduals of W^*-algebras. The next results summarize those theorems and also Corollaries 3.3 and 3.5 in terms of C^*-algebras.

Corollary 4.1. *Let X be a real or complex C^*-algebra. Then, the following are equivalent:*

(i) X has the Daugavet property.
(ii) The norm of X is extremely rough.
(iii) The norm of X is not Fréchet-smooth at any point.

Corollary 4.2. *Let X be a real or complex W^*-algebra and let X_* be its predual. Then, the following are equivalent:*

(i) X has the Daugavet property.
(ii) X_* has the Daugavet property.
(iii) Every weak-open subset of B_{X_*} has diameter 2.
(iv) B_{X_*} has no strongly exposed points.
(v) B_{X_*} has no extreme points.

Corollary 4.3.

(a) *Let X be the predual of some real or complex W^*-algebra. Then, either every weak-open subset of B_X has diameter 2 or B_X has slices of arbitrary small diameter.*
(b) *Neither the dual of a real or complex C^*-algebra nor a real or complex C^*-algebra which is the bidual of some space, has the Daugavet property.*

The algebraic characterization of the Daugavet property for JB^*-triples (Proposition 3.9) is of course valid for C^*-algebras, but it could be more convenient to write it in terms of atomic projections. Let us give the definitions and results.

If X is a real or complex C^*-algebra, then $u \in X$ is a tripotent if and only if it is a *partial isometry*, i.e., u satisfies that $uu^*u = u$. Recall that a *projection* in a C^*-algebra is an element $p \in X$ such that $p^* = p$ and $p^2 = p$. It is clear that projections are partial isometries (and so tripotents), but there are partial isometries which are not projections. A projection p in X is said to be *atomic* if $p \neq 0$ and

$$\{x \in X \ : \ px^*p = x\} = \mathbb{R}p,$$

i.e., p is minimal seen as a tripotent. Therefore, in the complex case this is equivalent to $p \neq 0$ and $pXp = \mathbb{C}p$. The C^*-algebra X is said to be *non-atomic* if it does not have any atomic projection.

If X has atomic projections, then it clearly has minimal tripotents. Conversely, if X has a minimal tripotent, say u, then the projection $d = u^*u$ (called the domain projection associated to u) is atomic. Indeed, we take $x \in X$ such that $dx^*d = x$. Then,

$$u(ux)^*u = (uu^*u)x^*u^*u = u(u^*ux^*u^*u) = u(dx^*d) = ux$$

so, since u is minimal, $ux = \lambda u$ for some $\lambda \in \mathbb{R}$. Then,

$$\begin{aligned}\lambda d = u^*(\lambda u) = u^*(ux) = u^*(u(dx^*d)) = \\ = u^*\big((uu^*u)x^*u^*u\big) = u^*ux^*u^*u = dx^*d = x.\end{aligned}$$

We have shown that *a real or complex C^*-algebra has no minimal tripotents if and only if it is non-atomic.* So, for C^*-algebras, Proposition 3.9 can be written in terms of atomic projections.

Corollary 4.4.

(a) *A real or complex C^*-algebra has the Daugavet property if and only if it is non-atomic.*
(b) *The predual of a real or complex W^*-algebra has the Daugavet property if and only if the algebra is non-atomic.*

The complex case of the above result appears in [36, Theorem 2.1].

As a JBW^*-triple, every real or complex W^*-algebra X admits a natural decomposition into the atomic and non-atomic parts which is originated by the natural decomposition of the predual X_*. I.e., $X_* = A \oplus_1 N$, where the unit ball of N does not have any extreme point, and B_A is the closed convex hull of the extreme points of B_{X_*}. Thus, $X = \mathcal{A} \oplus_\infty \mathcal{N}$, where the subtriple $\mathcal{A} = N^\perp \equiv A^*$ is norm-generated by the minimal tripotents of X, and the subtriple $\mathcal{N} = A^\perp \equiv N^*$ has no minimal tripotents. Moreover, $\mathcal{A}$ and $\mathcal{N}$ are w^*-closed subalgebras of X, the first one is generated by its atomic projections and the second one has no atomic projections.

The next results put Corollary 3.7 and Theorem 3.8 in terms of C^*-algebras.

Corollary 4.5. *Let X be a real or complex W^*-algebra. Then, in the natural decomposition $X_* = A \oplus_1 N$, A has the RNP and N has the Daugavet property. Therefore, in the decomposition $X = \mathcal{A} \oplus_\infty \mathcal{N}$, $\mathcal{A}$ is a w^*-Asplund space (i.e., the dual of a space having the RNP) and $\mathcal{N}$ has the Daugavet property.*

Corollary 4.6. *Let X be a real or complex C^*-algebra without atomic projections, and let $X^{**} = \mathcal{A} \oplus_\infty \mathcal{N}$ the natural decomposition of its bidual into atomic and non-atomic parts. Then, the decomposition of every $x \in X$ as $x = a^{**} + n^{**}$, with $a^{**} \in \mathcal{A}$, $n^{**} \in \mathcal{N}$ satisfies $\|x\| = \|a^{**}\| = \|n^{**}\|$.*

5. The uniform Daugavet property

Following [8], a Banach space X is said to have the *uniform Daugavet property* if

$$D_X(\varepsilon) := \inf\{n \in \mathbb{N} \ : \ \mathrm{conv}_n(l^+(x,\varepsilon)) \supset S_X \ \forall x \in S_X\}$$

is finite for every $\varepsilon > 0$, where

$$l^+(x,\varepsilon) := \big\{y \in X \ : \ \|y\| \leqslant 1+\varepsilon, \ \|x+y\| > 2-\varepsilon\big\}$$

and $\mathrm{conv}_n(A)$ is the set of all convex combination of all n-point collections of elements of A. By [8, Remark 6.3], X has the uniform Daugavet property if and only if

$$\lim_{n\to\infty} \mathrm{Daug}_n(X,\varepsilon) = 0$$

for every $\varepsilon > 0$, where

$$\mathrm{Daug}_n(X,\varepsilon) := \sup_{x,y\in S_X} \mathrm{dist}\big(y, \mathrm{conv}_n(l^+(x,\varepsilon))\big).$$

Since (Lemma 2.1) X has the Daugavet property if and only if

$$B_X \subset \overline{\mathrm{co}}\big(\{y \in X \ : \ \|y\| \leqslant 1+\varepsilon,\ \|x+y\| > 2-\varepsilon\}\big)$$

for every $x \in S_X$ and every $\varepsilon > 0$, the uniform Daugavet property implies the Daugavet property, and it can be view as a quantitative approach to it.

Examples of spaces satisfying the uniform Daugavet property are $L_1[0,1]$ and $C(K)$ for every perfect compact space K [8, §6]. On the other hand, in [29] it is shown an example of a Banach space with the Daugavet property which does not satisfy the uniform Daugavet property.

The uniform Daugavet property was introduced in [8] to study when the Daugavet property passes from a Banach space to its so-called ultrapowers.

Let us recall here the notion of (Banach) ultraproducts [21]. Let $\mathcal{U}$ be a free ultrafilter on a nonempty set I, and let $\{X_i\}_{i\in I}$ be a family of Banach spaces. We can consider the ℓ_∞-sum of the family, $[\oplus_{i\in I} X_i]_{\ell_\infty}$, together with its closed subspace

$$N_{\mathcal{U}} := \Big\{\{x_i\}_{i\in I} \in [\oplus_{i\in I} X_i]_{\ell_\infty} \ : \ \lim_{\mathcal{U}} \|x_i\| = 0\Big\}.$$

The quotient space $[\oplus_{i\in I} X_i]_{\ell_\infty}/N_{\mathcal{U}}$ is called the *ultraproduct* of the family $\{X_i\}_{i\in I}$ relative to the ultrafilter $\mathcal{U}$, and is denoted by $(X_i)_{\mathcal{U}}$. Let (x_i) stand for the element of $(X_i)_{\mathcal{U}}$ containing a given family $\{x_i\} \in [\oplus_{i\in I} X_i]_{\ell_\infty}$. It is easy to check that $\|(x_i)\| = \lim_{\mathcal{U}} \|x_i\|$. Moreover, the ultraproduct $(X_i^*)_{\mathcal{U}}$ can be seen as a subspace of $\big[(X_i)_{\mathcal{U}}\big]^*$ by identifying each element $(f_i) \in (X_i^*)_{\mathcal{U}}$ with the (well-defined) functional on $(X_i)_{\mathcal{U}}$ given by

$$(x_i) \longmapsto \lim_{\mathcal{U}} (f_i(x_i)) \qquad \big((x_i) \in (X_i)_{\mathcal{U}}\big).$$

If $\{Y_i\}_{i\in I}$ is another family of Banach spaces and for each $i \in I$ we take an operator $T_i \in L(X_i, Y_i)$ with $\sup_{i\in I} \|T_i\| < \infty$, we can define the *utraproduct of the family of operators* $\{T_i\}_{i\in I}$ with respect to the ultrafilter $\mathcal{U}$, denoted (T_i), as

$$(x_i) \longmapsto (T_i x_i) \qquad \big((x_i) \in (X_i)_{\mathcal{U}}\big).$$

This is a well defined operator from $(X_i)_{\mathcal{U}}$ to $(Y_i)_{\mathcal{U}}$ with

$$\|(T_i)\| = \lim_{\mathcal{U}} \|T_i\|.$$

If all the X_i are equal to some Banach space X, the ultraproduct of the family is called the $\mathcal{U}$*-ultrapower* of X and it is usually denoted by $X_{\mathcal{U}}$. For $T \in L(X)$, by (T) we denote the ultraproduct of the family $\{T_i\}_{i\in I}$ where $T_i = T$ for every $i \in I$.

In [8, Corollary 6.5], it is proved that a Banach space X has the uniform Daugavet property if and only if every ultrapower $X_{\mathcal{U}}$, $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$, has the Daugavet property, in which case $X_{\mathcal{U}}$ even has the uniform Daugavet property. Let us comment that it is routine to prove that a Banach X has the (usual) Daugavet property whenever $X_{\mathcal{U}}$ does, $\mathcal{U}$ a free ultrafilter on an arbitrary set I (we can use Lemma 2.1.*ii* or, alternatively, we can prove directly that every rank-one operator $T \in L(X)$ satisfies (DE) since its ultrapower $(T) \in L(X_{\mathcal{U}})$, which is also a rank-one operator on $X_{\mathcal{U}}$, does). On the other hand, as we have said before, there is a Banach space with the Daugavet property which does not have the uniform Daugavet property [29], thus the Daugavet property does not always pass to ultrapowers.

Our aim in this section is to prove that the Daugavet property and its uniform version are equivalent for real or complex JB^*-triples and their isometric preduals.

As we said before, this is true for $C(K)$ spaces and for $L_1[0,1]$. These facts were proved in [8, §6], where explicit estimations for $D_{C(K)}(\varepsilon)$ and $D_{L_1[0,1]}(\varepsilon)$ were done. Our approach is different: we will use Theorems 3.2 and 3.10 to show that, for JB^*-triples and their isometric preduals, the Daugavet property passes to arbitrary ultrapowers.

Since an ultrapower of a JB^*-triple is again a JB^*-triple (see [13]), the result for this class follows immediately from Theorem 3.10 and the following lemma, which can be of independent interest.

Lemma 5.1. *Let $\{X_i\}_{i\in I}$ be a family of Banach spaces, $\mathcal{U}$ a free ultrafilter of a set I, and $\delta > 0$. If the norm of each X_i is δ-rough, then so does the norm of $(X_i)_{\mathcal{U}}$.*

Proof. Given a norm-one element $x = (x_i) \in (X_i)_{\mathcal{U}}$ and a positive number $\alpha < 1$, we have to show that the slice $S\big(B_{[(X_i)_{\mathcal{U}}]^*}, (x_i), \alpha\big)$ of the unit ball of $\big[(X_i)_{\mathcal{U}}\big]^*$ has diameter greater than δ. Indeed, we can suppose that $\|x_i\| = 1$ for every $i \in I$ and, since the norm of each X_i is δ-rough, given a family $\{\varepsilon_i\}$ of positive number with $\lim_{\mathcal{U}} \varepsilon_i = 0$, we can find $f_i, g_i \in S_{X_i^*}$ such that

$$\|f_i - g_i\| > \delta - \varepsilon_i \qquad \text{and} \qquad \operatorname{Re} f_i(x_i) > 1-\alpha, \quad \operatorname{Re} g_i(x_i) > 1-\alpha.$$

Now, we consider the elements $f = (f_i)$ and $g = (g_i)$ of the unit ball of $(X_i^*)_{\mathcal{U}} \subseteq \big[(X_i)_{\mathcal{U}}\big]^*$, and we observe that, on one hand,

$$\big\|(f_i) - (g_i)\big\| = \lim_{\mathcal{U}} \|f_i - g_i\| \geqslant \delta$$

and, on the other hand,

$$\operatorname{Re} f(x) = \lim_{\mathcal{U}} f_i(x_i) > 1-\alpha, \qquad \operatorname{Re} g(x) = \lim_{\mathcal{U}} g_i(x_i) > 1-\alpha. \qquad \square$$

By using the above lemma and Theorem 3.10, we have that $X_{\mathcal{U}}$ has the Daugavet property whenever the JB^*-triple X does. But, as we already mentioned, the converse result is true in general.

Theorem 5.2. *Let X be a real or complex JB^*-triple and let $\mathcal{U}$ be a free ultrafilter on a set I. Then, X has the Daugavet property if and only if $X_{\mathcal{U}}$ does. Therefore, the Daugavet property and the uniform Daugavet property are equivalent for JB^*-triples.*

As a consequence of the above theorem and Proposition 3.9, we obtain the following result about JB^*-triples.

Corollary 5.3. *Let X be a real or complex JB^*-triple and $\mathcal{U}$ a free ultrafilter on a set I. Then, $X_{\mathcal{U}}$ has a minimal tripotent if and only if X does.*

Remark 5.4. It is also true that every ultraproduct of JB^*-triples is a JB^*-triple (see [13]). Then, by using Theorem 3.10 and Lemma 5.1, we also obtain that *the ultraproduct of a family of JB^*-triples with the Daugavet property also has the Daugavet property.* In other words (Proposition 3.9), *the ultraproduct of a family of JB^*-triples without minimal tripotents also has no minimal tripotent.*

The second part of the present section is devoted to preduals of JBW^*-triples.

Even though the ultrapower of the dual of a Banach space is not, in general, the dual of the ultrapower of the space (see [21, §7]), it can be proved that the ultrapower of a predual of a JBW^*-triple is again the predual of some JBW^*-triple. In the complex case, the proof is easy to state: the dual of the ultrapower $X_{\mathcal{U}}$ of a Banach space X is 1-complemented in another ultrapower $(X^*)_{\mathcal{M}}$ of X^* [21], and the contractive projection Theorem applies.

Since we have not find any reference to the above result in the literature, we give a detailed proof. Actually, a more general result can be state.

Proposition 5.5. *Let $\{X_i\}_{i\in I}$ be a family of Banach spaces such that each X_i^* is a (real or complex) JBW^*-triple, and let $\mathcal{U}$ be a free ultrafilter on I. Then, $(X_i)_\mathcal{U}$ is the predual of some (real or complex) JBW^*-triple.*

Proof. We start with the complex case. By [21, Corollary 7.6], there is another free ultrafilter $\mathcal{B}$ on an index set I', such that $\big[(X_i)_\mathcal{U}\big]^*$ is isometric to a 1-complemented subspace of $((X_i^*)_\mathcal{U})_\mathcal{B}$, which is a JB^*-triple. But 1-complemented subspaces of complex JB^*-triples are JB^*-triples (see [31]).

If each X_i^* is a real JBW^*-triple, then there is a conjugation τ_i on each X_i such that $\big(\widehat{X_i}\big)^*$ is a complex JBW^*-triple and $X_i = \widehat{X_i}^{\tau_i}$ [23]. On one hand, $\big[\big(\widehat{X_i}\big)_\mathcal{U}\big]^*$ is a JBW^*-triple by the complex case. On the other hand, we consider $\tau = (\tau_i)$, the ultraproduct of the family of the conjugations τ_i, and we observe that τ is a conjugation (routine) and that $\big[(\widehat{X_i})_\mathcal{U}\big]^\tau \equiv (X_i)_\mathcal{U}$. Indeed, $(\widehat{x_i}) \in \big[(\widehat{X_i})_\mathcal{U}\big]^\tau$ if and only if $\lim_\mathcal{U} \|\tau_i(\widehat{x_i}) - \widehat{x_i}\| = 0$. Thus, the image of the natural inclusion of $(X_i)_\mathcal{U}$ into $(\widehat{X_i})_\mathcal{U}$ falls into $\big[(\widehat{X_i})_\mathcal{U}\big]^\tau$, and it is onto since, for every $(\widehat{x_i}) \in \big[(\widehat{X_i})_\mathcal{U}\big]^\tau$, we have $(\widehat{x_i}) = \big(\tau_i(\widehat{x_i})\big) \in \big(\widehat{X_i}^{\tau_i}\big)_\mathcal{U} \equiv (X_i)_\mathcal{U}$. Now, the dual of $(X_i)_\mathcal{U} \equiv \big[(\widehat{X_i})_\mathcal{U}\big]^\tau$ is a real form (using τ^*, which is also a conjugation) of $\big[(\widehat{X_i})_\mathcal{U}\big]^*$, and hence it is a real JBW^*-triple. □

With this in mind, the equivalence of the Daugavet property and its uniform version for preduals of JBW^*-triples is a consequence of Theorem 3.2.

Theorem 5.6. *Let X be a real or complex JBW^*-triple and let $\mathcal{U}$ be a free ultrafilter on a set I. Then, X_* has the Daugavet property if and only if $(X_*)_\mathcal{U}$ does. Therefore, the Daugavet property and the uniform Daugavet property are equivalent for preduals of JBW^*-triples.*

In the proof we will use the following easy fact: if Y is a Banach space and $Z \subseteq Y^*$ is a norming subspace, then for every strongly exposed point $y \in S_Y$, the exposing functional belong to Z. Observe that this is the case of the ultraproduct of the duals of a family of Banach space seen as a norm-closed subspace of the dual of the ultraproduct of the spaces.

Proof. We only have to show that $(X_*)_\mathcal{U}$ has the Daugavet property whenever X_* does. Since $(X_*)_\mathcal{U}$ is the predual of some JBW^*-triple, it suffices to show that its unit ball has no strongly exposed points (Theorem 3.2). Therefore, we suppose, for the sake of contradiction, that the unit ball of $(X_*)_\mathcal{U}$ has a strongly exposed point, say (x_i). By the preceding remark, there exists (ϕ_i) in the unit sphere of $(X)_\mathcal{U}$ (which we can suppose to satisfy $\|\phi_i\| = 1$ for every i) which strongly expose (x_i). Let us fix $0 < \varepsilon_0 < 1$. Now, for every $\alpha > 0$, since X_* has the Daugavet property, we can apply Lemma 2.1.*ii* to get, for every $i \in I$, a point $y_i \in S_{X_*}$ such that

$$\|x_i - y_i\| \geqslant 2 - \varepsilon_0 \qquad \text{and} \qquad \mathrm{Re}\ \phi_i(y_i) > 1 - \alpha/2.$$

Now, (y_i) belong to the unit ball of $(X_*)_\mathcal{U}$,

$$\|(x_i) - (y_i)\| = \lim_\mathcal{U} \|x_i - y_i\| \geqslant 2 - \varepsilon_0,$$

and

$$\mathrm{Re}\,(\phi_i)[(y_i)] = \lim_\mathcal{U} \phi_i(y_i) > 1 - \alpha.$$

Since α is arbitrary, we conclude that every slice of the unit ball of $(X_*)_{\mathcal{U}}$ defined by (ϕ_i) has diameter greater or equal than $2-\varepsilon_0$ (recall that $\operatorname{Re}(\phi_i)[(x_i)]=1$). Hence, (ϕ_i) does not strongly expose (x_i), a contradiction. $\square$

As a consequence of the above theorem and Theorem 3.2, we obtain the following.

Corollary 5.7. *Let X be a real or complex JBW^*-triple and let $\mathcal{U}$ be a free ultrafilter on a set I. Then, the unit ball of $(X_*)_{\mathcal{U}}$ have extreme points if and only if B_{X_*} does.*

As a consequence of Theorems 3.2, 5.2 and 5.6, we obtain

Corollary 5.8. *Let X_* be the predual of a real or complex JBW^*-triple X. Then, X_* has the uniform Daugavet property if and only if X does.*

It is worth mentioning that it is not known whether the uniform Daugavet property passes from the dual of a Banach space to the space.

Remark 5.9. The proof of Theorem 5.6 can be straightforwardly adapted to show that *the ultraproduct of a family of preduals of JBW^*-triples with the Daugavet property also has the Daugavet property.* Therefore, Corollary 5.7 can be also adapted to show that *the unit ball of the ultraproduct of a family of preduals of JBW^*-triples has no extreme points, provided that the unit ball of each factor does not have any extreme point.*

It is worth mentioning that Corollary 5.7 can not be stated for general Banach spaces, as the following example shows.

Example 5.10. *There exists a Banach space X whose unit ball does not have any extreme point and a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that the unit ball of $X_{\mathcal{U}}$ has an extreme point* [22, Example 2.14].

Let us comment a particular case in which the conclusion of Corollary 5.7 can be easily stated.

Remark 5.11. *Let X be a Banach space. Suppose that there exists $\delta>0$ such that for every $x\in S_X$, there is $y\in X$ with $\|y\|\geqslant\delta$ such that $\|x\pm y\|\leqslant 1$ (in particular, B_X has no extreme points). Then, for every free ultrafilter $\mathcal{U}$ on a set I, the unit ball of $X_{\mathcal{U}}$ does not have any extreme point.* Indeed, let (x_i) be a norm-one element of $X_{\mathcal{U}}$, which we can suppose to satisfy $\|x_i\|=1$ for every i. Then, for every $i\in I$, take $y_i\in X$ with $\|y_i\|\geqslant\delta$ and $\|x_i\pm y_i\|\leqslant 1$. If we consider $(y_i)\in X_{\mathcal{U}}$, then

$$\|(y_i)\|\geqslant\delta \qquad \text{and} \qquad \|(x_i)\pm(y_i)\|\leqslant 1.$$

Therefore, (x_i) is not an extreme point of the unit ball of $X_{\mathcal{U}}$.

It is easy to show that the above situation is fulfilled by $L_1[0,1]$ with $\delta=1$.

Example 5.12. *For every $f\in L_1[0,1]$ with $\|f\|_1=1$, there is $g\in L_1[0,1]$ with $\|g\|_1=1$ and such that $\|f\pm g\|_1=1$.* Indeed, up to an isometric isomorphism, we can suppose $f(t)\geqslant 0$ for every $t\in[0,1]$ and, by continuity, we can find $t_0\in]0,1[$ such that

$$\int_0^{t_0} f(t)\ dt=\int_{t_0}^1 f(t)\ dt=\frac{1}{2}.$$

Then, if we consider $g = f\left(\chi_{[0,t_0]} - \chi_{[t_0,1]}\right) \in L_1[0,1]$, we clearly have $\|g\|_1 = 1$ and

$$\begin{aligned}\|f \pm g\|_1 &= \int_0^{t_0} \big(f(t) \pm f(t)\big)\, dt \;+\; \int_{t_0}^1 \big(f(t) \mp f(t)\big)\, dt \\ &= \left(\frac{1}{2} \pm \frac{1}{2}\right) + \left(\frac{1}{2} \mp \frac{1}{2}\right) = 1\end{aligned}$$

Actually, a very similar result (with δ arbitrarily closed to 1) can be stated for every $L_1(\mu)$ if μ does not have any atom.

For the sake of completeness, we finish the paper by summarizing the results of the present section in terms of C^*-algebras and preduals of W^*-algebras.

Corollary 5.13.

(a) *The ultraproduct of every family of real or complex C^*-algebras with the Daugavet property also has the Daugavet property. In particular, the Daugavet and the uniform Daugavet property are equivalent for real or complex C^*-algebras.*
(b) *The ultrapower of a real or complex C^*-algebra has atomic projections if and only if the algebra does.*
(c) *The ultraproduct of every family of preduals of real or complex W^*-algebras with the Daugavet property also has the Daugavet property. In particular, the Daugavet and the uniform Daugavet property are equivalent for preduals of real or complex W^*-algebras.*
(d) *Let X_* be the predual of a real or complex W^*-algebra X. Then, X_* has the uniform Daugavet property if and only if X does.*
(e) *Let Y be the predual of a real or complex W^*-algebra. Then, B_Y has an extreme point if and only if the unit ball of every ultrapower of Y does.*

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