AN ALTERNATIVE DAUGAVET PROPERTY

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ABSTRACT. We introduce a strictly weaker version of the Daugavet property as follows: a Banach space X has this alternative Daugavet property (ADP in short) if the norm identity

(aDE) $$\max_{|\omega|=1} \|Id + \omega T\| = 1 + \|T\|$$

holds for all rank-one operators $T : X \to X$. In such a case, all weakly compact operators on X also satisfy (aDE). We give some geometric characterizations of the alternative Daugavet property in terms of the space and its successive duals. We prove that the ADP is stable for c_0-, l_1- and l_∞-sums and characterize when some vector-valued function spaces have the property. Finally, we show that a C^*-algebra (or the predual of a von Neumann algebra) has the ADP if and only if its atomic projection (resp. the atomic projection of the algebra) are central. We also establish some geometric properties of JB^*-triples, and characterize JB^*-triples possessing the ADP and the Daugavet property.

1. INTRODUCTION

Given a real or complex Banach space X, we write X^* for the dual space and $L(X)$ for the Banach algebra of bounded linear operator on X.

We say that a Banach space X has the *Alternative Daugavet Property* (*ADP* in short), if the norm identity

(aDE) $$\max_{|\omega|=1} \|Id + \omega T\| = 1 + \|T\|$$

holds for all rank-one operators $T \in L(X)$. We will prove later that, in this case, actually all weakly compact operators on X satisfy (aDE) (see Theorem 2.2). It is clear that a Banach space X has the ADP whenever X^* has, but we shall show in this paper that the reverse result does not hold (see Remark 4.4).

The definition of the ADP is certainly related to the so called Daugavet property. A Banach space X has the *Daugavet property* [24] if every rank-one operator $T : X \to X$ satisfies the norm equality

(DE) $$\|Id + T\| = 1 + \|T\|,$$

which has become known as the Daugavet equation. In this case, all weakly compact operators on X also satisfy (DE) [24, Theorem 2.3]. Therefore, this definition of Daugavet property coincides with the one that appeared in [3]. It is a remarkable result due to I. Daugavet [10] that all compact operators on $C[0,1]$ satisfy (DE). Over the sixties, seventies and eighties, the validity of the Daugavet equation was proved for some classes of operators on various spaces by Y. Abramovich [1], C. Foias and I. Singer [15], J. Holub [19, 20], G. Lozanovskii [33], and others (see [2, 3, 43] for a detailed account of the subject). Let us state that all weakly compact operators

on $C(K)$ or $L_1(\mu)$ satisfy (DE) whenever K is a perfect compact space and μ is an atomless positive measure. In the nineties, new ideas were infused into the field by many papers (for instance, [2, 3, 23, 24, 38, 41, 44, 46]). The state-of-the-art information on the Daugavet Property can be found in the survey paper [45].

Observe that equation (aDE) for an operator T just means that there exists a modulus-one scalar ω such that ωT satisfies (DE). Therefore, the Daugavet property implies the ADP.

On the other hand, let us mention that equation (aDE) appears in several of the above cited papers, as [1, 2, 19, 20, 43]. In these papers it is proved that (aDE) is satisfied by all $T \in L(X)$ whenever $X = C(K)$ or $X = L_1(\mu)$. Actually, this result appeared in the 1970 paper [13, pp. 483], where the equation (aDE) is related to a constant introduced by G. Lumer in 1968, the numerical index of a Banach space. Let us give the necessary definitions. Given an operator $T \in L(X)$, the *numerical range* of T is the subset of the scalar field

$$V(T) = \{x^*(Tx) \; : \; x \in S_X, \; x^* \in S_{X^*}, \; x^*(x) = 1\}.$$

The *numerical radius* is the seminorm defined on $L(X)$ by

$$v(T) = \sup\{|\lambda| \; : \; \lambda \in V(T)\}$$

for each $T \in L(X)$. The *numerical index* of the space X is defined by

$$n(X) = \inf\{v(T) \; : \; T \in S_{L(X)}\},$$

or, equivalently, the greatest constant $k \geqslant 0$ such that $k\|T\| \leqslant v(T)$ for all $T \in L(X)$. We can find in the literature many examples of Banach spaces whose numerical indices have been computed. For instance, if H is a Hilbert space of dimension greater than 1, $n(H) = 0$ in the real case and $n(H) = 1/2$ in the complex case; the numerical index of a C^*-algebra A is equal to 1 or 1/2 depending on whether or not A is commutative; $n(X) = 1$ whenever $X = L_1(\mu)$ or $X^* = L_1(\mu)$ for any positive measure μ. For more information and background, we refer the interested reader to the monographs by F. Bonsall and J. Duncan [7, 8] and to the survey paper [35]. Recent results can be found in [14, 26, 32, 36, 37].

In [13], it was shown that, given $T \in L(X)$,

$$\max_{|\omega|=1} \|Id + \omega T\| = 1 + \|T\| \iff v(T) = \|T\|.$$

In particular, a Banach space X has numerical index 1 if and only if (aDE) is satisfied by all bounded operators on X. It is also true that

$$\|Id + T\| = 1 + \|T\| \iff \sup \operatorname{Re} V(T) = \|T\|.$$

Therefore, X has the Daugavet property if and only if $\sup \operatorname{Re} V(T) = \|T\|$ for all rank-one operators $T \in L(X)$. We shall prove these facts later in the paper (see Lemma 2.3).

We have shown that a Banach space X has the ADP if it has the Daugavet property or $n(X) = 1$. The reversed results are not true in general. For instance, $X = c_0 \oplus_1 C([0, 1], l_2)$ has the ADP but it does not have the Daugavet property, nor does it have numerical index 1 (see Example 3.2 for details). On the other hand, for spaces having the RNP and for Asplund spaces, the alternative Daugavet property and the numerical index 1 coincide (see Remark 2.4). No similar result can be expected for the Daugavet property. Indeed, by Corollary 3.3, every Banach space with the ADP can be renormed to still have the ADP, but to fail the Daugavet property.

The outline of the paper is as follows.

In Section 2 we give some geometric characterizations of the ADP in terms of the space and its successive duals, analogous to those given in [24, 45] for the Daugavet property. We use these characterizations to prove that all weakly compact operators on a space with the ADP satisfy (aDE). We then clarify the relationship between numerical ranges of operators and the equations (DE) and (aDE), and use this result to get new geometric characterizations of the ADP and the Daugavet property. Finally, some isomorphic implications of the ADP are established.

Section 3 is devoted to the study of the stability properties of the ADP. We show that the c_0-, l_1- or l_∞-sum of a family of Banach spaces has the ADP if and only if all the summands have. For spaces of vector-valued functions we have the following results. Let K be a compact Hausdorff space, let μ be a positive measure and let X be a Banach space. Then, $C(K, X)$ (resp. $L_1(\mu, X)$) has the ADP if and only if X has or K is perfect (resp. μ is atomless). If μ is σ-finite, then $L_\infty(\mu, X)$ has the ADP if and only if X has or μ is atomless. Also, we present examples showing that these results cannot be extended to arbitrary injective or projective tensor products. At the end of the section, we discuss the stability properties of the ADP for M-ideals.

Finally, Section 4 is devoted to the characterization of the C^*-algebras possessing the ADP. We will prove that a C^*-algebra has the ADP if and only if its atomic projections are central. Moreover, the predual of a von Neumann algebra A has the ADP if and only if A has it; in such a case, A can be written as the l_∞-sum of a non-atomic von Neumann algebra and a commutative von Neumann algebra (i.e., the l_∞-sum of a Banach space with the Daugavet property and a Banach space with numerical index 1). We also show that such decomposition is not possible for arbitrary C^*-algebras. To obtain these results, we actually work with the concept of a JB^*-triple, an algebraic structure which generalizes C^*-algebras and JB^*-algebras. The necessary definitions and basic results are presented in Section 4. We deduce the above results from a characterization of the JB^*-triples possessing the ADP. We also prove a characterization of the Daugavet property for JB^*-triples.

Throughout the paper, the symbols B_X and S_X denote, respectively, the closed unit ball and the unit sphere of a Banach space X. For a subset A of X, we write $\mathrm{co}(A)$ for the convex hull of A and $\overline{\mathrm{co}}(A)$ for the closed convex hull. The absolutely (resp. absolutely closed) convex hull of A is then $\mathrm{co}(\mathbb{T}A)$ (resp. $\overline{\mathrm{co}}(\mathbb{T}A)$), where $\mathbb{T}$ denotes the set of modulus-one scalars, that is, $\mathbb{T} = \{-1, 1\}$ for real spaces and $\mathbb{T} = \{\omega \in \mathbb{C} : |\omega| = 1\}$ for complex spaces. We use $\mathrm{ex}(B)$ to denote the set of extreme points of the convex set B. Finally, if $x^* \in X^*$, $x^{**} \in X^{**}$, we write $x^* \otimes x^{**}$ for the element of $L(X)^*$ given by $[x^* \otimes x^{**}](T) = x^{**}(T^*x^*)$ for all $T \in L(X)$.

2. Geometric characterizations and basic properties

Since the ADP is some kind of hybrid between the Daugavet property and the numerical index 1, it is natural that results on both topics can be carried to the ADP. This is the case, for instance, for some geometric characterizations of the Daugavet property given in [24, 45]. Some notation is required. A *slice* of B_X is a

set of the form
$$S(x^*, \alpha) = \{x \in B_X \; : \; \mathrm{Re}\, x^*(x) > 1 - \alpha\},$$
where $x^* \in S_{X^*}$ and $\alpha > 0$. If X is a dual space and x^* is taken from the predual, then $S(x^*, \alpha)$ is called a w^*-slice. For $x \in S_X$, we write
$$\Delta_\varepsilon(x) = \{y \in B_X \; : \; \|x - y\| \geqslant 2 - \varepsilon\}.$$

Proposition 2.1. *Let X be a Banach space. Then, the following are equivalent.*

(i) *X has the alternative Daugavet property.*

(ii) *For all $x_0 \in S_X$, $x_0^* \in S_{X^*}$ and $\varepsilon > 0$, there is some $x \in S_X$ such that $\mathrm{Re}\, x_0^*(x) \geqslant 1 - \varepsilon$ and $\max\limits_{|\omega|=1} \|x + \omega x_0\| \geqslant 2 - \varepsilon$.*

(ii*) *For all $x_0 \in S_X$, $x_0^* \in S_{X^*}$ and $\varepsilon > 0$, there is some $x^* \in S_{X^*}$ such that $\mathrm{Re}\, x^*(x_0) \geqslant 1 - \varepsilon$ and $\max\limits_{|\omega|=1} \|x^* + \omega x_0^*\| \geqslant 2 - \varepsilon$.*

(iii) *For any slice $S = S(x_0^*, \alpha_0)$ of B_X, $x_0 \in S_X$ and $\varepsilon > 0$, there exists a point $x \in S$ such that $\max\limits_{|\omega|=1} \|x + \omega x_0\| \geqslant 2 - \varepsilon$.*

(iii*) *For any w^*-slice $S^* = S(x_0, \alpha_0)$ of B_{X^*}, $x_0^* \in S_{X^*}$ and $\varepsilon > 0$, there is some $x^* \in S^*$ such that $\max\limits_{|\omega|=1} \|x^* + \omega x_0^*\| \geqslant 2 - \varepsilon$.*

(iv) *For any slice $S = S(x_0^*, \alpha_0)$ of B_X, $x_0 \in S_X$ and $\varepsilon > 0$, there exists a slice of B_X $S_1 \subseteq S$ such that $\max\limits_{|\omega|=1} \|x + \omega x_0\| \geqslant 2 - \varepsilon$ for all $x \in S_1$.*

(iv*) *For any w^*-slice $S^* = S(x_0, \alpha_0)$ of B_{X^*}, $x_0^* \in S_{X^*}$ and $\varepsilon > 0$, there exists a w^*-slice S_1^* of B_{X^*} contained in S^* such that $\max\limits_{|\omega|=1} \|x^* + \omega x_0^*\| \geqslant 2 - \varepsilon$ for all $x^* \in S_1^*$.*

(v) *$B_X = \overline{\mathrm{co}}\big(\mathbb{T}\Delta_\varepsilon(x)\big)$ for every $x \in S_X$ and every $\varepsilon > 0$.*

(v*) *$B_{X^*} = \overline{\mathrm{co}}^{w^*}\big(\mathbb{T}\Delta_\varepsilon(x^*)\big)$ for every $x^* \in S_{X^*}$ and every $\varepsilon > 0$.*

The proof is a straightforward adaptation of those given in [24, Lemma 2.1] and [45, Corollary 2.3], so we omit it. Using this result, we can prove an analogue of [24, Theorem 2.3].

Theorem 2.2. *If a Banach space X has the ADP, then (aDE) holds for all weakly compact operators.*

The proof is based on the one given in [24, Theorem 2.3] for the Daugavet property. We include it for the sake of completeness. Actually, the proof works equally well for strong Radon-Nikodým operators, that is, operators $T \in L(X)$ such that $\overline{T(B_X)}$ is a Radon-Nikodým set.

Proof. Let $T \in L(X)$ be weakly compact with $\|T\| = 1$. Then, the set $K = \overline{T(B_X)}$ is weakly compact and, therefore, it coincides with the closed convex hull of its denting points. Given $\varepsilon > 0$, take a denting point $y_0 \in K$ with $\|y_0\| > 1 - \varepsilon$. Then, for some $0 < \delta < \varepsilon$ there is a slice $S = \{y \in K \; : \; \mathrm{Re}\, y_0^*(y) \geqslant 1 - \delta\}$ of K containing y_0 and having diameter less than ε; here $y_0^* \in X^*$ and $\sup_{y \in K} y_0^*(y) = 1$. If we write $x_0^* = T^* y_0^*$, then $\|x_0^*\| = 1$ and
$$x \in B_X, \; \mathrm{Re}\, x_0^*(x) > 1 - \delta \quad \Longrightarrow \quad \|Tx - y_0\| < \varepsilon.$$
Now, we use (ii) in Proposition 2.1 to get $x \in S_X$ and $\omega \in \mathbb{T}$ such that
$$\mathrm{Re}\, x_0^*(x) > 1 - \delta \qquad \text{and} \qquad \|x + \omega y_0/\|y_0\|\| > 2 - \varepsilon.$$
Then $\|Tx - y_0\| < \varepsilon$ and $\|x + \omega y_0\| > 1 - 2\varepsilon$, so
$$\|Id + \omega T\| \geqslant \|x + \omega Tx\| \geqslant \|x + \omega y_0\| - \|\omega y_0 - \omega Tx\| > 2 - 3\varepsilon.$$
Letting $\varepsilon \downarrow 0$, we conclude that T satisfies (aDE). $\qquad\square$

On the other hand, some numerical range techniques can be used to study the ADP. We need two lemmas. The first one clarifies the relationship between numerical ranges and the equations (DE) and (aDE) cited in the introduction. Although it is essentially known, we include a proof for the sake of completeness. The second lemma is a new result on numerical radius of operators, which can be of independent interest.

Lemma 2.3. *Let X be a Banach space and $T \in L(X)$. Then:*

(a) *T satisfies (DE) if and only if $\sup \mathrm{Re}\, V(T) = \|T\|$.*
(b) *T satisfies (aDE) if and only if $v(T) = \|T\|$.*

Therefore, X has the Daugavet property (resp. the ADP) if and only if all rank-one operators $T \in L(X)$ satisfy $\sup \mathrm{Re}\, V(T) = \|T\|$ (resp. $v(T) = \|T\|$).

Proof. The result follows easily from the fact given in [6] and [34] (see [7, §9]) that

$$(1) \qquad \sup \mathrm{Re}\, V(T) = \lim_{\alpha \to 0^+} \frac{\|Id + \alpha T\| - 1}{\alpha}.$$

Indeed, if $\sup \mathrm{Re}\, V(T) = \|T\|$, since the function $\alpha \longmapsto \|Id + \alpha T\|$ is convex, the limit in (1) is an infimum, so

$$\|Id + T\| - 1 \geqslant \sup \mathrm{Re}\, V(T) = \|T\|$$

and T satisfies (DE). Conversely, if (DE) holds for T, then it also holds for αT for every $\alpha > 0$ (see [3, Lemma 2.1]). Then, (1) implies that $\sup \mathrm{Re}\, V(T) = \|T\|$. This gives us (a). To prove (b), we just use the facts that $v(T) = v(\omega T)$ for every $\omega \in \mathbb{T}$ and that $v(T) = \max_{|\omega|=1} \sup \mathrm{Re}\, V(\omega T)$. $\qquad \square$

Remark 2.4. By Theorem 2.2 and Lemma 2.3, the statements "X has the ADP" and "$n(X) = 1$" are equivalent if X is reflexive. In fact, by [32, Remark 6], these two statements are equivalent if X satisfies the RNP or if X is an Asplund space. In general, a space with the ADP need not have numerical index 1. For instance, $X = C([0, 1], H)$ (H is a Hilbert space of dimension greater than one) satisfies the Daugavet property (see [24]) and hence the ADP, but $n(X) = n(H) < 1$ by [36, Theorem 5].

Lemma 2.5. *Let X be a Banach space and let*

$$\mathcal{B} = \{x^* \otimes x^{**} \ : \ x^* \in \mathrm{ex}\,(B_{X^*}),\ x^{**} \in \mathrm{ex}\,(B_{X^{**}}),\ x^{**}(x^*) = 1\}.$$

Then, for every $T \in L(X)$, we have

(a) *$v(T) = \sup\{|x^{**}(T^*x^*)| \ : \ x^* \otimes x^{**} \in \mathcal{B}\}$,*
(b) *$\sup \mathrm{Re}\, V(T) = \sup\{\mathrm{Re}\, x^{**}(T^*x^*) \ : \ x^* \otimes x^{**} \in \mathcal{B}\}$.*

Proof. It is well known that

$$\overline{\mathrm{co}}\big(V(T)\big) = \{\varphi(T) \ : \ \varphi \in L(X)^*,\ \|\varphi\| = \varphi(Id) = 1\}$$

(see [7, Theorem 9.4], for example). By [29, Theorem 8], we have

$$\{\varphi \in L(X)^* \ : \ \|\varphi\| = \varphi(Id) = 1\} = \overline{\mathrm{co}}^{w^*}(\mathcal{B}),$$

so the result follows easily. $\qquad \square$

The first result on the ADP proved by using numerical range techniques is a new geometric characterization. We write $X \oplus_1 Y$ (resp. $X \oplus_\infty Y$) for the l_1-sum (resp. l_∞-sum) of the spaces X and Y.

Proposition 2.6. *Let X be a Banach space. Then X has the ADP if and only if $B_{X^* \oplus_\infty X^{**}} = \overline{\mathrm{co}}^{w^*}(\Gamma)$, where*

$$\Gamma = \left\{ (x^*, x^{**}) \ : \ x^* \in \mathrm{ex}\,(B_{X^*}),\, x^{**} \in \mathrm{ex}\,(B_{X^{**}}),\, |x^{**}(x^*)| = 1 \right\}.$$

Proof. Write $Y = X \oplus_1 X^*$, so $Y^* = X^* \oplus_\infty X^{**}$. The assumption of the proposition is clearly equivalent to $\|y_0\| = \sup\{|y^*(y_0)| \ : \ y^* \in \Gamma\}$ for all $y_0 \in Y$, that is,

$$(2) \qquad \|x_0\| + \|x_0^*\| = \sup\left\{|x^*(x_0) + x^{**}(x_0^*)| \ : \ (x^*, x^{**}) \in \Gamma\right\}$$

for all $x_0 \in X$, $x_0^* \in X^*$. Suppose first that X has the ADP. Given $x_0 \in X$ and $x_0^* \in X^*$, we consider the rank-one operator $T \in L(X)$, $Tx = x_0^*(x)x_0$ for all $x \in X$. By Lemma 2.3, $v(T) = \|T\| = \|x_0\|\,\|x_0^*\|$ so, for every $\varepsilon > 0$, Lemma 2.5.(a) gives us a pair $(x^*, x^{**}) \in \Gamma$ such that

$$(1 - \varepsilon)\|x_0\|\,\|x_0^*\| \leqslant |x^{**}(T^*x^*)| = |x^{**}(x_0^*)|\,|x^*(x_0)|.$$

By choosing suitable $\omega_1, \omega_2 \in \mathbb{T}$, we get

$$\mathrm{Re}\,\omega_1\,x^*(x_0) = |x^*(x_0)| \geqslant (1 - \varepsilon)\|x_0\|$$
$$\mathrm{Re}\,\omega_2\,x^{**}(x_0^*) = |x^{**}(x_0^*)| \geqslant (1 - \varepsilon)\|x_0^*\|.$$

Now, $(\omega_1 x^*, \omega_2 x^{**}) \in \Gamma$ and

$$\left|\omega_1 x^*(x_0) + \omega_2 x^{**}(x_0^*)\right| \geqslant (1 - \varepsilon)\left(\|x_0\| + \|x_0^*\|\right).$$

Conversely, take a rank-one operator $T \in L(X)$, which has the form

$$Tx = x_0^*(x)x_0 \qquad (x \in X),$$

where $x_0 \in X$ and $x_0^* \in X^*$. By using (2), for every $\varepsilon > 0$ we may find $(x^*, x^{**}) \in \Gamma$ such that

$$|x^*(x_0) + x^{**}(x_0^*)| \geqslant \left(\|x_0\| + \|x_0^*\|\right) - \varepsilon.$$

Therefore,

$$|x^*(x_0)| \geqslant \|x_0\| - \varepsilon \qquad \text{and} \qquad |x^{**}(x_0^*)| \geqslant \|x_0^*\| - \varepsilon,$$

and Lemma 2.5.(a) gives us that

$$v(T) \geqslant |x^{**}(T^*x^*)| = |x^{**}(x_0^*)|\,|x^*(x_0)| \geqslant (\|x_0^*\| - \varepsilon)(\|x_0\| - \varepsilon).$$

Letting $\varepsilon \downarrow 0$, we have $v(T) \geqslant \|x_0^*\|\,\|x_0\| = \|T\|$. Thanks to Lemma 2.3, this means that T satisfies (aDE). $\square$

The above argument can be adapted to get a new geometric characterization of the Daugavet property.

Proposition 2.7. *Let X be a Banach space. Then X has the Daugavet property if and only if $B_{X^* \oplus_\infty X^{**}} = \overline{\mathrm{co}}^{w^*}(\Upsilon)$, where*

$$\Upsilon = \left\{ (x^*, x^{**}) \ : \ x^* \in \mathrm{ex}\,(B_{X^*}),\, x^{**} \in \mathrm{ex}\,(B_{X^{**}}),\, x^{**}(x^*) = 1 \right\}.$$

Proof. The result follows by repeating the proof of Proposition 2.6 using part (b) of Lemma 2.5 and real parts instead of modulus. $\square$

Let us mention that we do not know of any characterization of Banach spaces with numerical index 1 in terms of the space and its successive duals, without using operators.

To finish the section, we show that not every Banach space can be renormed to have the ADP. In [32, Theorem 3], it is proved that if X is an infinite-dimensional real Banach space with $n(X) = 1$, then $X \supseteq l_1$ if X has the RNP and $X^* \supseteq l_1$ if X is an Asplund space. But spaces with RNP and Asplund spaces have numerical index 1 when they have the ADP. Inspecting the proofs, one realizes that the

condition $n(X) = 1$ was only used to show that $\|T\| = v(T)$ for rank 1 $T \in L(X)$, and the RNP (resp. Asplund assumption) is only needed to get infinitely many denting (resp. w^*-denting) points. This implies:

Remark 2.8. Let X be an infinite-dimensional real Banach space with the ADP.

    (a) If the set of denting points of B_X is infinite, then $X \supset c_0$ or $X \supset l_1$.
    (b) If the set of w^*-denting points of B_{X^*} is infinite, then $X^* \supset l_1$.

Consequently, any real Banach space X for which X^{**}/X is separable fails the ADP (by [12, page 219], X^* and X^{**} have the RNP whenever X^{**}/X is separable).

3. Stability properties

Our first goal in this section is to show that the ADP is stable by c_0-, l_1-, and l_∞-sums. Given an arbitrary family $\{X_\lambda : \lambda \in \Lambda\}$ of Banach spaces, we denote by $[\oplus_{\lambda \in \Lambda} X_\lambda]_{c_0}$ (resp. $[\oplus_{\lambda \in \Lambda} X_\lambda]_{l_1}$, $[\oplus_{\lambda \in \Lambda} X_\lambda]_{l_\infty}$) the c_0-sum (resp. l_1-sum, l_∞-sum) of the family. For infinite countable sums of copies of a space X we write $c_0(X)$, $l_1(X)$ or $l_\infty(X)$.

Proposition 3.1. *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of Banach spaces and let Z be the c_0-, l_1- or l_∞-sum of the family. Then Z has the ADP if and only if X_λ has the ADP for every $\lambda \in \Lambda$.*

Proof. We start by proving that X_λ has the ADP when Z has, and we first work with the c_0- or l_∞-sums. In both cases, we can write $Z = X_\lambda \oplus_\infty Y$ for suitable Y. Now, we fix a rank-one operator $S \in L(X_\lambda)$ with $\|S\| = 1$ and $0 < \varepsilon < 1$. Let $T \in L(Z)$ be the operator given by $T(x, y) = (Sx, 0)$ for all $(x, y) \in Z$. Then T is a rank-one operator with $\|T\| = 1$. The ADP of Z gives us $x \in B_{X_\lambda}$, $y \in B_Y$ and $\omega_1 \in \mathbb{T}$ such that

$$\max\{\|x + \omega_1 Sx\|, \|y\|\} = \|(x, y) + \omega_1 T(x, y)\| > 2 - \varepsilon.$$

But $\|y\| \leqslant 1 < 2 - \varepsilon$, so

$$\max_{|\omega|=1} \|Id + \omega S\| \geqslant \|x + \omega_1 Sx\| > 2 - \varepsilon.$$

Letting $\varepsilon \downarrow 0$, we get that S satisfies (aDE) and X_λ has the ADP. The argument for the l_1-sum is the same, using that T^* satisfies the (aDE).

The proof for the converse result can be easily adapted from the one given in [46, Theorem 1]. $\square$

We can now easily obtain examples of Banach spaces with the ADP which do not have the Daugavet property and whose numerical index is not 1.

Example 3.2. Let $X = c_0 \oplus_1 C([0, 1], l_2)$. Then X has the ADP since $n(c_0) = 1$ and $C([0, 1], l_2)$ has the Daugavet property. But, on one hand, $n(X) = n(l_2) < 1$ by [36, Proposition 1 and Theorem 5] and, on the other hand, X does not have the Daugavet property since c_0 does not have it.

Another consequence of Proposition 3.1 is that it is not possible to find an isomorphic property which ensures that the ADP and the Daugavet property are equivalent.

Corollary 3.3. *Let X be a Banach space with the ADP. Then there exists a Banach space Y isomorphic to X such that Y has the ADP but not the Daugavet property.*

Proof. If X fails the Daugavet property, we are done. So, suppose that X has the Daugavet property and take a one-dimensional subspace Z of X. Then $X = Z \oplus W$ for suitable W. Let $Y = Z \oplus_\infty W$, which is clearly isomorphic to X. Now $n(Z) = 1$ and W has the Daugavet property by [24, Theorem 2.14], so Y has the ADP by Proposition 3.1. But, since Y has a finite-dimensional M-summand, Y does not have the Daugavet property by [24, Proposition 2.10]. □

Proposition 3.1 also implies that the space $c_0(X)$ (respectively, $l_1(X)$ or $l_\infty(X)$) has the ADP if and only if X has the ADP. This result cannot be extended to arbitrary vector-valued functions spaces. Indeed, ℓ_2^2 does not have the ADP, since $n(\ell_2^2) < 1$. However, $C([0,1], \ell_2^2)$, $L_1([0,1], \ell_2^2)$, and $L_\infty([0,1], \ell_2^2)$ have the Daugavet property (see [24, 37]) and hence the ADP.

Let us recall some notation. Given a compact Hausdorff space K and a Banach space X, we write $C(K, X)$ for the Banach space of all continuous functions from K into X, endowed with the supremum norm. If (Ω, Σ, μ) is a positive measure space, $L_1(\mu, X)$ is the Banach space of all Bochner-integrable functions $f : \Omega \to X$ with

$$\|f\|_1 = \int_\Omega \|f(t)\| \, d\mu(t).$$

If μ is σ-finite, then $L_\infty(\mu, X)$ stands for the space of all essentially bounded Bochner-measurable functions f from Ω into X, endowed with its natural norm

$$\|f\|_\infty = \inf\{\lambda \geqslant 0 \ : \ \|f(t)\| \leqslant \lambda \text{ a.e.}\}.$$

We refer to [12] for background.

The following result describes vector-valued function spaces with the ADP in a manner similar to the description of function spaces with the Daugavet property in [36, Remarks 6 and 9] and [37, Theorem 5]. The proofs are straightforward adaptations of the ones given there, so we omit them.

Theorem 3.4. *Let X be a Banach space, K a compact Hausdorff space and μ a positive measure. Then:*

    (a) *$C(K, X)$ has the ADP if and only if K is perfect or X has the ADP.*
    (b) *$L_1(\mu, X)$ has the ADP if and only if μ is atomless or X has the ADP.*
    (c) *If μ is σ-finite, then $L_\infty(\mu, X)$ has the ADP if and only if μ is atomless or X has the ADP.*

Recall that $C(K, X) = C(K) \otimes_\varepsilon X$ and $L_1(\mu, X) = L_1(\mu) \otimes_\pi X$ where $\otimes_\varepsilon$ and $\otimes_\pi$ denote, respectively, the injective and projective tensor products. So, one may ask if (i) and (ii) in the above theorem might be special cases of a general result for tensor products. But [36, Example 10] shows that this is not the case. Indeed, let $X = l_1^4$ and $Y = l_\infty^4$, that is, the real four-dimensional l_1 and l_∞ spaces respectively. As $n(X \otimes_\varepsilon X) < 1$, $n(Y \otimes_\pi Y) < 1$, and $X \otimes_\varepsilon X$, $Y \otimes_\pi Y$ are finite-dimensional, neither $X \otimes_\varepsilon X$ nor $Y \otimes_\pi Y$ has the ADP, in spite of the fact that $X, Y, X \otimes_\pi X$, and $Y \otimes_\varepsilon Y$ have it.

Let us mention that in [23, §4] there are some negative results on the stability of the Daugavet property for injective or projective tensor products. The authors prove that there exists a two-dimensional complex space Y such that $L_1^{\mathbb{C}}[0,1] \otimes_\varepsilon Y$ and $L_\infty^{\mathbb{C}}[0,1] \otimes_\pi Y^*$ do not satisfy the Daugavet property, in contrast to the fact that the complex spaces $L_1^{\mathbb{C}}[0,1]$ and $L_\infty^{\mathbb{C}}[0,1]$ have it [23, Theorem 4.2 and Corollary 4.3].

We finish this section by proving some results on the stability of the ADP by M-ideals. Recall that a closed subspace J of a Banach space X is called an *M-ideal*

in X if $X^* = Y \oplus_1 J^\perp$ for some Banach space Y (here $J^\perp = \{x^* \in X^* : x^*|_J = 0\}$). In this case, $\{x^*|_J : x^* \in Y\}$ is isometric to J^*, and $X^* = J^* \oplus_1 J^\perp$ ([18], p. 11).

Proposition 3.5. *Suppose J is an M-ideal in a Banach space X.*

(a) *If both J and X/J have the (alternative) Daugavet property, then X has the (alternative) Daugavet property.*

(b) *If X has the (alternative) Daugavet property, then so does J.*

Proof. We shall consider only the alternative Daugavet property. The Daugavet part of the proposition was established in [24], via a different technique. Below, we shall denote by $\Gamma(Z)$ the set appearing in Proposition 2.6 for a Banach space Z, that is, the set of of all pairs (z^*, z^{**}) where $z^* \in \mathrm{ex}\,(B_{Z^*})$, $z^{**} \in \mathrm{ex}\,(B_{Z^{**}})$, and $|z^{**}(z^*)| = 1$.

(a) We can write $X^* = Y \oplus_1 J^\perp$, with Y isometric to J^*. Then, $\Gamma(X)$ is the collection of points of the form

$$(a_1^* \oplus_1 0) \oplus_\infty (a_1^{**} \oplus_\infty a_2^{**}) \qquad \text{and} \qquad (0 \oplus_1 a_2^*) \oplus_\infty (a_1^{**} \oplus_\infty a_2^{**}),$$

where a_1^*, a_2^*, a_1^{**}, and a_2^{**} are extreme points of B_Y, $B_{J^\perp}$, B_{Y^*}, and $B_{J^{\perp*}}$ respectively, for which $|a_1^{**}(a_1^*)| = |a_2^{**}(a_2^*)| = 1$. By Proposition 2.6, it suffices to show that, for $x_1^* \in S_Y$, $x_2^* \in S_{J^\perp}$, $x_1^{**} \in S_{Y^*}$, and $x_2^{**} \in S_{J^{\perp*}}$, $(x_1^* \oplus_1 0) \oplus_\infty (x_1^{**} \oplus_\infty x_2^{**})$ and $(0 \oplus_1 x_2^*) \oplus_\infty (x_1^{**} \oplus_\infty x_2^{**})$ belong to $\overline{\mathrm{co}}^{w^*}(\Gamma(X))$.

Since J has the ADP, Proposition 2.6 says that there exists a net $\{(b_\alpha^*, b_\alpha^{**})\}$ in the convex hull of $\Gamma(J)$, converging to (x_1^*, x_1^{**}) in the $\sigma(J^* \oplus_\infty J^{**}, J \oplus_1 J^*)$ topology. In other words, $\{b_\alpha^*\}$ converges to x_1^* in $\sigma(J^*, J)$, and $\{b_\alpha^{**}\}$ converges to x_1^{**} in $\sigma(J^{**}, J^*)$. By [18, Remark I.1.13], B_J is $\sigma(X, J^*)$ dense in B_X, hence the net $\{b_\alpha^*\}$ converges to x_1^* in $\sigma(X^*, X)$. Combining this with Krein-Milman Theorem, we see that $(x_1^* \oplus_1 0) \oplus_\infty (x_1^{**} \oplus_\infty x_2^{**})$ belongs to the $\sigma(X^* \oplus_\infty X^{**}, X \oplus_1 X^*)$ closure of the convex hull of $\Gamma(X)$.

The case of $(0 \oplus_1 x_2^*) \oplus_\infty (x_1^{**} \oplus_\infty x_2^{**})$ is dealt with in the same way, except that here we simply observe that, whenever a net $\{b_\alpha^*\}$ converges to x_2^* in $\sigma(J^\perp, X/J)$, then it also converges in $\sigma(J^\perp, X)$.

(b) Suppose $x^* \in J^*$, $x^{**} \in J^{**}$, and $\|x^*\| = \|x^{**}\| = 1$. Another application of Proposition 2.6 shows that there exists a net $\{x_\alpha^* \oplus_\infty x_\alpha^{**}\}$ in the convex hull of $\Gamma(X)$, converging to $x^* \oplus_\infty x^{**}$ in the weak* topology. We can write

$$x_\alpha^* \oplus_\infty x_\alpha^{**} = \sum_{i=1}^{M(\alpha)} c_{\alpha i}(y_{\alpha i}^* \oplus_1 0) \oplus_\infty (y_{\alpha i1}^{**} \oplus_\infty y_{\alpha i2}^{**}) +$$

$$+ \sum_{j=1}^{N(\alpha)} d_{\alpha j}(0 \oplus_1 z_{\alpha j}^*) \oplus_\infty (z_{\alpha j1}^{**} \oplus_\infty z_{\alpha j2}^{**}),$$

where $c_{\alpha i}, d_{\alpha j} \geqslant 0$, $\sum_i c_{\alpha i} + \sum_j d_{\alpha j} = 1$, $y_{\alpha i}^* \in J^*$, $z_{\alpha j}^* \in J^\perp$, $y_{\alpha i1}^{**}, z_{\alpha j1}^{**} \in J^{**}$, $y_{\alpha i2}^{**}, z_{\alpha j2}^{**} \in J^{\perp*}$, and $|y_{\alpha i1}^{**}(y_{\alpha i}^*)| = |z_{\alpha j2}^{**}(z_{\alpha j}^*)| = 1$. We identify $x^* \oplus_\infty x^{**}$ with $(x^* \oplus_1 0) \oplus_\infty (x^{**} \oplus_\infty 0)$. Then, the net $\left\{\sum_j d_{\alpha j} z_{\alpha j}^*\right\}$ converges to 0 in the $\sigma(J^\perp, X)$ topology. Since J is a subspace of X, the net $\{\sum_i c_{\alpha i} y_{\alpha i}^*\}$ converges to x^* in the $\sigma(J^*, J)$ topology. Since we assume that $\|x^*\| = 1$, $\lim_\alpha \sum_i c_{\alpha i} = 1$, and therefore, $\lim_\alpha \sum_j d_{\alpha j} = 0$. Thus, $\lim_\alpha \|\sum_j d_{\alpha j} z_{\alpha j1}^{**}\| = 0$, and the net $\sum_i c_{\alpha i} y_{\alpha i1}^{**}$ converges to x^{**} in the $\sigma(X^{**}, X^*)$ topology, hence also in the $\sigma(J^{**}, J^*)$ topology. By the above, the net $\sum_{i=1}^{M(\alpha)} c_{\alpha i} y_{\alpha i}^* \oplus_\infty y_{\alpha i1}^{**}$ converges in the $\sigma(J^* \oplus_\infty J^{**}, J \oplus_1 J^*)$ topology to $x^* \oplus x^{**}$. $\qquad \square$

Remark 3.6. A quotient of a space with the ADP by an M-ideal need not have the ADP. Indeed, consider the C^*-algebra $X = C([0, 1], M_2)$ (here $M_2 = L(l_2^2)$ is the space of 2×2 matrices). By Theorem 3.4, X has the ADP (in fact, X has the Daugavet property). $J = \{f \in X : f(1/2) = 0\}$ is a closed two-sided ideal in X, hence, by [18, Theorem V.4.4], it is an M-ideal in X. However, $M_2 = X/J$ fails the ADP. Indeed, M_2 is finite-dimensional so it has the ADP if and only if $n(M_2) = 1$, but this is not the case because it is a noncommutative C^*-algebra (see [26]).

4. C^*-ALGEBRAS AND JB^*-TRIPLES

The main goal of this section is to prove the following two results. A definition is needed: if A is a C^*-algebra, a non-zero projection $p \in A$ is called *atomic* (or *minimal*) if $pAp = p\,\mathbb{C}$.

Theorem 4.1. *A C^*-algebra has the alternative Daugavet property if and only if all of its atomic projections are central.*

Theorem 4.2. *The predual A_* of a von Neumann algebra A has the ADP if and only if the algebra A has.*

From these theorems, we are able to get two nice consequences.

Corollary 4.3. *Let A be a von Neumann algebra having the ADP. Then, there exists a commutative von Neumann algebra C and a non-atomic von Neumann algebra N such that $A = C \oplus_\infty N$. Moreover, $n(C) = 1$ and N has the Daugavet property.*

Proof. It is known (see e.g. Chapter 6 of [25]) that a von Neumann algebra A can be written as $A = B \oplus_\infty N$, where N is a non-atomic von Neumann algebra and B is the weak*-closure of its atomic projections. By [38, Theorem 2.1], N has the Daugavet property, so it has the ADP. Therefore, by Proposition 3.1, A has the ADP if and only if B has. But, thanks to Theorem 4.1 and B being the weak*-closure of its atomic projections, B has the ADP if and only if it is commutative. $\qquad\square$

We will prove later (Remark 4.9) that no such decomposition is possible for general C^*-algebras.

The second consequence is an example showing that the ADP does not pass from the space to the dual. The existence of such an example is known for the Daugavet property ($C[0, 1]$, see [24]) and it is an open problem for the numerical index (see [35]).

Example 4.4. Consider the (non-commutative) CAR C^*-algebra U (see e.g. Chapter III of [11] for the definition). Since U has no atomic projections, it has the Daugavet property (and hence the ADP) by [38, Theorem 2.1]. On the other hand, U^{**} is a non-commutative von Neumann algebra. By Theorems I.9.6 and I.9.8 of [11], U has a non-commutative faithful representation. Therefore, by Proposition III.6.36 of [42], U^{**} contains as an M-summand a non-commutative von Neumann algebra which is the weak* closed span of its atomic projections (see [16] for a generalization of this result). Thus, U^* and U^{**} fail the ADP by Proposition 3.1 and Theorems 4.1 and 4.2.

We will deduce the proofs of Theorems 4.1 and 4.2 from the corresponding ones for JB^*-triples. The so-called JB^*-triples are certain normed Jordan triple systems which have been studied because of their connection to bounded symmetric domains in Banach spaces and to C^*-algebras. JB^*-triples generalize Jordan C^*-algebras

and, therefore, also C^*-algebras. A brief introduction into JB^*-triples is given below. Interested readers are referred to [30, 31, 39, 40] for further information about this class of objects. Additional references will be given in the text.

A JB^*-*triple* is a complex Banach space U equipped with a triple product $(a, b, c) \longmapsto \{abc\}$ mapping $U \times U \times U$ into U and satisfying the five conditions below.

(1) The triple product $(a, b, c) \longmapsto \{abc\}$ is linear in a and c, and conjugate linear in b.

(2) The triple product is symmetric – that is, $\{abc\} = \{cba\}$.

(3) For any $x \in U$, the operator $D_x : U \longrightarrow U$, defined by $D_x u = \{xxu\}$, is Hermitian (that is, $\exp(itD_x)$ is an isometry for any $t \in \mathbb{R}$) with non-negative spectrum.

(4) The "main identity" is satisfied:
$$D_x\{abc\} = \{D_x a, b, c\} - \{a, D_x b, c\} + \{a, b, D_x c\}$$
for any $x, a, b, c \in U$.

(5) For any $x \in U$, $\|x\|^2 = \|D_x\| = \sup\{\lambda : \lambda \in \sigma(D_x)\}$. By [27], this condition is equivalent to

(5') For any $x \in U$, $\|\{xxx\}\| = \|x\|^3$.

It follows from [17] that $\|\{abc\}\| \leqslant \|a\| \|b\| \|c\|$ for any $a, b, c \in U$.

We say that a map ϕ between JB^*-triples U_1 and U_2 is a *triple isomorphism* if $\phi(\{abc\}) = \{\phi(a), \phi(b), \phi(c)\}$. By [22] and [27], ϕ is a surjective isometry iff it is a surjective triple isomorphism with trivial kernel. A (*triple*) *ideal* of a JB^*-triple U is a closed subspace V of U such that $\{VUU\} \subset V$ and $\{UVU\} \subset V$.

As we have already mentioned, C^* algebras are JB^*-triples. More generally, a closed subspace of a C^*-algebra A is called a JC^*-*triple* if it is closed under the triple product
$$\{abc\} = (ab^*c + cb^*a)/2.$$
It is easy to verify conditions (1) – (5). Also, JB^*-algebras become JB^*-triples with the product
$$\{abc\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*.$$
Numerous examples of JB^*-triples can be found in [40].

Suppose x is an element of a JB^*-triple U. In addition to the operator D_x, we define $Q_x : u \longmapsto \{xux\}$. An element $e \in U$ is called a *tripotent* if $\{eee\} = e$. For any tripotent e, we define *Peirce projections*:
$$P_2(e) = Q_e^2, \quad P_1(e) = 2(D_e - Q_e^2), \quad P_0(e) = Id - 2D_e + Q_e^2.$$
The range of $P_j(e)$ is denoted by $U_j(e)$ $(j = 0, 1, 2)$. By definition, $\sum_j P_j(e) = Id$, hence
$$U = \text{span}[U_2(e), U_1(e), U_0(e)] \qquad (\textit{Peirce decomposition}).$$
A tripotent e is called *minimal* if $U_2(e) = \mathbb{C}e$. e is called *diagonalizing* if $U_1(e) = 0$.

To clarify the above concepts, let us give an example. If U is a JC^*-triple, then $e \in U$ is a tripotent iff it is a partial isometry. In this case, let $d = e^*e$ and $r = ee^*$ be the domain and range projections of e. Then $P_2(e)x = rxd$, $P_1(e)x = rx(1 - d) + (1 - r)xd$, $P_0(e)x = (1 - r)x(1 - d)$ (see [40, §3.2]). If, more concretely, U is a C^*-algebra, then the tripotent e is minimal iff its range projection (or, equivalently, domain projection) is atomic. Indeed, suppose d is atomic. Then $dexd = \lambda d$ (with $\lambda \in \mathbb{C}$ depending on $x \in U$), and therefore,

$rxd = edexd = \lambda ed = \lambda e$. On the other hand, if e is a minimal tripotent, then $rxd = \lambda e$ ($\lambda \in \mathbb{C}$) for any $x \in U$, hence $dexd = e^* rxd = \lambda e^* e = \lambda d$.

We show that a JB^*-triple has the ADP iff every minimal tripotent in it is diagonalizing. Observe that, then, Theorem 4.1 follows as a corollary.

Theorem 4.5. *A JB^*-triple has the alternative Daugavet property if and only if all of its minimal tripotents are diagonalizing.*

If a JB^*-triple can be thought of as an analogue of a C^*-algebra, then a JBW^*-*triple* corresponds to a von Neumann algebra: it is a JB^*-triple which is a dual Banach space. As in the von Neumann algebra case, the predual of a JBW^*-triple is unique (see [5]). The following result implies Theorem 4.2 as a corollary.

Theorem 4.6. *The predual of a JBW^*-triple has the alternative Daugavet property if and only if the JBW^*-triple has.*

Results similar to Theorems 4.5 and 4.6 can also be proved about the Daugavet property. The C^*-versions are contained in [38, Theorem 2.1 and Corollary 2.3].

Theorem 4.7.   (a) *A JB^*-triple has the Daugavet property if and only if it has no minimal tripotents.*
  (b) *The predual to a JBW^*-triple U has the Daugavet property if and only if U has no minimal tripotents.*

The decomposition of a JBW^*-triple into a direct sum of its "atomic" and "non-atomic" parts is similar to the corresponding decomposition of a von Neumann algebra. Let U be a JBW^*-triple. By [17], U is triple isomorphic to a direct sum of a JBW^*-triple N, having no minimal tripotents, and a JBW^*-triple A which is the weak*-closed linear span of its minimal tripotents (A is atomic). Equivalently, $U = A \oplus_\infty N$. Moreover, A is isometric (hence triple isomorphic) to $\left[\bigoplus_{i \in I} C_i \right]_{l_\infty}$, where C_i are Cartan factors (see [17] and [21]). By the above theorem, N has the Daugavet property, so it has the ADP. Therefore, Proposition 3.1 says that U has the ADP if and only if all the Cartan factors C_i have. Suppose now that U has the ADP. For each $i \in I$, take a minimal tripotent e in C_i (Cartan factors always have minimal tripotents, see e.g. [17]). Then, by Theorem 4.5 e is diagonalizing, that is, $C_i = U_2(e) \oplus U_0(e)$ (the l_∞-sum of ideals). By definition of a factor, $U_0(e) = 0$ (see [22]), thus each C_i is 1-dimensional. This yields:

Corollary 4.8. *Let U be a JBW^*-triple with the alternative Daugavet property. Then, U is triple isomorphic to a direct sum of a JBW^*-triple N with no minimal tripotents, and a commutative von Neumann algebra C (viewed as a JC^*-triple). Equivalently, $U = C \oplus_\infty N$, where $n(C) = 1$ and N has the Daugavet property.*

Remark 4.9. No such decomposition exists for JB^*-triples which are not duals of Banach spaces. Even more, the decomposition is not possible even for general C^*-algebras. Indeed, consider the C^*-algebra $A = c_0 \oplus_\infty L_\infty([0,1], K)$ and its unitization A_1 (here K is the space of compact operators on l_2). c_0 is commutative, and $L_\infty([0,1], K)$ has the Daugavet property; so, by Proposition 3.1, A has the ADP. By [18, Theorem V.4.4], A is an M-ideal in A_1, hence, by Proposition 3.5, A_1 has the ADP (observe that $\dim(A_1/A) = 1$, so $n(A_1/A) = 1$).

Suppose, for the sake of contrdiction, that A_1 is triple isomorphic to a direct sum of triple ideals B_1 and B_2, where B_1 is a commutative C^*-algebra and B_2 has no minimal tripotents (i.e., it is "non-atomic"). Then B_1 and B_2 are complementary M-summands in A_1 (that is, $A_1 = B_1 \oplus_\infty B_2$). Indeed, if $b_i \in B_i$ ($i = 1, 2$), then

$\{b_1 b_2 A_1\} = 0$, and therefore, $(b_1 + b_2)^{3^n} = b_1^{3^n} + b_2^{3^n}$. This, in turn, implies that

$$\|b_1 + b_2\| \leqslant \lim_{n \to \infty} \left(\|b_1\|^{3^n} + \|b_2\|^{3^n} \right)^{1/3^n} = \max\{\|b_1\|, \|b_2\|\}.$$

To prove the inequality in the other direction, suppose $\|b_1\| \geqslant \|b_2\|$. By the above, $\|b_1 + b_2\|, \|b_1 - b_2\| \leqslant \|b_1\|$, and $b_1 = ((b_1 + b_2) + (b_1 - b_2))/2$, hence the inequalities above are in fact equalities.

By [18, Propositions I.1.11 and I.1.17], $A \cap B_1$ and $A \cap B_2$ are also M-ideals (hence, by [5], triple ideals) in both A_1 and A. Since $A \cap B_1$ and $A \cap B_2$ are a commutative C^*-algebra and a non-atomic JB^*-triple, respectively, we see that $A \cap B_1 = c_0$ and $A \cap B_2 = L_\infty([0,1], K)$. Moreover, $A_1^* = A^\perp \oplus_1 B = B_1^\perp \oplus_1 B_2^\perp$. Applying [18, Theorem I.1.10], we obtain that the one-dimensional space $A^\perp$ is a subspace of either $B_1^\perp$ or $B_2^\perp$. If $A^\perp \subset B_1^\perp$, then $B_1 \subset A$, hence $B_1 = c_0$, and therefore, $\mathbf{1} \in B_2$ (here $\mathbf{1}$ stands for the identity in A_1). However, then B_2 is not an ideal, which yields a contradiction. The possibility of $A^\perp \subset B_2$ is ruled out in the same way.

Remark 4.10. From the arguments preceding Corollary 4.8, we deduce that the unique Cartan factor which satisfies the ADP is $\mathbb{C}$.

It remains to prove Theorems 4.5, 4.6, and 4.7. In order to do that, we begin by recalling some facts about JB^*-triples, and proving a few lemmas. Suppose U is a JB^*-triple. If $e \in U$ is a tripotent, it is known that $P_k(e)P_j(e) = 0$ if $k \neq j$ (see e.g. Formulas JP3, JP23, and JP25 in [31]). By [39],

$$x \in U_j(e) \quad \Longleftrightarrow \quad D_e x = (j/2)x.$$

In [31] and [39], we find *multiplication rules* (also called *Peirce calculus*):

- $\{U_i(e)U_j(e)U_k(e)\} \subset U_{i-j+k}(e)$ if $i - j + k \in \{0, 1, 2\}$,
  $\{U_i(e)U_j(e)U_k(e)\} = 0$ otherwise.
- Hence, $U_j(e)$ $(j = 0, 1, 2)$ are JB^*-triples.
- $\{U_2(e)U_0(e)U\} = \{U_0(e)U_2(e)U\} = 0$.

Elements $a, b \in U$ are called *orthogonal* if $\{abU\} = 0$. By [31, Lemma 3.9] or [39], for tripotents e and f the following four statements are equivalent:

$$(1)\, \{efU\} = 0 \; \Leftrightarrow \; (2)\, \{feU\} = 0 \; \Leftrightarrow \; (3)\, \{eef\} = 0 \; \Leftrightarrow \; (4)\, \{eff\} = 0.$$

For $\lambda \in \mathbb{C}$, define the operator $S_\lambda(e) = \sum_{j=0}^{2} \lambda^j P_j(e)$. It was shown in [16], that $S_\lambda(e)$ is an isometry whenever $|\lambda| = 1$. As a consequence, we formulate a folklore lemma:

Lemma 4.11. *Suppose e is a tripotent in a JB^*-triple U.*

- (a) *The Peirce projections $P_0(e)$, $P_1(e)$, and $P_2(e)$ are contractive. Moreover, the projection $P_0(e) + P_2(e)$ is contractive.*
- (b) *$\|P_0(e) + \lambda P_1(e)\|$, $\|P_2(e) + \lambda P_1(e)\| \leqslant 1$ whenever $|\lambda| \leqslant 1/\sqrt{2}$.*

Proof. Part (a) follows directly from the fact that $S_\lambda(e)$ is an isometry. To prove (b), note that

$$\left\| P_0(e) + \frac{1+i}{2} P_1(e) \right\| = \frac{1}{2} \|Id + S_i(e)\| \leqslant 1.$$

But

$$\left(P_0(e) + \frac{1+i}{2} P_1(e) \right) S_\omega(e) = P_0(e) + \frac{(1+i)\omega}{2} P_1(e).$$

Since $\|S_\omega(e)\| = 1$ if $|\omega| = 1$, $P_0(e) + \lambda P_1(e)$ is contractive whenever $|\lambda| = 1/\sqrt{2}$. A simple convexity argument completes the proof of the lemma. $\square$

Lemma 4.12. *Suppose e is a minimal tripotent in a JB^*-triple U, and $x \in U_1(e)$. Then $\|e + x\| \geqslant 1 + \|x\|^2/4$ whenever $\|x\| \leqslant 1$.*

Proof. By [5], U^{**} is again a JB^*-triple (in fact, a JBW^*-triple), and the triple product is weak* continuous in each variable. Thus, $U_j(e)^{**}$ can be identified with $U_j^{**}(e) = P_j(e)U^{**}$ (the "multiplication" operators D_e and Q_e, defined on U^{**}, coincide with D_e^{**} and Q_e^{**}). In particular, if e is a minimal tripotent in U, it is also minimal in U^{**}.

By the above, we can assume that U is a JBW^*-triple, and $x \neq 0$. By [4], $x = \sum_{j \in J} c_j f_j$ (weak* convergence), where $\sup_j |c_j| = 1$ and the tripotents (f_j) are mutually orthogonal. By [9], the cardinality of J does not exceed 2. Moreover, one of the two cases takes place:

(1) $x = cf$, where $c \in \mathbb{C}$ and f is a minimal tripotent in $U_1(e)$, but not in U, and $e \in U_2(f)$.

(2) $x = c_1 f_1 + c_2 f_2$, where $|c_2| \leqslant |c_1| = \|x\|$, the mutually orthogonal tripotents f_1 and f_2 are minimal in U (and therefore, minimal in $U_1(e)$), and f_2 may be equal to 0 (if $f_2 = 0$, let $c_2 = 0$).

In the first case, by [9], U is (triple) isomorphic (and therefore, linearly isometric) to $J(e) \oplus_\infty U'$. Here $J(e)$ is the ideal in U generated by e, and U' is also an ideal. Moreover, $J(e)$ is triple isomorphic to the JC^*-algebra $S(H)$ – the space of symmetric matrices on a Hilbert space H, equipped with the triple product $\{abc\} = (ab^*c + cb^*a)/2$. Let $d = e^*e$ and $r = ee^*$ be the domain and range projections of e. Since $\{eex\} = x/2$, $x \in J(v)$. Write $x = x_1 + x_2$, where $x_1 = dx(1-r)$ and $x_2 = (1-d)xr$. Then $\|x\| = \max\{\|x_1\|, \|x_2\|\}$, and the desired estimate follows from simple matrix computations.

Now consider the second case. By [9], $e \in U_1(f_i)$ ($i = 1, 2$, $f_i \neq 0$). Since $U_1(f_i)$ is the 1/2-eigenspace of the operator D_{f_i}, $e = 2\{ef_if_i\}$. By the multiplication rules,

$$P_2(e)\big((e+x)^3\big) = e + 2\{exx\} = \left(1 + \sum_{i=1}^{2} |c_i|^2\right) e,$$

and therefore, $\|e + x\|^3 \geqslant \big\|P_2(e)\big((e+x)^3\big)\big\| \geqslant 1 + \|x\|^2$. $\qquad\square$

Lemma 4.13. *Suppose e is a minimal tripotent in a JB^*-triple U, and $x \in U_1(e)$ with $\|x\| = 1$. Then $\|\lambda e + x + x_0\| \geqslant 1 + |\lambda|^2/8$ whenever $x_0 \in U_0(e)$ and $|\lambda| \leqslant 1$.*

Proof. As in the previous lemma, we can assume that U is a JBW^* triple, and consider two cases. In the first case, $U = J(v) \oplus U'$, where $J(v)$ is triple isomorphic to $S(H)$. In this situation, simple matrix computations yield the result.

We concentrate on the second case – namely, $x = f_1 + c_2 f_2$, where f_1 and f_2 are mutually orthogonal minimal tripotents in U (f_2 may be zero), and $|c_2| \leqslant 1$. By [9], $e \in U_1(f_i)$ ($i = 1, 2$). Consider $x' = -S_{-i}(e)x = \lambda e + ix - x_0$. $P_j(f) = P_j(\lambda f)$ whenever $|\lambda| = 1$ and f is a tripotent, hence, by Lemma 4.11,

$$\|x\| = \|x'\| \geqslant \frac{1}{2}\left(\left\|f_1 + \frac{1}{\sqrt{2}}P_1(f_1)(\lambda e + x_0)\right\| + \left\|f_1 + \frac{1}{\sqrt{2}}P_1(f_1)(\lambda e - x_0)\right\|\right)$$

$$\geqslant \left\|f_1 + \frac{\lambda}{\sqrt{2}}P_1(f_1)\,e\right\| = \left\|f_1 + \frac{\lambda}{\sqrt{2}}\,e\right\|.$$

Using Lemma 4.12, we conclude that $\|f_1 + (\lambda/\sqrt{2})e\| \geqslant 1 + |\lambda|^2/8$. $\qquad\square$

To proceed, we need to introduce the *simultaneous Peirce decomposition* of a JB^*-triple U. Suppose $e_1, e_2, \ldots, e_n$ are mutually orthogonal tripotents in U. For $1 \leqslant i \leqslant n$, let $U_{ii} = U_2(e_i)$. If $1 \leqslant i, j \leqslant n$ and $i \neq j$, let $U_{ij} = U_1(e_i) \cap U_1(e_j)$ (then $U_{ij} = U_{ji}$). We set

$$U_{i0} = U_{0i} = U_1(e_i) \cap (\cap_{j \neq i} U_0(e_j)) \qquad \text{and} \qquad U_{00} = \cap_j U_0(e_j).$$

Then $U = \text{span}[U_{ij} : 0 \leqslant i, j \leqslant n]$. As with the "standard" Peirce decomposition, we have multiplication rules: $\{U_{ij} U_{jk} U_{k\ell}\} \subset U_{i\ell}$, and the triple product is zero otherwise (taking permutations of indices into account). For more information on the simultaneous Peirce decomposition, the reader is referred to Chapter 5 of [30], Chapter 3 of [31], or [39].

Proposition 4.14. *In the above notation, suppose $x_j \in U_{1j}$ for $j = 0, 2, 3, \ldots, n$. Then $\left\| \sum_j x_j \right\|^2 \leqslant 2 \sum_j \|x_j\|^2$.*

Proof. Let $y_0 = \sum_j x_j$, and $y_{n+1} = y_n^3$. We shall show that, for $n \in \mathbb{N}$, $y_n = \sum_{j \neq 1} y_{nj}$, where $y_{nj} \in U_{1j}$, and

$$(3) \qquad \|y_{nj}\| \leqslant \left(2 \sum_{\ell \neq 1} \|x_\ell\|^2 \right)^{(3^n - 1)/2} \|x_j\|.$$

Once the equation above is proved, we are done. Indeed, then

$$\|y_1\| = \|y_n\|^{1/3^n} \leqslant \left(\sum_j \|y_{nj}\| \right)^{1/3^n} \leqslant \left(2 \sum_{\ell \neq 1} \|x_\ell\|^2 \right)^{1/2} \left(\sum_j \|x_j\| \right)^{1/3^n}.$$

Passing to the limit as $n \to \infty$ completes the proof.

For $n = 0$, the equation (3) is obviously true. Suppose it holds for n, and prove it for $n + 1$. Note that $y_{n+1} = \sum_{j,k,\ell} \{y_{nj} y_{nk} y_{n\ell}\}$. By the multiplication rules, the triple product above is non-zero only in the following two situations: (1) $j = k$, $\{y_{nj} y_{nk} y_{n\ell}\} \in U_{1\ell}$; (2) $k = \ell$, $\{y_{nj} y_{nk} y_{n\ell}\} \in U_{1j}$. Thus,

$$y_{n+1,j} = 2 \sum_k \{y_{nj} y_{nk} y_{nk}\} = 2 \sum_k D_{y_{nk}} y_{nj},$$

and, by the induction hypothesis,

$$\|y_{n+1,j}\| \leqslant 2 \sum_{k \neq 1} \|y_{nk}\|^2 \|y_{nj}\|$$

$$\leqslant 2 \sum_{k \neq 1} \left(\left(2 \sum_{\ell \neq 1} \|x_\ell\|^2 \right)^{(3^n - 1)/2} \|x_k\| \right)^2 \cdot \left(2 \sum_{\ell \neq 1} \|x_\ell\|^2 \right)^{(3^n - 1)/2} \|x_j\|$$

$$= \left(2 \sum_{\ell \neq 1} \|x_\ell\|^2 \right)^{(3^{n+1} - 1)/2} \|x_j\|.$$

This establishes (3). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The next corollary follows from the proposition above by a simple duality argument (we keep the notation of the previous lemma).

Corollary 4.15. *Suppose $x^* \in U^*$ with $\|x^*\| \leqslant 1$, and $\varepsilon > 0$. Then $\|x^*|_{U_{ij}}\| \leqslant \varepsilon$ for $j \in \{1, \ldots, n\} \setminus J$, where the set J has a cardinality not exceeding $1 + 2/\varepsilon^2$.*

Lemma 4.16. *For every $\varepsilon \in (0, 1)$ there exists $N(\varepsilon) \in \mathbb{N}$ with the following property: if $N \geqslant N(\varepsilon)$ and $e_1, e_2, \ldots, e_N$ are mutually orthogonal tripotents in a JB^*-triple U and x^* is a norm one functional on U, then there exists $i \in \{1, 2, \ldots, N\}$ such that $\|x^*|_{V_i}\| \leqslant \varepsilon$, where $V_i = \text{span}[U_{ij} : 0 \leqslant j \leqslant N] = U_2(e_i) + U_1(e_i)$.*

Proof. Suppose $M = \lceil 2/\varepsilon \rceil + 1$ and $N > 16M^2 2^{2M}/\varepsilon^2$. For the sake of contradiction, suppose that for any i there exists $x_i \in V_i$ s.t. $\|x_i\| = 1$ and $\operatorname{Re} x^*(x_i) > \varepsilon$. We shall arrive at a contradiction. Write $x_i = \sum_{j=0}^N x_{ij}$, with $x_{ij} \in U_{ij}$. Denote by P_{ij} the "coordinate" projection onto U_{ij}. Let $\delta = \varepsilon/4$ and $x_{ij}^* = x^*|_{U_{ij}}$. Fix $i_1 \in \{1, \ldots, N\}$, and let $y_1^1 = x_{i_1}$. Let $J_{11} = \{j : \|x_{ij}^*\| \geqslant \delta\}$ (by the previous corollary, $|J_{11}| \leqslant 4/\delta^2$). Fix $i_2 \in \{1, \ldots, N\} \setminus J_{11}$, and let $y_2^2 = x_{i_2} - P_{i_2 i_1} x_{i_2}$, $y_1^2 = y_1^1 - P_{i_2 i_1} y_1^1$. Since, by [39], $\|P_{ij}\| \leqslant 1$ for any (i, j), and since $\|x_{i_1 i_2}^*\| < \varepsilon$, $\operatorname{Re} x^*(y_2^2), \operatorname{Re} x^*(y_1^2) \geqslant \varepsilon - \delta$. Moreover, $y_1^1 \in U_0(e_{i_2})$ and $P_{i_2 i_1} y_1^1 = x_{i_1 i_2} \in U_1(e_{i_2})$. Thus, $y_1^2 = P_0(e_{i_2}) y_1^1$, and $\|y_1^2\| \leqslant 1$. Similarly, $\|y_2^2\| \leqslant 1$.

Now let's fix $n \leqslant M$, and suppose we have already selected:

(1) a sequence $i_1, \ldots, i_n$, sets $J_{ki} = \{j : \|x_{i_k j}^*\| \geqslant \delta/2^{k-1}\}$ ($1 \leqslant k \leqslant n-1$, $1 \leqslant i \leqslant k$, $|J_{ki}| \leqslant 2^{2k}/\delta^2$), s.t. $i_{k+1} \notin \cup_{1 \leqslant i \leqslant j \leqslant k} J_{ji}$;

(2) $y_1^n, \ldots, y_n^n$ s.t. $\operatorname{Re} x^*(y_j^n) > \varepsilon - \sum_{k=1}^n \delta/2^{k-1}$, $\|y_j^n\| \leqslant 1$ for $1 \leqslant j \leqslant n$, and $P_{i_k i_j} y_j^n = 0$ whenever $j \neq k$.

We make the $(n+1)$-st step. Let $J_{ni} = \{j : \|x_{i_j}^*\| \geqslant \delta/2^n\}$ ($1 \leqslant i \leqslant n$). We know that $|J_{ni}| \leqslant 2^{2n}/\delta^2$. Thus, if $N > n^2 2^{2n}/\delta^2$, we can find

$$i_{n+1} \in \{1, \ldots, N\} \setminus \big(\cup_{1 \leqslant i \leqslant k \leqslant n} J_{ki} \big).$$

For $1 \leqslant j \leqslant n$, let $y_j^{n+1} = y_j^n - P_1(e_{i_{n+1}}) y_j^n = P_0(e_{i_{n+1}}) y_i^n$. By [39], $P_1(e_{i_{n+1}})$ acts on V_{i_j} as the "coordinate" projection onto $V_{i_j i_{n+1}}$ This implies that $P_{i_k i_j} y_j^{n+1} = 0$ for $k \in \{1, 2, \ldots, n+1\} \setminus \{j\}$. Since $P_0(e_{i_{n+1}})$ is contractive, $\|y_j^{n+1}\| \leqslant 1$. Moreover, $|x^*(P_1(e_{i_{n+1}}) y_i^n)| \leqslant \delta/2^n$, hence $\operatorname{Re} x^*(y_i^{n+1}) \geqslant \varepsilon - \sum_{k=1}^{n+1} \delta/2^{k-1}$.

y_{n+1}^{n+1} is defined as $x_{i_{n+1}} - \sum_{j=1}^n x_{i_{n+1} i_j}$ (hence $P_{i_j i_{n+1}} y_{n+1}^{n+1} = 0$ for $j \leqslant n$). Thus, $\operatorname{Re} x^*(x_{i_{n+1}}) \geqslant \varepsilon - n\delta/2^n \geqslant \varepsilon - \sum_{k=1}^{n+1} \delta/2^{k-1}$. Using [39], one can show that $y_{n+1}^{n+1} = P_0(e) x_{i_{n+1}}$, where $e = \sum_{j=1}^n e_{i_j}$. Hence $\|y_{n+1}^{n+1}\| \leqslant 1$.

Therefore, the conditions (1) and (2) above are satisfied with $n+1$ instead of n. Continuing in this fashion, we construct a sequence $y_1^M, \ldots, y_M^M$, satisfying (1) and (2) (with $n = M$). Let $z_1 = \sum y_j^M$. By definition of M, $\operatorname{Re} x^*(z) > 1$. To achieve a contradiction, we shall define $z_{n+1} = \{z_n z_n z_n\}$ ($n \in \mathbb{N}$), and show that $\|z_{n+1}\| = \|z\|^{3^n} \leqslant M$.

For $1 \leqslant j \leqslant M$ let

$$W_j = \sum_{k \in \{0, 1, \ldots, N\} \setminus \{i_1, \ldots, i_M\} \cup \{i_j\}} U_{jk}.$$

We claim that $z_n = \sum_{j=1}^M z_{nj}$, with $z_{nj} \in W_j$ and $\|z_{nj}\| \leqslant 1$. Indeed, by the multiplication rules of [39], $\{W_{j_1} W_{j_2} W_{j_3}\} \subset W_{j_1}$ if $j_1 = j_2 = j_3$, and 0 otherwise. To illustrate this, consider $\{U_{i_1 j_1} U_{i_2 j_2} U_{i_3 j_3}\}$, where $j_m \neq i_n$ if $m \neq n$. If the triple product is non-zero, then $j_1 = j_2$. In this case, either i_3 or j_3 must be equal to i_1, which is only possible if $j_3 = i_1$. Then the only non-zero products are $\{U_{i_1 j_1} U_{j_1 i_1} U_{i_1 j_3}\} \subset U_{i_1 j_3}$ and $\{U_{j_1 i_1} U_{i_1 j_2} U_{j_2 i_1}\} \subset U_{i_1 j_2}$.

Note that $z_1 = \sum_{j=1}^M z_{1j}$ with $z_{1j} = y_j^M$. Suppose $z_n = \sum_{j=1}^M z_{nj}$ for some $n \in \mathbb{N}$, with $z_{nj} \in W_j$ and $\|z_{nj}\| \leqslant 1$. By the reasoning above, $\{z_{nj} z_{nk} z_{n\ell}\} = 0$ unless $j = k = \ell$. Therefore, $z_{n+1} = \sum_{j=1}^M z_{n+1, j}$, with $z_{n+1, j} = z_{nj}^3 \in W_j$. Furthermore $\|z_{n+1, j}\| = \|z_{nj}\|^3 \leqslant 1$.

Therefore, $\|z_n\| \leqslant \sum_{j=1}^M \|z_{nj}\| \leqslant M$, and $\|z\| \leqslant \lim_n M^{1/3^n} = 1$. This yields a contradiction. $\square$

We can now proceed with the proofs of the main theorems of the section.

Proof of Theorem 4.5. Suppose first that a JB^*-triple U has a minimal non-diagonalizing tripotent e. Find $x^* \in U^*$ s.t. $\|x^*\| = 1$ and $P_1(e)^*x^* = x^*$. Consider an operator $T \in L(U)$, defined by $Tx = x^*(x)e$. We shall show that

(4) $$\|Id + T\| < 2.$$

Since the same inequality holds for ωT instead of T (that is, for ωx^* instead of x^*) whenever $|\omega| = 1$, equation (4) proves one direction of the theorem.

Suppose, for the sake of contradiction, that $\|Id + T\| = 2$. Then for any $\delta \in (0, 1/2)$ there exists $x \in U$ such that $\|x\| = 1$ and $\|x + x^*(x)e\| > 2 - \delta$. We can write $x = \lambda e + x_1 + x_0$, with x_1 and x_0 in $U_1(e)$ and $U_0(e)$, respectively. Note that $\|x_1\| \geqslant |x^*(x)| > 1 - \delta$. By Lemma 4.13,

$$1 + \frac{|\lambda|^2}{8\|x_1\|^2} \leqslant \left\| \frac{\lambda}{\|x_1\|}e + \frac{x_1}{\|x_1\|} + \frac{x_0}{\|x_1\|} \right\| \leqslant \frac{1}{\|x_1\|}.$$

Thus, $|\lambda|^2 \leqslant 8\|x_1\|(1 - \|x_1\|) < 8\delta$, and $\|(x_1 + x_0) + x^*(x_1 + x_0)\| > 2 - 3\sqrt{\delta} - \delta$.

Therefore, for any $\varepsilon > 0$ there exists $x = x_1 + x_0$ ($x_j \in U_j(e)$ for $j = 0, 1$) s.t. $\|x\| \leqslant 1$ and $\|x + x^*(x)e\| > 2 - \varepsilon$. By a simple extreme point argument, there exists ω with $|\omega| = 1$ s.t. $\|x + \omega e\| > 2 - \varepsilon$. Since ωe is a tripotent with the same Peirce projections as e, we can assume without loss of generality that $\omega = 1$.

By the triangle inequality,

$$\|e + tx\| \geqslant \|e + x\| - (1 - t)\|x\| > (2 - \varepsilon) - (1 - t) = 1 + t - \varepsilon$$

for any $t \in (0, 1)$. On the other hand, $e + tx = ((1 - t)e + tx_1) + t(e + x_0)$. By Lemma 4.11, $e + x_0 \leqslant 1$. By [16], $\|e + y\| \leqslant 1 + \|y\|/2$ whenever $y \in U_1(e)$ and $\|y\| \leqslant c$ (c is a universal constant). Thus,

$$\|(1 - t)e + tx_1\| = \left\| (1 - t)\left(e + \frac{t}{1 - t}x_1\right) \right\| \leqslant (1 - t)\left(1 + \frac{t}{2(1 - t)}\right) \leqslant 1 - \frac{t}{2}$$

if $t < c$. Putting it all together: $1 + t - \varepsilon < \|e + tx\| \leqslant 1 - t/2$ whenever $\varepsilon > 0$ and $t \in (0, c)$. This is the contradiction proving equation (4).

To prove the opposite implication of the theorem, we need to use the "odd functional calculus" developed in [4] and [28]. Suppose x is an element of a JB^*-triple U. We define odd powers of x: $x^1 = x$, $x^{2n+1} = \{xx^{2n-1}x\}$. By standard triple identities, $x^{\ell+m+n} = \{x^\ell x^m x^n\}$ if ℓ, m, and n are odd. Define the *spectrum* of x:

$$Sp(x) = \{\lambda \in \mathbb{C} : x \notin (Q_x - \lambda^2)U\}.$$

If $x \neq 0$, then $Sp(x)$ is a non-empty compact subset of $\mathbb{R}$, for which $-Sp(x) = Sp(x)$. Moreover, there exists a unique surjective triple isomorphism between $C^-(Sp(x))$ (the space of continuous odd functions on $Sp(x)$) and the smallest closed subtriple of U containing x (we denote it by U_x). This isomorphism extends naturally to a triple isomorphism from the set of all odd Borel functions on $Sp(x)$ to $U_x^{**} \subset U^{**}$.

Let $\Lambda(x) = Sp(x) \cap (0, \infty)$. Then there exists an orthogonal family $(e_\lambda)_{\lambda \in \Lambda(x)}$ of tripotents in U^{**} s.t. $x = \sum_\lambda \lambda e_\lambda$ (the sum converges in the weak* topology). This implies that $\|x\| = \max\{t : t \in \Lambda(x)\}$. If λ is an isolated point of $\Lambda(x)$, then $e_\lambda \in U$.

We also need to mention the following classical result: if a tripotent e in U is not minimal, then there exist non-trivial orthogonal tripotents $e_1, e_2 \in U^{**}$ s.t. $e = e_1 + e_2$. For the convenience of a reader, we outline the proof below.

Consider (following [39]) a *real* space $Z = \{x \in U_2(e) \,|\, Q_e x = x\}$. $Q_e^2 = Id$ on $U_2(e)$, hence $U_2(e) = Z \oplus iZ$. If $U_2(e) \neq \mathbb{C}e$, then there exists $z \in Z \backslash \mathbb{R}e$. Then $p(z) \in Z$ for any odd polynomial p. Moreover, by the weak* continuity of Q_e and by [28], $\{ef(z)e\} = f(z)$ for any odd Borel function f. Thus we produce orthogonal tripotents $u, v \in U_2^{**}(e) \backslash \mathbb{C}e$ s.t. $\{eue\} = u$ and $\{eve\} = v$. By (6.18) of [39], $e_1 = \{ueu\}$ and $e_2 = e - e_1$ are orthogonal tripotents. By the identity (J) of [39], $\{eve_1\} = 0$ and $\{uee_1\} = u$, hence $e_1 \notin \mathbb{C}e$.

To complete the proof of Theorem 4.5, it suffices to show that, for any JB^*-triple U without minimal non-diagonalizing tripotents,

$$\sup_{|\omega|=1} \|Id + \omega T\| = 1 + \|T\|$$

for any $T \in L(U)$ of rank 1 and norm 1 (see [3, Lemma 2.1]). Such an operator T is given by $Tx = x^*(x)a$, with $\|x^*\| = \|a\| = 1$, $a \in U$, and $x^* \in X^*$.

Suppose first that 1 is an isolated point of $\Lambda(a)$, and $U_2(e)$ is finite dimensional (e is the tripotent corresponding to the point 1). Then $e = e_1 + e_2$, where the tripotents e_1 and e_2 are orthogonal, and e_1 is minimal in $U_2(e)$. Then e_1 is also minimal in U. This follows, for instance, from the theory of joint Peirce decompositions, sketched above and described in greater detail in [39].

Since e_1 is diagonalizing, every $x \in U$ can be written as $x = \lambda e_1 + x_0$, with $\lambda \in C$ and $x_0 \in U_0(e_1)$. By an extreme point argument, for every $\varepsilon > 0$ there exists $x = \lambda e_1 + x_0$ with $|\lambda| = \|x_0\| = 1$ s.t. Re $x^*(x) > 1 - \varepsilon$. Then

$$\|\overline{\lambda} Id + T\| \geqslant \|\overline{\lambda} x + x^*(x)a\| \geqslant \|P_2(e_1)\left(\overline{\lambda} x + x^*(x)a\right)\|.$$

By definition, $P_2(e_1)x = \lambda e_1$. Moreover, $a = e_1 + e_2 + (a - e)$, and $e_2 + (a - e) \subset U_0(e_1)$. Therefore, $P_2(e_1)a = e_1$. This implies that $\|P_2(e_1)(\overline{\lambda} x + x^*(x)a)\| \geqslant 2 - \varepsilon$.

Now suppose that the above condition (1 is an isolated point of $\Lambda(a)$, and $U_2(e)$ is finite dimensional) is not satisfied. Fix $\varepsilon > 0$ and $N = N(\varepsilon) \in \mathbb{N}$ (in the notation of Lemma 4.16). If 1 is an isolated point of $\Lambda(a)$, then there exist N mutually orthogonal tripotents $e_1, e_2, \ldots, e_N \in U^{**}$ s.t. $e = \sum_i e_i$. If 1 is not an isolated point of $\Lambda(a)$, find a sequence $1 - \varepsilon < \lambda_1 < \lambda_2 < \ldots < \lambda_N < 1$, and let $e_i = e_{\lambda_i} \in U^{**}$ (these tripotents are mutually orthogonal). In either case, one easily sees that $\{e_i a e_i\} = c_i e_i$ for $1 \leqslant i \leqslant N$, with $c_i \in (1 - \varepsilon, 1)$.

By Lemma 4.16, there exists i for which $\|x^*|_{(U_1(e_i) + U_2(e_i))}\| < \varepsilon$. Fix $x^{**} \in U^{**}$ s.t. $\|x^{**}\| = x^{**}(x^*) = 1$. Let $\tilde{x}^{**} = P_0(e_i)x^{**} + e_i$. Then $|(x^{**} - \tilde{x}^{**})x^*| < 2\varepsilon$ and $\|\tilde{x}^{**}\| = 1$. Note that $Q_{e_i}a = c_i e_i$ and $Q_{e_i}\tilde{x}^{**} = e_i$. Since Q_{e_i} is a contraction,

$$\sup_{|\omega|=1} \|Id_{U^{**}} + \omega T^{**}\| \geqslant \sup_{|\omega|=1} \|Q_{e_i}(\tilde{x}^{**} + \omega x^*(\tilde{x}^{**})a)\|$$

$$= \sup_{|\omega|=1} |1 + \omega x^*(\tilde{x}^{**})c_i| \geqslant 2 - 3\varepsilon.$$

Since ε is arbitrarily small and $\|Id_{U^{**}} + \omega T^{**}\| = \|Id_U + \omega T\|$, we are done. $\square$

Proof of Theorem 4.6. If all minimal tripotents in a JBW^*-triple are diagonalizing, then the triple has the ADP, and so does its predual.

Conversely, suppose a JBW^*-triple U has a minimal tripotent e, and $U_1(e) \neq 0$. By [5], triple product in a JBW^*-triple is separately weak* to weak* continuous. Thus, Peirce projections $P_j(e)$ are weak* to weak* continuous, and $U_j(e)$ are JBW^*-triples ($j = 0, 1, 2$). We can find a norm 1 weak* continuous functional $x_* \in U_*$ s.t. $x_* = P_1(e)^*x_*$. Define a linear operator $T : x \longmapsto x(x_*)e$. In the proof of Theorem 4.5, we showed that $\sup_{|\omega|=1} \|Id + \omega T\| < 2$. Since T is weak* to weak* continuous, we are done. $\square$

Sketch of the proof of Theorem 4.7. (b) follows from (a) in the same manner as Theorem 4.6 follows from Theorem 4.5.

To prove that a JB^*-triple without minimal tripotents has the Daugavet property, we use essentially the same technique as in the proof of Theorem 4.5 (cf. [38]): suppose $T \in L(U)$ is an operator given by $Tx = ax^*(x)$, with $\|a\| = \|x^*\| = 1$. Fix $\varepsilon > 0$ and find mutually orthogonal tripotents $e_1, \ldots, e_N \in U^{**}$ s.t. $Q_{e_i}a = c_i e_i$ with $c_i \in (1 - \varepsilon, 1)$. Find $x^{**} \in U^{**}$ s.t. $\|x^{**}\| = x^{**}(x^*) = 1$. Let $\tilde{x}^{**} = P_0(e_i)x^{**} + e_i$, for a suitable i. Then $\|\tilde{x}^{**}\| = 1$ and $\operatorname{Re} \tilde{x}^{**}(x^*) > 1 - 2\varepsilon$. Therefore,

$$\|Id_U + T\| \geqslant \|(Id_{U^{**}} + T^{**})\tilde{x}^{**}\| \geqslant \|Q_{e_i}(\tilde{x}^{**} + \tilde{x}^{**}(x^*)e_i)\| > 2 - 3\varepsilon.$$

Now suppose e is a minimal tripotent in a JB^*-triple U. Define $T \in L(U)$ by setting $Tx = -P_2(e)x$ (if U is a JBW^*-triple, then, by [5], T is weak* to weak* continuous). We shall show that $\|Id + T\| < 2$. Indeed, suppose, for the sake of contradiction, that for every $\varepsilon > 0$ there exists $x \in U$ with $\|x\| = 1$ and $\|x - P_2(e)x\| > 2 - \varepsilon$. Write $x = \lambda e + x_1 + x_0$, with $x_j \in U_j(e)$ $(j = 0, 1)$ and $\lambda \in (1 - \varepsilon, 1)$. Then $\|x_1 + x_0\| > 2 - \varepsilon$. On the other hand, $\|x_0\|, \|x_1\| \leqslant \|x\| = 1$. By the triangle inequality,

$$\|x_0 + tx_1\| \geqslant \|x_0 + x_1\| - (1 - t)\|x_1\| \geqslant 1 - \varepsilon + t$$

for any $t \in (0, 1)$. On the other hand, by [16], $\|x_0 + tx_1\| < 1 + t/2$ whenever $t \in (0, c)$ (c is an absolute constant). This yields a contradiction. $\qquad \square$

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