ON CL-SPACES AND ALMOST-CL-SPACES

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Abstract. We find some necessary conditions for a real Banach space to be an almost-CL-space. We also discuss the stability of CL-spaces and almost-CL-spaces by c_0- and l_1-sums. Finally, we address the question if a space of vector-valued continuous functions can be a CL-space or an almost-CL-space.

1. Introduction

A real or complex Banach space is said to be a *CL-space* if its unit ball is the absolutely convex hull of every maximal convex subset of the unit sphere. If the unit ball is the closed absolutely convex hull of every maximal convex subset of the unit sphere, we say that the space is an *almost-CL-space*.

The concept of CL-space was introduced by R. Fullerton in 1960 [6]. Later on, Å. Lima [12] introduced the almost-CL-spaces, though the property specifying them had been previously used by J. Lindenstrauss [14]. Both definitions appeared only for real spaces, but they extend easily to the complex case. For general information on CL-spaces and almost-CL-spaces, including the connection with intersection properties of balls, we refer to [8, 9, 11, 13] and the already cited [6, 12, 14]. More recent results can be found in [19].

Examples of real CL-spaces are $L_1(\mu)$ for an arbitrary measure μ, and its isometric preduals, in particular $C(K)$, where K is a compact Hausdorff space (see [11, §3]). We do not know of any example of a real almost-CL-space which is not a CL-space.

For complex Banach spaces, the situation is different. CL-spaces and almost-CL-spaces have not received much attention in the complex case. The only result we are able to find in the literature is the characterization of finite-dimensional (real or complex) CL-spaces given by Å. Lima in [12, Corollary 3.7].

In section 2 we fix our notation and give some examples, mainly in the complex case. We show that the complex spaces $C(K)$ are always CL-spaces, whereas complex $L_1(\mu)$ are just almost-CL-spaces. Actually, l_1 and $L_1[0,1]$ are not CL-spaces.

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Section 3 contains some isomorphic results. We show that the dual of an infinite-dimensional real almost-CL-space always contains (an isomorphic copy of) l_1. If, in addition, the dual is separable, then the space contains c_0.

In section 4 we discuss the stability of the classes of CL-spaces and almost-CL-spaces by c_0- and l_1-sums. Our results are as expected, the sum is in one of the classes if and only if all the summands are, with one remarkable exception: an infinite l_1-sum of nonzero complex Banach spaces is never a CL-space.

Finally, in section 5 we deal with the space $C(K,X)$ of continuous functions from a compact Hausdorff space K into a Banach space X. We show that $C(K,X)$ is an almost-CL-space if and only if X is. The analog for CL-spaces remains open.

2. Notation and examples

Throughout the paper, X denotes a real or complex Banach space, B_X is its closed unit ball, S_X its unit sphere, and X^* the dual space. We will denote by $\mathbb{T}$ the unit sphere of the scalar field. Thus, $\mathbb{T}=\{-1,1\}$ when dealing with real spaces, while $\mathbb{T}=\{\lambda\in\mathbb{C}\ :\ |\lambda|=1\}$ in the complex case. Given a subset $A\subset X$, we write $\mathrm{ex}(A)$ for the set of extreme points in A and $\mathrm{co}(A)$ for the convex hull of A. Note that $\mathrm{co}(\mathbb{T}\,A)$ is the absolutely convex hull of A. The closed convex hull of A is denoted by $\overline{\mathrm{co}}(A)$. Finally, for a set $B\subset X^*$, we denote by $\overline{B}^{w^*}$ and $\overline{\mathrm{co}}^{w^*}(B)$ the w^*-closure and the w^*-closed convex hull of B.

It is worth pointing out basic facts on the definitions of CL-spaces and almost-CL-spaces. By using the Hahn-Banach and Krein-Milman theorems, one can easily prove that every maximal convex subset F of S_X has the form

$$F=\{x\in B_X\ :\ x^*(x)=1\}$$

for some $x^*\in \mathrm{ex}\,(B_{X^*})$. We denote by $\mathrm{mex}\,(B_{X^*})$ the set of those $x^*\in \mathrm{ex}\,(B_{X^*})$ with the property that the set $\{x\in B_X\ :\ x^*(x)=1\}$ is a maximal convex subset of S_X. It is easy to see that $\mathrm{mex}\,(B_{X^*})$ is a boundary for X, that is, for every $x\in X$, there is $x^*\in \mathrm{mex}\,(B_{X^*})$ such that $x^*(x)=\|x\|$. It follows that

$$B_{X^*}=\overline{\mathrm{co}}^{w^*}\big(\mathrm{mex}\,(B_{X^*})\big).$$

With the above facts in mind, X is a CL-space (resp. an almost-CL-space) if and only if, for every $x^*\in \mathrm{mex}\,(B_{X^*})$, B_X is the convex hull (resp. closed convex hull) of the set $\{x\in B_X\ :\ |x^*(x)|=1\}$. Let us also mention that maximal convex subsets of S_X are nothing but maximal faces of B_X. This accounts for the relation of CL-spaces and almost-CL-spaces to facial structure of balls. More information on this topic may be found in [1, 11].

We shall now comment on some examples of CL-spaces and almost-CL-spaces. Real CL-spaces are related to an intersection property of balls, namely the *3.2-Intersection property* (3.2.I.P.). A Banach space is said to have this property if every set of three mutually intersecting balls has non-empty intersection. The 3.2.I.P. was first investigated by O. Hanner [8] and systematically studied by J. Lindenstrauss [14] and Å. Lima [11]. More references on the 3.2.I.P. are [9,

12, 13]. In 1977, Å. Lima showed that every real Banach space with the 3.2.I.P. is a CL-space [11, Corollary 3.6], but the converse is false even in the finite-dimensional case (see [8, Remark 3.6] or [19, §3]). The real space $L_1(\mu)$ and its isometric preduals satisfy the 3.2.I.P. Therefore, they are CL-spaces.

To end up with the real examples, let us pose an open question. We do not know if there exists a real almost-CL-space which is not a CL-space.

The situation in the complex case is quite different. Not much attention has been paid to complex CL-spaces and almost-CL-spaces in the literature, so we provide the basic examples. As a matter of fact, there are complex almost-CL-spaces which are not CL-spaces.

Proposition 1. (*i*) *For every compact Hausdorff space K, the complex space $C(K)$ is a CL-space.*
(*ii*) *For every finite measure μ, the complex space $L_1(\mu)$ is an almost-CL-space.*
(*iii*) *The complex spaces l_1 and $L_1[0,1]$ are not CL-spaces.*

We need a lemma on the field $\mathbb{C}$ whose proof is straightforward. We write $\mathbb{D} = \{\lambda \in \mathbb{C} \ : \ |\lambda| \leqslant 1\}$.

Lemma 2. *For every $z_0 \in \mathbb{D}$, there exist two continuous functions $\varphi, \psi : \mathbb{D} \longrightarrow \mathbb{D}$ satisfying $|\varphi(z_0)| = |\psi(z_0)| = 1$ and $z = \frac{1}{2}\big(\varphi(z) + \psi(z)\big)$ for every $z \in \mathbb{D}$.*

Proof of Proposition 1. (*i*). By the well-known characterization of extreme points in $B_{C(K)^*}$, it suffices to show, for each $t_0 \in K$, that $B_{C(K)}$ is the convex hull of the set $\{f \in B_{C(K)} \ : \ |f(t_0)| = 1\}$. Indeed, given $t_0 \in K$ and $f \in B_{C(K)}$, we take $z_0 = f(t_0)$ and find functions φ, ψ as in the above lemma. Then $f = \frac{1}{2}(f_1 + f_2)$ where $f_1 = \varphi \circ f$ and $f_2 = \psi \circ f$ satisfy that $|f_1(t_0)| = |f_2(t_0)| = 1$.

(*ii*). Let F be a maximal convex subset of $S_{L_1(\mu)}$. Up to an isometric isomorphism, we can suppose that

$$F = \{f \in L_1(\mu) \ : \ \|f\| = 1, \ f \geqslant 0 \text{ a.e.}\}.$$

But the absolutely convex hull of F contains a dense subset of the unit ball, namely the set of simple functions, so $B_{L_1(\mu)} = \overline{\text{co}}(\mathbb{T}\, F)$ as desired.

(*iii*). It is clear that $\text{ex}\big(B_{L_1[0,1]^*}\big) = \text{mex}\big(B_{L_1[0,1]^*}\big)$, so the set

$$F = \{f \in L_1[0,1] \ : \ \|f\| = 1, \ f \geqslant 0 \text{ a.e.}\}$$

is a maximal convex subset of $S_{L_1[0,1]}$. Then, it suffices to find a function $f \in B_{L_1[0,1]}$ which cannot be obtained as an absolutely convex combination of elements in F. Indeed, let $f(t) = e^{2\pi i t}$ for every $t \in [0,1]$, and suppose that we could write $f = \sum_{j=1}^{n} a_j \lambda_j f_j$ a.e. in $[0,1]$, where $a_j > 0$, $|\lambda_j| = 1$, $f_j \in F$ for $j = 1, 2, \ldots, n$,

and $\sum_{j=1}^n a_j = 1$. Then, from

$$1 = \int_0^1 f(t)e^{-2\pi it}\,dt = \sum_{j=1}^n a_j \int_0^1 \lambda_j\, e^{-2\pi it} f_j(t)\,dt$$
$$\leqslant \sum_{j=1}^n a_j \int_0^1 f_j(t)\,dt = \sum_{j=1}^n a_j = 1,$$

we would get $\lambda_1 e^{-2\pi it} f_1(t) = f_1(t)$ for almost every $t \in [0,1]$, which is clearly impossible. The case of l_1 can be treated similarly and it will become a particular case of Corollary 10. □

3. Some isomorphic results

Our aim here is to obtain some isomorphic properties of real almost-CL-spaces. We start with an easy but useful lemma.

Lemma 3. *Let X be an almost-CL-space. Then $|x^{**}(x^*)| = 1$ for every $x^{**} \in \operatorname{ex}(B_{X^{**}})$ and every $x^* \in \operatorname{mex}(B_{X^*})$.*

Proof. If $x^* \in \operatorname{mex}(B_{X^*})$, we have $B_X = \overline{\operatorname{co}}\big(\{x \in B_X \ : \ |x^*(x)| = 1\}\big)$ and Goldstine's Theorem gives

$$B_{X^{**}} = \overline{\operatorname{co}}^{w^*}\big(\{x \in B_X \ : \ |x^*(x)| = 1\}\big).$$

Now, by the 'reversed' Krein-Milman Theorem we get

$$\operatorname{ex}(B_{X^{**}}) \subseteq \overline{\{x \in B_X \ : \ |x^*(x)| = 1\}}^{w^*},$$

which clearly implies $|x^{**}(x^*)| = 1$ for every $x^{**} \in \operatorname{ex}(B_{X^{**}})$. □

Remark 4. In the real case, the converse of the above lemma holds. This follows from [12, Theorems 3.1 and 3.4].

A useful sufficient condition for a real Banach space X to contain (a subspace isomorphic to) c_0 or l_1 was found in [16], namely the existence of an infinite set $A \subset S_X$ where all extreme points in B_{X^*} can only take the values ± 1. This fact combined with Lemma 3 yields the main result in this section.

Theorem 5. *Let X be an infinite-dimensional real almost-CL-space. Then X^* contains l_1. If in addition X^* is separable, then X contains c_0.*

Proof. Being a boundary for the infinite-dimensional space X, the set $\operatorname{mex}(B_{X^*})$ must be infinite. Then, we may apply Lemma 3 and [16, Proposition 2] to get that X^* contains either c_0 or l_1. But a dual space contains l_∞ (hence l_1) as soon as it contains c_0 (see [15, Proposition 2.e.8], for example).

To prove the second part of the proposition, observe that Lemma 3 implies that $\|x^* - y^*\| = 2$ for distinct $x^*, y^* \in \operatorname{mex}(B_{X^*})$. Therefore, if X^* is separable, $\operatorname{mex}(B_{X^*})$ has to be countable. But a real infinite-dimensional space

which admits a countable boundary contains c_0 by a result due to V. Fonf [5, Remark 2]. □

Remark 6. The above result improves those given in [16] for real Banach spaces with numerical index 1. A Banach space has *numerical index 1* if every bounded linear operator $T : X \longrightarrow X$ satisfies

$$\|T\| = \sup\{|x^*(Tx)| \ : \ x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1\}.$$

For more references and background we refer the reader to [2, 3, 4, 17, 18]. Almost-CL-spaces have numerical index 1 [17, §4], and it is proved in [16] that the dual of an infinite-dimensional real Asplund space with numerical index 1 contains l_1.

The behaviour of CL-spaces or almost-CL-spaces under duality is unclear to us. It was shown by Å. Lima [12, Corollary 3.6] that a real Banach space X is an almost-CL-space provided that X^* is a CL-space. We are not aware of any relevant result in the opposite direction. Next we get such a result, under an additional assumption.

Proposition 7. *Let X be a Banach space not containing l_1. If X is an almost-CL-space, then X^* is an almost-CL-space.*

Proof. For $x^{**} \in \operatorname{mex}(B_{X^{**}})$, we are to show that $B_{X^*} = \overline{\operatorname{co}}(H)$, where $H = \{x^* \in B_{X^*} \ : \ |x^{**}(x^*)| = 1\}$. We have $\operatorname{mex}(B_{X^*}) \subseteq H$ by Lemma 3. Moreover, since $\operatorname{mex}(B_{X^*})$ is a boundary for a space X not containing l_1, we can use [7, Theorem III.1] to get that $B_{X^*} = \overline{\operatorname{co}}\big(\operatorname{mex}(B_{X^*})\big)$, and we are done. □

4. Sums of CL-spaces and almost-CL-spaces

This section is devoted to study the stability of the classes of CL-spaces and almost-CL-spaces by c_0- and l_1-sums. Given an arbitrary family $\{X_\lambda \ : \ \lambda \in \Lambda\}$ of Banach spaces, we denote by $[\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}$ (resp. $[\oplus_{\lambda\in\Lambda} X_\lambda]_{l_1}$) the c_0-sum (resp. l_1-sum) of the family. In case Λ has just two elements, we use the simpler notation $X \oplus_\infty Y$ or $X \oplus_1 Y$.

With respect to c_0-sums, we have the following.

Proposition 8. *Let $\{X_\lambda \ : \ \lambda \in \Lambda\}$ be a family of Banach spaces and let $X = [\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}$. Then, X is a CL-space (resp. an almost-CL-space) if and only if X_λ is a CL-space (resp. an almost-CL-space) for every $\lambda \in \Lambda$.*

Proof. For each $\lambda \in \Lambda$, we write I_λ for the injection of X_λ into X, and P_λ will be the projection of X onto X_λ. Under the natural identification $X^* = [\oplus_{\lambda\in\Lambda} X_\lambda^*]_{l_1}$, P_λ^* becomes the injection of X_λ^* into X^*. Using the well-known fact that

$$\operatorname{ex}(B_{X^*}) = \bigcup_{\lambda\in\Lambda} P_\lambda^*\big(\operatorname{ex}\left(B_{X_\lambda^*}\right)\big),$$

one can easily check that a convex subset F of S_X is maximal if and only if there exist $\lambda \in \Lambda$ and a maximal convex subset F_λ of S_{X_λ} such that

$$F_\lambda = P_\lambda(F) \qquad \text{and} \qquad F = P_\lambda^{-1}(F_\lambda) \cap B_X. \tag{1}$$

Suppose first that every X_λ is an almost-CL-space, and fix a maximal convex subset F of S_X. Then, we can find $\lambda \in \Lambda$ and a maximal convex subset F_λ of S_{X_λ} satisfying (1), so $B_{X_\lambda} = \overline{\text{co}}(\mathbb{T}F_\lambda)$. Now, given $x \in B_X$ and $\varepsilon > 0$, take $y_\lambda \in \text{co}(\mathbb{T}F_\lambda)$ such that $\|y_\lambda - P_\lambda(x)\| < \varepsilon$, and consider $y = x - I_\lambda(P_\lambda(x)) + I_\lambda(y_\lambda)$. It is clear that $y \in B_X$ and $P_\lambda(y) = y_\lambda$, so $y \in \text{co}(\mathbb{T}F)$. Moreover, $\|y - x\| < \varepsilon$, so we have shown that $B_X = \overline{\text{co}}(\mathbb{T}F)$ and X is an almost-CL-space. If every X_λ is a CL-space, then we can repeat the above argument with $\varepsilon = 0$ to get that X is a CL-space.

Conversely, suppose that X is an almost-CL-space. Fix $\lambda \in \Lambda$ and let F_λ be a maximal convex subset of S_{X_λ}. Then, the set F given by (1) is a maximal convex subset of S_X, so $B_X = \overline{\text{co}}(\mathbb{T}F)$. Now, given $x_\lambda \in B_{X_\lambda}$ and $\varepsilon > 0$, there exists $y \in \text{co}(\mathbb{T}F)$ such that $\|y - I_\lambda(x_\lambda)\| < \varepsilon$. Then

$$P_\lambda(y) \in \text{co}(\mathbb{T}F_\lambda) \qquad \text{and} \qquad \|P_\lambda(y) - x_\lambda\| < \varepsilon,$$

so $B_{X_\lambda} = \overline{\text{co}}(\mathbb{T}F_\lambda)$ and X_λ is an almost-CL-space. The same argument with $\varepsilon = 0$ gives that X_λ is a CL-space if X is. □

The examples in Proposition 1 tell us that the situation for l_1-sums cannot be so tidy. Actually, we have:

Proposition 9. *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of nonzero Banach spaces and let $X = [\oplus_{\lambda\in\Lambda} X_\lambda]_{l_1}$. Then:*

- *(i) X is an almost-CL-space if and only if every X_λ is.*
- *(ii) In the real case, X is a CL-space if and only if every X_λ is.*
- *(iii) In the complex case, X is a CL-space if and only if every X_λ is, and the set Λ is finite.*

Proof. Once again we write I_λ and P_λ for the natural injections and projections. Now, $X^* = [\oplus_{\lambda\in\Lambda} X_\lambda^*]_{l_\infty}$, I_λ^* is the natural projection of X^* onto X_λ^*, and the relationship between extreme points in B_{X^*} and $B_{X_\lambda^*}$ is the following:

$$x^* \in \text{ex}\,(B_{X^*}) \iff I_\lambda^*(x^*) \in \text{ex}\left(B_{X_\lambda^*}\right) \ \ \forall\, \lambda \in \Lambda.$$

With this in mind, it is easy to check that maximal convex subsets of S_X are exactly the sets of the form

$$F = \{x \in S_X \ : \ P_\lambda(x) \in \|P_\lambda(x)\| F_\lambda \ \ \forall \lambda \in \Lambda\}, \tag{2}$$

where F_λ is a maximal convex subset of S_{X_λ} for all $\lambda \in \Lambda$.

(i). Suppose first that X is an almost-CL-space and, for fixed $\mu \in \Lambda$, let F_μ be a maximal convex subset of S_{X_μ}. For every $\lambda \neq \mu$ we choose an arbitrary

maximal convex subset F_λ of S_{X_λ} and consider the maximal convex subset F of S_X given by (2). Note that

$$P_\mu(x) \in \|P_\mu(x)\|F_\mu \subset \mathrm{co}(\mathbb{T}F_\mu)$$

for all $x \in F$, so

$$P_\mu\big(\mathrm{co}(\mathbb{T}F)\big) \subseteq \mathrm{co}(\mathbb{T}F_\mu).$$

Now, using that $B_X = \overline{\mathrm{co}}(\mathbb{T}F)$, we have

$$B_{X_\mu} = P_\mu(B_X) = P_\mu\big(\overline{\mathrm{co}}(\mathbb{T}F)\big) \subseteq \overline{P_\mu(\mathrm{co}(\mathbb{T}F))} \ \subseteq \overline{\mathrm{co}}(\mathbb{T}F_\mu),$$

and we have shown that X_μ is an almost-CL-space. Note that if X is a CL-space, then

$$B_{X_\mu} = P_\mu\big(\mathrm{co}(\mathbb{T}F)\big) \subseteq \mathrm{co}(\mathbb{T}F_\mu)$$

and X_μ is a CL-space.

Conversely, suppose that every X_λ is an almost-CL-space, and let F be a maximal convex subset of S_X. Then F has the form given in (2) where F_λ is a maximal convex subset of S_{X_λ} for each $\lambda \in \Lambda$. Since $B_{X_\lambda} = \overline{\mathrm{co}}(\mathbb{T}F_\lambda)$ and $I_\lambda(F_\lambda) \subseteq F$ for every $\lambda \in \Lambda$, we have

$$\bigcup_{\lambda\in\Lambda} I_\lambda\big(B_{X_\lambda}\big) \subseteq \bigcup_{\lambda\in\Lambda} \overline{\mathrm{co}}\big(\mathbb{T}I_\lambda(F_\lambda)\big) \subseteq \overline{\mathrm{co}}(\mathbb{T}F).$$

Now, just recall that B_X is the closed convex hull of $\bigcup_{\lambda\in\Lambda} I_\lambda(B_{X_\lambda})$, so $B_X \subseteq \overline{\mathrm{co}}(\mathbb{T}F)$ and X is an almost-CL-space. Observe that, if Λ is finite, then $B_X = \mathrm{co}\left(\bigcup_{\lambda\in\Lambda} I_\lambda(B_{X_\lambda})\right)$ and the above argument allows to obtain $B_X = \mathrm{co}(\mathbb{T}F)$ from $B_{X_\lambda} = \mathrm{co}(\mathbb{T}F_\lambda)$ for all λ. Therefore, finite l_1-sums of CL-spaces are CL-spaces.

(*ii*). One implication has already been proved. For the converse, suppose that each X_λ is a real CL-space, let F be a maximal convex subset of S_X and write F as in (2). Fix $x \in S_X$ and use that $B_{X_\lambda} = \mathrm{co}(F_\lambda \cup -F_\lambda)$ to write

$$P_\lambda(x) = \|P_\lambda(x)\|\big(t_\lambda y_\lambda - (1-t_\lambda)z_\lambda\big) \qquad (\lambda \in \Lambda)$$

where $y_\lambda,\, z_\lambda \in F_\lambda$ and $0 \leqslant t_\lambda \leqslant 1$ for all λ. It follows that

$$x = \sum_{\lambda\in\Lambda} I_\lambda(P_\lambda(x)) = y - z$$

where

$$y = \sum_{\lambda\in\Lambda} \|P_\lambda(x)\| t_\lambda I_\lambda(y_\lambda) \quad \text{and} \quad z = \sum_{\lambda\in\Lambda} \|P_\lambda(x)\|(1-t_\lambda) I_\lambda(z_\lambda)$$

satisfy that

$$\|y\| + \|z\| = \sum_{\lambda\in\Lambda} \|P_\lambda(x)\| = 1.$$

For each λ we have $P_\lambda(y) = \|P_\lambda(y)\|y_\lambda \in \|P_\lambda(y)\|F_\lambda$ and it clearly follows that we may write $y = \|y\|y_0$ with $y_0 \in F$. Similarly, $z = \|z\|z_0$ with $z_0 \in F$, and we get

$$x = \|y\|y_0 - \|z\|z_0 \in \mathrm{co}(F \cup -F).$$

We have shown that $S_X \subset \mathrm{co}(F \cup -F)$, so $B_X = \mathrm{co}(F \cup -F)$ and X is a CL-space.

(*iii*). We have already seen that every X_λ is a CL-space if X is, and that the converse holds for finite Λ. Thus, it only remains to show that X cannot be a CL-space if Λ is infinite. Without loss of generality we just consider the case $\Lambda = \mathbb{N}$. Indeed, choose an infinite countable subset Λ_0 of Λ and write $X = Y \oplus_1 Z$ with $Y = [\oplus_{\lambda\in\Lambda_0} X_\lambda]_{l_1}$ and suitable Z. Were X a CL-space, our above results show that Y would be a CL-space as well. Thus, suppose that $X = [\oplus_{k\in\mathbb{N}} X_k]_{l_1}$ is a CL-space to get a contradiction.

For each $k \in \mathbb{N}$, we fix $e_k^* \in \operatorname{mex}\big(B_{X_k^*}\big)$ and build up the set

$$F = \{x \in S_X \ : \ e_k^*(P_k(x)) = \|P_k(x)\| \ \forall k \in \mathbb{N}\},$$

a maximal convex subset of S_X. Now, let

$$x = \sum_{k=1}^{\infty} \frac{\alpha_k}{2^k}\, I_k(e_k),$$

where (α_k) is a sequence of distinct elements in $\mathbb{T}$, $e_k \in S_{X_k}$ and $e_k^*(e_k) = 1$ for every $k \in \mathbb{N}$. Since $B_X = \operatorname{co}(\mathbb{T}F)$, we can write x in the form

$$x = \sum_{j=1}^{m} \beta_j x^{(j)} \qquad \text{where} \quad \begin{cases} \beta_1, \beta_2, \ldots, \beta_m \in \mathbb{C}, \\ \sum_{j=1}^m |\beta_j| = 1, \ \text{ and} \\ x^{(1)}, x^{(2)}, \ldots, x^{(m)} \in F. \end{cases}$$

We deduce that, for every $k \in \mathbb{N}$,

$$\frac{\alpha_k}{2^k} = e_k^*\big(P_k(x)\big) = \sum_{j=1}^{m} \beta_j e_k^*\big(P_k(x^{(j)})\big) = \sum_{j=1}^{m} \beta_j \|P_k(x^{(j)})\|,$$

so

$$\begin{aligned} 1 = \sum_{k=1}^{\infty} \frac{1}{2^k} &= \sum_{j=1}^{m}\sum_{k=1}^{\infty} \overline{\alpha_k}\, \beta_j \|P_k(x^{(j)}\| \\ &\leqslant \sum_{j=1}^{m}\sum_{k=1}^{\infty} |\beta_j|\, \|P_k(x^{(j)})\| \\ &= \sum_{j=1}^{m} |\beta_j|\, \|x^{(j)}\| = 1. \end{aligned}$$

It follows that

$$\overline{\alpha_k}\ \beta_j \|P_k(x^{(j)})\| = |\beta_j|\, \|P_k(x^{(j)})\| \tag{3}$$

for every $j \in \{1, 2, \ldots, m\}$ and every $k \in \mathbb{N}$. Since

$$0 \neq P_k(x) = \sum_{j=1}^{m} \beta_j P_k(x^{(j)}),$$

for each $k \in \mathbb{N}$, there must exist some $j \in \{1, 2, \ldots, m\}$ such that $\beta_j P_k(x^{(j)}) \neq 0$, and (3) gives $\alpha_k = \dfrac{\beta_j}{|\beta_j|}$. This contradicts the choice of the sequence (α_k). $\square$

Let us point out the following consequence.

Corollary 10. *Let μ be a measure. The complex space $L_1(\mu)$ is a CL-space if and only* $\dim(L_1(\mu)) < \infty$.

Proof. By [10, pp. 136], we have

$$L_1(\mu) \equiv l_1(\Gamma) \oplus_1 \left[\bigoplus_{\alpha\in A} L_1\big([0,1]^{m_\alpha}\big)\right]_{l_1}$$

for some index set Γ and some set A of cardinal numbers $m_\alpha \geqslant \aleph_0$. If $L_1(\mu)$ is a CL-space, Proposition 9 gives us that Γ and A are finite and $L_1\left([0,1]^{m_\alpha}\right)$ is a CL-space for every $\alpha \in A$. Now, since Theorem 14.10 of [10] gives

$$L_1\left([0,1]^{m_\alpha}\right) \equiv \left[\bigoplus_{k=1}^{\infty} L_1\big([0,1]^{m_k}\big)\right]_{l_1}$$

where $m_k = m_\alpha$ for every $k \in \mathbb{N}$, another application of Proposition 9 tell us that $L_1\left([0,1]^{m_\alpha}\right)$ is not a CL-space. Therefore, $A = \emptyset$, $L_1(\mu) \equiv l_1(\Gamma)$ for suitable finite set Γ, and $\dim(L_1(\mu)) < \infty$. The converse result is clear. □

5. The space $C(K, X)$

Given a compact Hausdorff space K and a Banach space X, we consider the Banach space $C(K, X)$ of all continuous functions from K into X, endowed with its natural supremum norm.

Proposition 11. *Let K be a compact Hausdorff space and let X be a Banach space. Then:*

(*i*) *$C(K, X)$ is an almost-CL-space if and only if X is.*
(*ii*) *If $C(K, X)$ is a CL-space, then X is also a CL-space.*

Proof. Let us write $Y = C(K, X)$ and recall that the extreme points in B_{Y^*} are functionals of the form $f \longmapsto x^*(f(t))$ where $x^* \in \operatorname{ex}(B_{X^*})$ and $t \in K$ (see [20, Theorem 1.1]). With this in mind, it is easy to check that the maximal convex subsets of S_Y are just the sets of the form

$$\mathcal{F} = \{f \in S_Y \ : \ f(t) \in F\}, \tag{4}$$

where $t \in K$ and F is a maximal convex subset of S_X.

Suppose first that X is an almost-CL-space. Fix a maximal convex subset $\mathcal{F}$ of S_Y and let t and F be as in (4). Since $B_X = \overline{\operatorname{co}}(\mathbb{T}F)$, given $f \in S_Y$ and $\varepsilon > 0$ we may find $x \in \operatorname{co}(\mathbb{T}F)$ such that

$$\|x - f(t)\| < \varepsilon$$

and build a continuous function $\varphi : K \longrightarrow [0, 1]$ such that

$$\varphi(t) = 1 \qquad \text{and} \qquad \varphi(s) = 0 \ \text{ if } \ \|x - f(s)\| \geqslant \varepsilon.$$

If we define $g \in S_Y$ by

$$g(s) = \varphi(s)x + (1 - \varphi(s))f(s) \quad (s \in K),$$

it is easy to see that $g \in \mathrm{co}(\mathbb{T}\mathcal{F})$ and $\|f - g\| \leqslant \varepsilon$. Therefore, $B_Y = \overline{\mathrm{co}}(\mathbb{T}\mathcal{F})$ and Y is an almost-CL-space.

Now, suppose that Y is an almost-CL-space. Fix a maximal convex subset F of S_X, and consider the maximal convex subset $\mathcal{F}$ of S_Y given by (4) for an arbitrary $t \in K$. Now, fix $x \in B_X$ and $\varepsilon > 0$, consider $f = x\,\chi_K \in B_Y$ and use that $B_Y = \overline{\mathrm{co}}(\mathbb{T}\mathcal{F})$ to find $g \in \mathrm{co}(\mathbb{T}\mathcal{F})$ such that $\|f - g\| < \varepsilon$. Then $g(t) \in \mathrm{co}(\mathbb{T}F)$ and $\|g(t) - x\| < \varepsilon$, so $B_X = \overline{\mathrm{co}}(\mathbb{T}F)$ and X is an almost-CL-space.

In the case when Y is a CL-space, we can repeat the above argument with $\varepsilon = 0$ to prove that X is also a CL-space. □

To finish the paper, let us mention that we do not know whether $C(K, X)$ is a CL-space whenever X is.

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References

[1] E. Alfsen and E. Effros, Structure in real Banach spaces, *Ann. of Math.* **96** (1972), 98–173.
[2] F. F. Bonsall and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*, London Math. Soc. Lecture Note Ser. **2** (Cambridge University Press, 1971).
[3] F. F. Bonsall and J. Duncan, *Numerical Ranges II*, London Math. Soc. Lecture Note Ser. **10** (Cambridge University Press, 1973).
[4] J. Duncan, C. M. McGregor, J. D. Pryce, and A. J. White, The numerical index of a normed space, *J. London Math. Soc.* **2** (1970), 481–488.
[5] V. P. Fonf, Weakly extremal properties of Banach spaces, *Math. Notes* **45** (1989), 488–494.
[6] R. E. Fullerton, Geometrical characterization of certain function spaces. In: *Proc. Inter. Sympos. Linear spaces* (Jerusalem 1960), pp. 227–236 (Pergamon, Oxford 1961).
[7] G. Godefroy, Boundaries of a convex set an interpolation sets, *Math. Ann.* **277** (1987), 173–184.
[8] O. Hanner, Intersections of translates of convex bodies, *Math. Scand.* **4** (1956), 65–87.
[9] A. B. Hansen and Å. Lima, The structure of finite dimensional Banach spaces with the 3.2. intersection property, *Acta Math.* **146** (1981), 1–23.
[10] H. E. Lacey, *The Isometric Theory of Classical Banach Spaces*, Springer-Verlag, Berlin, 1974.
[11] Å. Lima, Intersection properties of balls and subspaces in Banach spaces, *Trans. Amer. Math. Soc.* **227** (1977), 1–62.
[12] Å. Lima, Intersection properties of balls in spaces of compact operators, *Ann. Inst. Fourier, Grenoble* **28** (1978), 35–65.
[13] Å. Lima, On extreme operators on finite-dimensional Banach spaces whose unit balls are polytopes, *Ark. Mat.* **19** (1981), 97–116.
[14] J. Lindenstrauss, *Extension of compact operators*, Memoirs Amer. Math. Soc. **48** (Providence 1964).

[15] J. Lindenstrauss and L. Tzafriri, *Clasical Banach Spaces I: Sequence Spaces* (Springer-Verlag, Berlin 1977).
[16] G. López, M. Martín, and R. Payá, Real Banach spaces with numerical index 1, *Bull. London Math. Soc.* **31** (1999), 207–212.
[17] M. Martín, A survey on the numerical index of a Banach space, *Extracta Math.* **15** (2000), 265–276.
[18] M. Martín and R. Payá, Numerical index of vector-valued function spaces, *Studia Math.* **142** (2000), 269–280.
[19] S. Reisner, Certain Banach spaces associated with graphs and CL-spaces with 1-unconditonal bases, *J. London Math. Soc* **43** (1991), 137–148.
[20] W. M. Ruess and C. P. Stegall, Extreme points in duals of operator spaces, *Math. Ann.* **261** (1982), 535–546.

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