BANACH SPACES HAVING THE RNP AND NUMERICAL INDEX 1

MIGUEL MARTÍN

ABSTRACT. Let X be a Banach space with the RNP. Then, the following are equivalent.
(*i*) X has numerical index 1.
(*ii*) $|x^{**}(x^*)| = 1$ for all $x^* \in \mathrm{ex}\,(B_{X^*})$ and $x^{**} \in \mathrm{ex}\,(B_{X^{**}})$.
(*iii*) X is an almost-CL-space.
(*iv*) There are a compact Hausdorff space K and a linear isometry $J : X \to C(K)$ such that $|x^{**}(J^*\delta_s)| = 1$ for all $s \in K$ and $x^{**} \in \mathrm{ex}\,(B_{X^{**}})$.

If X is a real space, the above conditions are equivalent to be semi nicely embedded in some space $C(K)$.

The numerical index of a Banach space is a constant relating the norm and the numerical radius of operators on the space. Let us present the relevant definitions. For a Banach space X, we write B_X for the closed unit ball and S_X for the unit sphere. We denote by X^* the dual space and by $L(X)$ the Banach algebra of all bounded linear operators on X. For such an operator T, the *numerical radius* of T is

$$v(T) = \sup\{|x^*(Tx)| \ : \ x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\}.$$

The *numerical index* of the space X is then given by

$$n(X) = \max\{k \geq 0 \ : k\,\|T\| \leq v(T) \ \ \forall\, T \in L(X)\}.$$

We refer the reader to the books [3, 4] and to the expository paper [13] for general information and background. Recent results can be found in [7, 12, 14, 15].

Let us mention here some facts concerning the numerical index which will be relevant to our discussion. First, one has $v(T^*) = v(T)$ for every $T \in L(X)$, where T^* is the adjoint operator of T (see [3, §9]), and it clearly follows that $n(X^*) \leq n(X)$. The question whether this is actually an equality seems to be open. Second, it is a classical result that L- and M-spaces have numerical index 1 [6].

The aim of this paper is to characterize Banach spaces with numerical index 1 among those having the Radon-Nikodỳm property (RNP in short, see [5] for background). To this end, we prove that some general

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sufficient conditions for a Banach space to have numerical index 1 are also necessary if the space has the RNP.

Let us introduce some definitions and comments. A Banach space X is an *almost-CL-space* if B_X is the absolutely closed convex hull of every maximal convex subset of S_X. This notion was introduced by Å. Lima [10], generalizing the concept of CL-space given by R. Fullerton [8] in 1960. Real and complex almost-CL-spaces have numerical index 1 (see [13, §4] or [1]), but it is not known whether the reciprocal result is true. As usual, we write $C(K)$ for the Banach space of all continuous functions from the compact Hausdorff space K into the scalar field. Given $s \in K$, δ_s stands for the functional $f \longmapsto f(s)$ on $C(K)$. Finally, we write $\mathrm{co}(B)$ for the convex-hull of B and, given a convex subset A of X, $\mathrm{ex}(A)$ and $\mathrm{dent}(A)$ are, respectively, the set of extreme points and the set of denting points of A.

The main result of the paper is the following.

Theorem 1. *Let X be a Banach space having the RNP. Then the following are equivalent.*

(*i*) $n(X) = 1$.

(*ii*) $|x^{**}(x^*)| = 1$ *for all* $x^* \in \mathrm{ex}\,(B_{X^*})$ *and* $x^{**} \in \mathrm{ex}\,(B_{X^{**}})$.

(*iii*) *X is an almost-CL-space.*

(*iv*) *There are a compact Hausdorff space K and a linear isometry $J : X \to C(K)$ such that* $|x^{**}(J^*\delta_s)| = 1$ *for all* $s \in K$ *and* $x^{**} \in \mathrm{ex}\,(B_{X^{**}})$.

Proof. The implication (*ii*) $\Rightarrow$ (*i*) is straightforward. Just use that $v(T) = v(T^*)$ for every $T \in L(X)$. The implication (*iii*) $\Rightarrow$ (*i*) has been commented above.

(*iv*) $\Rightarrow$ (*i*). We fix $T \in L(X)$ and we have to prove that $v(T) = \|T\|$. For each $s \in K$, we take $x^{**} \in \mathrm{ex}\,(B_{X^{**}})$ such that

$$\left|x^{**}\big(T^*(J^*\delta_s)\big)\right| = \|T^*(J^*\delta_s)\|.$$

Since $|x^{**}(J^*\delta_s)| = 1$, we have

$$v(T) = v(T^*) \geq \|T^*(J^*\delta_s)\|$$

for each $s \in K$. It is clear that

$$\|T\| = \|T^*\| = \sup\{\|T^*(J^*\delta_s)\| \ : \ s \in K\}$$

and so, $v(T) = \|T\|$ and $n(X) = 1$.

(*i*) $\Rightarrow$ (*ii*). We use [12, Lemma 1] to get $|x^*(x)| = 1$ for all $x \in \mathrm{dent}\,(B_X)$ and $x^* \in \mathrm{ex}\,(B_{X^*})$. Since X has the RNP, it may be concluded that

$$\mathrm{ex}\,(B_{X^{**}}) \subseteq \overline{\mathrm{dent}\,(B_X)}^{w^*},$$

so $|x^{**}(x^*)| = 1$ for all $x^* \in \mathrm{ex}\,(B_{X^*})$ and $x^{**} \in \mathrm{ex}\,(B_{X^{**}})$.

(*i*) $\Rightarrow$ (*iii*). Let F be a maximal convex subset of S_X. By using Hahn-Banach and Krein-Milman Theorems, it is easily checked that there exists $x^* \in \mathrm{ex}\,(B_{X^*})$ such that

$$F = \{x \in B_X \ : \ x^*(x) = 1\}.$$

So, by [12, Lemma 1], $\mathrm{dent}\,(B_X)$ is contained in the absolutely convex hull of F. On the other hand, since X has the RNP, $B_X = \overline{\mathrm{co}}(\mathrm{dent}\,(B_X))$. Thus, B_X is the closed absolutely convex hull of F and X is an almost-CL-space.

$(i) \Rightarrow (iv)$. For $x^* \in B_{X^*}$, we have

$$(1) \qquad x^* \in \mathrm{ex}\,(B_{X^*}) \quad \Leftrightarrow \quad |x^*(x)| = 1 \text{ for each } x \in \mathrm{dent}\,(B_X)\,.$$

One implication is [12, Lemma 1]; the other one arises from the fact that $\mathrm{dent}\,(B_X)$ is norming for X^*.

It follows from (1) that $K = \mathrm{ex}\,(B_{X^*})$ with the w^* topology is a Hausdorff compact space. Let J be the canonical injection from X into $C(K)$. Now, since $J^*\delta_s = s \in \mathrm{ex}\,(B_{X^*})$, we deduce from (ii) that $|x^{**}(J^*\delta_s)| = 1$ for all $s \in K$ and $x^{**} \in \mathrm{ex}\,(B_{X^{**}})$. □

Let us give some remarks on the above theorem.

Condition (ii) of the theorem is called E.P.I.P. in [10] and appeared in [11, §4]. In [10, Corollary 3.6], Å. Lima proved $(ii) \Rightarrow (iii)$ for arbitrary real Banach spaces. Nevertheless, we do not know if Lima's result is valid in the complex case without the RNP assumption.

Observe that, when proving $(iv) \Rightarrow (i)$, we do not need the RNP and that the result still holds if we change $C(K)$ by $C_b(\Omega)$, the Banach space of all bounded continuous functions from the topological Hausdorff space Ω into the scalar field. Therefore, we have the following by-product of the proof of Theorem 1.

Corollary 2. *Let Ω be a Hausdorff topological space and let X be a Banach space. If there exists a linear isometry $J : X \to C_b(\Omega)$ such that*

$$|x^{**}(J^*\delta_s)| = 1 \qquad\qquad (s \in \Omega,\ x^{**} \in \mathrm{ex}\,(B_{X^{**}})),$$

then $n(X) = 1$.

The above corollary improves one result recently given by D. Werner [16, Corollary 2.2]. Some definitions are required. Following [16], a Banach space X is said to be *nicely embedded* in $C_b(\Omega)$ if there exists a linear isometry $J : X \longrightarrow C_b(\Omega)$ such that for all $s \in \Omega$ the following properties are satisfied:

(N1) $\|J^*\delta_s\| = 1$.
(N2) $\mathrm{span}(J^*\delta_s)$ is an L-summand in X^*.

It is clear that nicely embedded spaces fulfill the conditions in Corollary 2, so they have numerical index 1. This is precisely Corollary 2.2 of [16] (see also the introduction of [14]). But the converse result is false even in the finite-dimensional setting. As a matter of fact, in the case when X is the 3-dimensional L-space, $n(X) = 1$ and X^* does not have any non-trivial L-summand.

In the real case, Corollary 2 can be written in a more suitable form by using a similar notation than Werner's result. Following Å. Lima [9, 10], a closed subspace Y of a Banach space X is said to be a *semi L-summand* if for every $x \in X$ there exists a unique $y \in Y$ such that $\|x - y\| = d(x, Y)$, and moreover this y satisfies $\|x\| = \|y\| + \|x - y\|$. A Banach space X is said to be *semi nicely embedded* in $C_b(\Omega)$ if there exists a linear isometry $J : X \to C_b(\Omega)$ such that for all $s \in \Omega$ the following properties are satisfied:

(N1) $\|J^*\delta_s\| = 1$.
(N2') $\operatorname{span}(J^*\delta_s)$ is a semi L-summand in X^*.

Given a real Banach space X and a point $x \in S_X$, [10, Theorem 3.1] says that $\operatorname{span}(x)$ is a semi L-sukmand if and only if $|x^*(x)| = 1$ for all $x^* \in \operatorname{ex}(B_{X^*})$. Therefore, condition (iv) in Theorem 1 is equivalent to the property of being semi nicely embedded in some $C(K)$. Therefore, for rcal spaces, Theorem 1 reads as follows.

Corollary 3. *Let X be a rewl Banach space having the RNP. Then $n(X) = 1$ if and only if X is semi nicely embedded in some $C(K)$.*

It is now easy to give examples of Banach spaces semi nicely embedded in some $C(K)$ which are not nicely embedded in any $C_b(\Omega)$. For instance, since the real space space l_1 has the RNP and $n(l_1) = 1$, it is semi nicely embedded in $C(\Delta)$, where Δ is the Cantor set. But l_∞ does not have any non-trivial L-summand (see [2, Theorem 6.15], for instance).

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Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain, *e-mail :* mmartins@ugr.es