STRUCTURE OF NESTED SEQUENCES OF BALLS IN BANACH SPACES

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Abstract. In this paper, we study the structure of the union of unbounded nested sequences of balls, and use them to characterize some geometric properties of X^*. We show that the union of an unbounded nested sequence of balls is a cone if the centers of the balls lie in a finite dimensional subspace. However, in general, such a union need not be a cone. In fact, examples can be constructed, up to renorming, in any infinite dimensional Banach space. We also study when such an union is the intersection of at most k half-spaces, and relate it with the number of extreme points of any face of the dual ball.

1. Introduction

We work with *real* Banach spaces. Let X be a Banach space. We denote by $B(x,r)$ (resp. $B[x,r]$) the open (resp. closed) ball of radius $r > 0$ around $x \in X$ and use $B(X)$ instead of $B[0,1]$. The unit sphere of X will be $S(X)$. We write D for the duality mapping for X, that is, the set-valued map from $S(X)$ to $\mathcal{P}(S(X^*))$ defined by

$$D(x) = \{x^* \in S(X^*) : x^*(x) = 1\}, \quad x \in S(X).$$

A Banach space X is said to be rotund if every point of $S(X)$ is an extreme point of $B(X)$. Vlasov [15] showed that X^* is rotund if and only if the union of any unbounded nested sequence of balls in X is either the whole of X or an open affine half-space.

Definition 1.1. A sequence $\{B_n = B(x_n, r_n)\}$ of open balls in X is *nested* if for all $n \geq 1$, $B_n \subseteq B_{n+1}$.

A nested sequence $\{B_n = B(x_n, r_n)\}$ of balls in X is *unbounded* if $r_n \uparrow \infty$.

It was proved by Beauzamy and Maurey ([6, p. 126]; see also [5, p. 183]) that a Banach space X is smooth if and only if the union of any unbounded nested

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sequences of balls whose centers lie on a straight line through the origin is either the whole of X or an open affine half-space.

Let $\{B_n\}$ be an unbounded nested sequence of balls. Let $B = \cup B_n$. Then B is an open convex set. In this paper, we study the structure of sets B that arise as the union of unbounded nested sequences of balls, and use them to characterize some geometric properties of X^*.

For example, as we saw above, if X^* is rotund, then B is either the whole of X or an open affine half-space. In particular, B is a cone.

Definition 1.2. A convex subset $C \subseteq X$ is a *cone* with vertex 0 if it is closed under multiplication by positive scalars. If C is a cone with vertex 0 and $x \in X$, then $x + C$ is a cone with vertex x.

In particular, we are interested in the question : Is B a cone? We show that if X is finite dimensional, then the answer is yes for every equivalent norm on X (Proposition 2.7). However, in general, B need not be a cone. In fact, we prove (see Theorem 2.16 and Corollary 2.17) that a Banach space X is finite dimensional if and only if for every equivalent norm, the union of each unbounded nested sequence of balls is a cone.

The answer, therefore, depends on the norm or on the nature of the unbounded nested sequence of balls. We show that if $\{B_n\}$ is an unbounded nested sequence of balls with centers in a finite dimensional subspace (Corollary 2.14), then $B = \cup B_n$ is a cone. On the other hand, if X^* is k-rotund for some k, then for every unbounded nested sequence $\{B_n\}$ of balls, $B = \cup B_n$ is a cone (Corollary 2.18). However, a complete isometric characterization of Banach spaces in which the union of every unbounded nested sequence of balls is a cone, remains open.

Observe that any cone has a vertex and an associated cone with vertex 0. We show that for B as above, there is a naturally associated cone C with vertex 0 (possibly empty) such that whenever B is a cone, C is nonempty and B is a translate of C. This brings us to the question : is C always nonempty? Here our isomorphic result is : a Banach space is reflexive if and only if for every equivalent norm and every unbounded nested sequence of balls, C is nonempty (see Lemma 2.20 (b), Theorem 2.21 and Corollary 2.22). Coming to the isometric question, we show, for example, that if X is an M-ideal in X^{**}, then for every unbounded nested sequence $\{B_n\}$ of balls, C is nonempty (Lemma 2.20 (d)). However, a complete isometric characterization of Banach spaces in which the cone C associated with an unbounded nested sequence of balls is always nonempty, again remains open.

Properties of Banach spaces characterized by the structure of unbounded nested sequence of balls like the above have also been studied in [12, 13], and have been localized and extended in [1, 2, 4]. See also the survey [3].

The third section of the paper will be devoted to the study of some geometric properties of X^* in terms of unbounded nested sequences of balls in X, generalizing the results of Vlasov and Beauzamy–Maurey mentioned above. Specifically, we show that the union of such a sequence is the intersection of at most k half-spaces if and only if any face of the dual ball is the convex hull of at most k extreme points.

2. Structure of $\bigcup B_n$

Let us first observe that, thanks to Hahn-Banach Theorem, for a Banach space X, every open convex set $B \neq X$ can be written as

$$B = \bigcap_{x^* \in A} \{x \in X : x^*(x) > \inf x^*(B)\},$$

where $A = \{x^* \in S(X^*) : x^* \text{ is bounded below on } B\}$. Observe that, since $B \neq X$, $A \neq \emptyset$. Now, such a set B is a cone with vertex $v \in X$ (not necessarily unique) if and only if $\inf x^*(B) = x^*(v)$ for every $x^* \in A$. That is, B is a cone with vertex v if and only if it can be written as

$$B = v + \bigcap_{x^* \in A} \{x \in X : x^*(x) > 0\} = v + C \text{ (say)}.$$

If $B(x, r)$ (or $B[x, r]$) is an open (closed) ball in X, then it is easy to prove that, for every $x^* \in S(X^*)$,

$$\inf x^*(B(x, r)) = \inf x^*(B[x, r]) = x^*(x) - r.$$

We will use these facts without reference in the sequel.

Our first result is about unbounded nested sequences of balls whose centers lie on a line.

Definition 2.1. An unbounded nested sequence of balls $\{B(x_n, r_n)\}$ in a Banach space X is called *straight* if there exist $x_0 \in S(X)$ and $\{\lambda_n\} \subseteq \mathbb{R}$ such that $x_n = \lambda_n x_0$ for all $n \geq 1$. Such x_0 is called the direction of this sequence.

Theorem 2.2. *The union of a straight unbounded nested sequence of balls is always a cone.*

Proof. Let $\{B_n = B(\lambda_n x_0, r_n)\}$ be a straight unbounded nested sequence of balls, $B = \cup B_n$, and $A = \{x^* \in S(X^*) : x^* \text{ is bounded below on } B\}$.

Translating the balls if necessary, we may assume $0 \in B[x_n, r_n]$. That is, $|\lambda_n| \leq r_n$. Observe that $x^* \in A$ if and only if there exists $c \in \mathbb{R}$ such that for all $n \geq 1$,

$$\inf x^*(B_n) = \lambda_n x^*(x_0) - r_n \geq c, \tag{1}$$

or, equivalently, for all $n \geq 1$,

$$\frac{\lambda_n}{r_n} x^*(x_0) \geq 1 + c/r_n.$$

Since $r_n \longrightarrow \infty$, the right hand side is eventually positive. It follows that eventually λ_n and $x^*(x_0)$ has the same sign. Thus if $\{\lambda_n\}$ has both a positive subsequence and a negative subsequence, then $A = \emptyset$, that is, $B = X$. If $\lambda_n > 0$ eventually, it follows that

$$\lim_{n\to\infty} \frac{\lambda_n}{r_n} = 1 \quad \text{and} \quad x^*(x_0) = 1,$$

that is, $A \subseteq D(x_0)$. It now follows from (1) that the sequence $\{\lambda_n - r_n\}$ is bounded below. And therefore, from (1) again, $A = D(x_0)$. Similarly, if $\lambda_n < 0$ eventually, then $A = -D(x_0)$.

Thus, we may assume $\lambda_n > 0$ for all $n \geq 1$, and $A = D(x_0)$. Put

$$C = \bigcap_{x^* \in A} \{x \in X : x^*(x) > 0\}$$

and observe that $x_0 \in C$, C is a cone with vertex 0, and for every $x \in B$, $x + C \subseteq B$. Moreover, for any $x^* \in A$,

$$\inf x^*(B) = \inf_n \{x^*(x_n) - r_n\} = \inf_n \{\lambda_n - r_n\} = \alpha \text{ (say) } > -\infty,$$

and then

$$B = \bigcap_{x^* \in A} \{x \in X : x^*(x) > \inf x^*(B) = \alpha = x^*(\alpha x_0)\} = \alpha x_0 + C$$

is a cone with vertex αx_0. □

Let us now isolate some general facts contained in the above proof. Fix an unbounded nested sequence of balls $\{B_n = B(x_n, r_n)\}$ in a Banach space X, and let $B = \cup B_n$. Let

$$A = \{x^* \in S(X^*) : x^* \text{ is bounded below on } B\}$$

and

$$C = \bigcap_{x^* \in A} \{x \in X : x^*(x) > 0\}.$$

Then

$$B = \bigcap_{x^* \in A} \{x \in X : x^*(x) > \inf x^*(B)\},$$

and if C is nonempty then it is a cone with vertex 0, and for every $x \in B$, $x + C \subseteq B$.

As before, we may assume $0 \in \overline{B}_n$, or, equivalently, $\|x_n\| \le r_n$ eventually. Now, $x^* \in A$ if and only if there exists $c \in \mathbb{R}$ such that for all $n \ge 1$,

$$x^*(x_n) - r_n \ge c,$$

that is,

$$x^*\left(\frac{x_n}{r_n}\right) \ge 1 + \frac{c}{r_n} \tag{2}$$

Let $x_0^{**} \in B(X^{**})$ be a w*-cluster point of the sequence $\{x_n/r_n\} \subseteq B(X)$. It follows from (2) that for every $x^* \in A$, $x_0^{**}(x^*) = 1$. And hence,

$$\lim \left\|\frac{x_n}{r_n}\right\| = \|x_0^{**}\| = 1$$

Therefore, if $\liminf \|x_n/r_n\| < 1$, then $B = X$. And if $B \ne X$, then $x_0^{**} \in S(X^{**})$ and $A \subseteq D(x_0^{**}) \cap X^*$. In this case, we can say more about the set A.

Lemma 2.3. *If $B \ne X$, the set A is a nonempty convex extremal subset (a face) of $S(X^*)$. Moreover,*

$$A = \{x^* \in S(X^*) : \{x^*(x_n) - r_n\} \text{ converges}\}.$$

Proof. Let $x^*, y^* \in A$ and $\lambda \in (0,1)$. Choose $c \in \mathbb{R}$ such that (2) holds for both x^* and y^*. Then,

$$(\lambda x^* + (1-\lambda)y^*)\left(\frac{x_n}{r_n}\right) \ge 1 + \frac{c}{r_n}$$

Since, $\{x_n/r_n\} \subseteq B(X)$, it follows that $\lambda x^* + (1-\lambda)y^* \in S(X^*)$ and therefore, $\lambda x^* + (1-\lambda)y^* \in A$. This proves A is convex.

To show that A is an extremal subset of $S(X^*)$, let $x^*, y^* \in S(X^*)$ and $t \in (0,1)$ such that $tx^* + (1-t)y^* \in A$. Then $tx^* + (1-t)y^* \in S(X^*)$ and therefore,

$$\begin{aligned} \inf(tx^* + (1-t)y^*)(B_n) &= (tx^* + (1-t)y^*)(x_n) - r_n \\ &= t(x^*(x_n) - r_n) + (1-t)(y^*(x_n) - r_n) \\ &= t \inf x^*(B_n) + (1-t) \inf y^*(B_n). \end{aligned}$$

Since $0 \in B[x_n, r_n]$, for any $z^* \in S(X^*)$, $\inf z^*(B_n) \le 0$. Thus, $tx^* + (1-t)y^* \in A$ implies $x^*, y^* \in A$.

Further, since the balls are nested, for all $x^* \in A$, the sequence $\{x^*(x_n) - r_n\}$ is decreasing and bounded below and hence converges. □

As an immediate consequence of the above result, we can prove one direction of Vlasov's Theorem [15]. Observe that if X^* is rotund, then the only convex subsets of $S(X^*)$ are singletons. So, A is a singleton and therefore, B is an affine half-space.

In our earlier work [2], we observed that locally Vlasov's theorem is actually a consequence of the fact that if X^* is rotund, then every point of $S(X^*)$ is a *rotund point* of $B(X^*)$.

Definition 2.4. [8] Let X be a Banach space. We say that $x \in S(X)$ is a *rotund point* of $B(X)$ (or, X is rotund at x) if $\|y\| = \|(x+y)/2\| = 1$ implies $x = y$.

In fact, the result of [2] can be reformulated in our notation as : $x^* \in S(X^*)$ is a rotund point of $B(X^*)$ if and only if for every unbounded nested sequence $\{B_n\}$ of balls such that $x^* \in A$, we have $A = \{x^*\}$. Again, one direction of the argument follows immediately.

Observe that if $x \in S(X)$ is a rotund point then the only convex subset of $S(X)$ that contains x is singleton $\{x\}$. Therefore, if A contains a rotund point of $S(X^*)$, then A is a singleton and B is an affine half-space.

The following result was established in [15] as a part of the study of convexity of Chebyshev sets. Here we give a much simpler proof using Theorem 2.2.

Proposition 2.5. [15, Proposition 3.4] *Let $\{B_n = B(x_n, r_n)\}$ be an unbounded nested sequence of balls. Let $B = \cup B_n$ and $M = X \setminus B$. Suppose* w-lim $x_n/\|x_n\| = x_0$. *Then for every $x \in B$,*

$$\bigcup_{r \geq 0} B\big(x + rx_0, r + d(x, M)\big) \subseteq B$$

where $d(x, M) = \inf\{\|x - m\| : m \in M\}$ is the distance from x to M.

Proof. By the above observations, we may assume

$$\lim \left\| \frac{x_n}{r_n} \right\| = 1 \text{ and } \|x_0\| = 1.$$

Let A and C be as above. Then $A \subseteq D(x_0)$.

Fix $x \in B$. Since the balls are nested,

$$\bigcup_{r \geq 0} B\big(x + rx_0, r + d(x, M)\big) = \bigcup_{n \geq 1} B\big(x + nx_0, n + d(x, M)\big).$$

By the proof of Theorem 2.2,

$$\bigcup_{n \geq 1} B\big(nx_0, n + d(x, M)\big) = -d(x, M)x_0 + \bigcap_{x^* \in D(x_0)} \{x \in X : x^*(x) > 0\},$$

and therefore,

$$\begin{aligned} \bigcup_{n \geq 1} B\big(x + nx_0, n + d(x, M)\big) &= x - d(x, M)x_0 + \bigcap_{x^* \in D(x_0)} \{x \in X : x^*(x) > 0\} \\ &\subseteq x - d(x, M)x_0 + C. \end{aligned}$$

Clearly,

$$x - d(x,M)x_0 \in \overline{B} = \bigcap_{x^* \in A} \{x \in X : x^*(x) \geq \inf x^*(B)\}.$$

And therefore,

$$\begin{aligned} x - d(x,M)x_0 + C &\subseteq \bigcap_{x^* \in A} \{x \in X : x^*(x) \geq \inf x^*(B)\} + C \\ &= \bigcap_{x^* \in A} \{x \in X : x^*(x) > \inf x^*(B)\} = B. \end{aligned}$$

□

The rest of this section is devoted to study whether or not B is a cone, and whether or not C is nonempty.

We start by proving that in finite dimensional spaces, B is always a cone. We need the following lemma, which we will use again in Section 3.

Lemma 2.6. *If for $x_1^*, x_2^*, \ldots, x_n^* \in A$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$, $y^* = \sum_{i=1}^{n} \lambda_i x_i^* \in A$, then $\sum_{i=1}^{n} \lambda_i = 1$ and $\inf y^*(B) = \sum_{i=1}^{n} \lambda_i \inf x_i^*(B)$.*

Proof. As $y^* \in A \subseteq S(X^*)$, we have

$$\begin{aligned} \inf y^*(B_m) &= y^*(x_m) - r_m = \sum_{i=1}^{n} \lambda_i x_i^*(x_m) - r_m \\ &= \sum_{i=1}^{n} \lambda_i (x_i^*(x_m) - r_m) + (\sum_{i=1}^{n} \lambda_i - 1) r_m \\ &= \sum_{i=1}^{n} \lambda_i \inf x_i^*(B_m) + (\sum_{i=1}^{n} \lambda_i - 1) r_m. \end{aligned}$$

Observe that $\inf z^*(B_m) \leq 0$ for every $z^* \in S(X^*)$, and since $y^* \in A$, $\inf_m \inf y^*(B_m) > -\infty$. So we get $\sum_{i=1}^{n} \lambda_i = 1$. □

Proposition 2.7. *Let $\{B_n = B(x_n, r_n)\}$ be an unbounded nested sequence of balls in a finite dimensional space X. Then $B = \cup B_n$ is a cone.*

Proof. Let us first note that B is a cone if and only if there is a $v \in X$ such that for all $x^* \in A$, $x^*(v) = \inf x^*(B)$.

Let $\{x_1^*, x_2^*, \ldots, x_m^*\}$ be a maximal linearly independent subset of A. By independence, there is a $v \in X$ such that $x_i^*(v) = \inf x_i^*(B)$ for all $i = 1, 2, \ldots, m$. It now suffices to note that this v works for all $x^* \in A$.

Indeed, let $x^* \in A$. By maximality, there is $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}$ such that $x^* = \sum_{i=1}^m \lambda_i x_i^*$. By Lemma 2.6, it follows that $\inf x^*(B) = \sum_{i=1}^{m} \lambda_i \inf x_i^*(B) = \sum_{i=1}^{n} \lambda_i x_i^*(v) = x^*(v)$. □

Coming to infinite dimensions, we quickly note an easy necessary condition for B to be a cone.

Lemma 2.8. *If B is a cone, then there exists $c \in \mathbb{R}$ such that for every $x^* \in A$ and $n \geq 1$,*

$$x^*(x_n) - r_n \geq c.$$

And in this case, A is w-closed.*

Proof. Observe that if B is a cone with vertex v, then for all $x^* \in A$,

$$\inf_n [x^*(x_n) - r_n] = \inf x^*(B) = x^*(v) \geq -\|v\| = c \text{ (say)}.$$

Moreover, if B is a cone with vertex v, then $B - v$ is a cone with vertex 0 and $x^* \in A$ if and only if x^* is non-negative on $B - v$. It follows that the set A has to be w*-closed. □

Question 2.9. *We do not know whether the above condition is also sufficient for B to be a cone.*

Now we are in a position to extend Theorem 2.2 to prove that if the centers of the balls belong to a finite dimensional subspace, then B is a cone. For this, we need a tool to extend the phenomena under study from a subspace Y to the whole space X. We observe :

Lemma 2.10. *Let $\{B_n\}$ be an unbounded nested sequence of balls in X with centers in a subspace Y such that $0 \in \overline{B}_1$. Then $\{B_n \cap Y\}$ is an unbounded nested sequence of balls in Y. Let $B = \cup B_n$ and let*

$$\begin{aligned} A_X &= \{x^* \in S(X^*) : x^* \text{ is bounded below on } B\} \text{ and} \\ A_Y &= \{y^* \in S(Y^*) : y^* \text{ is bounded below on } B \cap Y\} \end{aligned}$$

Then for any $y^* \in A_Y$, every norm preserving extension of y^* is in A_X and for any $x^* \in A_X$, $x^*|_Y \in A_Y$. Moreover, $\inf x^*(B) = \inf x^*|_Y(B \cap Y)$.

Proof. It suffices to show that if $x^* \in A_X$, then $\|x^*|_Y\| = 1$. Let $\{B_n = B(y_n, r_n)\}$. Put $z_n = y_n/r_n$. It follows that $\|z_n\| \leq 1$. By (2) and the fact that $r_n \longrightarrow \infty$, we conclude $\lim_n x^*(z_n) = 1$. And hence, $\|x^*|_Y\| = 1$. The rest is clear. □

Proposition 2.11. *Let $\{B_n\}$ be an unbounded nested sequence of balls in X with centers in a subspace Y and $B = \cup B_n$. If $B \cap Y$ is a cone in Y, then B is a cone.*

Proof. As noted before B is a cone if and only if there is a $v \in X$ such that for all $x^* \in A$, $x^*(v) = \inf x^*(B)$.

Since $B \cap Y$ is a cone in Y, there is a $u \in Y$ such that for all $y^* \in A_Y$, $y^*(u) = \inf y^*(B \cap Y)$. It suffices to note that this u also works in X.

Indeed, let $x^* \in A_X$. Then $x^*|_Y \in A_Y$ and therefore, $x^*(u) = x^*|_Y(u) = \inf x^*|_Y(B \cap Y) = \inf x^*(B)$. □

Question 2.12. *It is an open question whether the converse is true.*

Remark 2.13. Observe that if the answer to Question 2.9 is yes, then so is the answer to this one. This is clear as if the bound on $B \cap Y$ is not uniform over A_Y, the bound on B cannot be uniform over A_X.

Corollary 2.14. *Let $\{B_n\}$ be an unbounded nested sequence of balls in a Banach space X with centers in a finite dimensional subspace. Then $B = \cup B_n$ is a cone.*

We now show that for every infinite dimensional Banach space, there are an equivalent norm and an unbounded nested sequence of balls (in the new norm) such that the union is not a cone. For this, we need the following lemma.

Lemma 2.15. (*a*) *Let X be an infinite dimensional Banach space. Then for any $\varepsilon > 0$, there are an ε-equivalent norm $|||\cdot|||$ on X and sequences $\{z_i\} \subseteq X$ and $\{z_i^*\} \subseteq X^*$ such that $|||z_i||| = |||z_i^*||| = 1$ and*

$$z_k^*(z_n) \begin{cases} = 1 & \text{if} \quad 1 \le k \le n, \\ < 1 & \text{if} \quad k > n. \end{cases}$$

(*b*) *Let X be separable and non-reflexive. Then there are an equivalent norm $|||\cdot|||$ on X and sequences $\{z_i\} \subseteq X$ and $\{z_i^*\} \subseteq X^*$ satisfying the above properties and with the additional property that $\overline{\{z_i^*\}}^{w^*}$ contains some ball centered at the origin.*

Proof. (*a*). Let $x_0 \in S(X)$ and $x_0^* \in S(X^*)$ with $x_0^*(x_0) = 1$. Let $r > 0$, $r_1 > 0$. Define

$$\begin{aligned} G &= [(1+r)x_0^* + \ker(x_0)] \cap B((1+r)x_0^*, r_1) \quad \text{and} \\ V &= \mathrm{co}\{\pm(1+r)x_0^*, B(X^*)\}. \end{aligned}$$

Clearly, $V \cap G = \{(1+r)x_0^*\}$. Choose $z_1^* \in G \setminus \{(1+r)x_0^*\}$ and (by the Hahn-Banach separation theorem) find a $z_1 \in X$ with $z_1^*(z_1) = 1 > \sup z_1(V)$. Next find

$z_2^* \in [G \setminus \{(1+r)x_0^*\}] \cap \{x^* \in X^* : x^*(z_1) < 1\}$ such that for

$$L_2 = \left\{ \sum_{i=1}^{2} \alpha_i z_i^* \ : \sum_{i=1}^{2} \alpha_i = 1 \right\},$$

$L_2 \cap V = \emptyset$, i.e., $(1+r)x_0^* \notin L_2$.

Why does such an z_2^* exist? Let us do the following: first take $z_2^* \in G \cap \{x^* \in X^* : x^*(z_1) = 1\}$. Clearly, $(1+r)x_0^* \notin L_2$. Hence the distance between $(1+r)x_0^*$ and L_2 is positive. Next move z_2^* a little in G into the half-space $\{x^* \in X^* : x^*(z_1) < 1\}$ in such a way that the line L_2 in the new position still does not contain $(1+r)x_0^*$. This is exactly what we need.

Observe that a functional separating L_2 from V must be constant on L_2. Thus, there exists a $z_2 \in X$ with $z_2(L_2) = 1 > \sup z_2(V)$. Notice that $[G \setminus \{(1+r)x_0^*\}] \cap \bigcap_{i=1}^{2} \{x^* \in X^* : x^*(z_i) < 1\}$ is a nonempty open set, since $G \cap \bigcap_{i=1}^{2} \{x^* \in X^* : x^*(z_i) < 1\}$ is a nonempty [it contains $(1+r)x_0^*$] open set. As before, we can find $z_3^* \in [G \setminus \{(1+r)x_0^*\}] \cap \bigcap_{i=1}^{2} \{x^* \in X^* : x^*(z_i) < 1\}$ such that for

$$L_3 = \left\{ \sum_{i=1}^{3} \alpha_i z_i^* : \sum_{i=1}^{3} \alpha_i = 1 \right\},$$

$L_3 \cap V = \emptyset$, i.e., $(1+r)x_0^* \notin L_3$. The rest of the inductive construction is clear.

Put $U = \overline{co}^{w^*}[\{\pm z_i^*\} \cup B(X^*)]$ and

$$|||x||| = \sup\{x^*(x) : x^* \in U\}, \quad x \in X.$$

For $r > 0$ and $r_1 > 0$ small enough, the new norm $|||\cdot|||$ is ε-isomorphic to the original one. All the properties of the sequences $\{z_i\}$ and $\{z_i^*\}$ are clear from the construction.

(b). We use the same idea of construction as above, but with some changes. First of all, we take $V = B(X^*)$ and instead of x_0, we take $x_0^{**} \in X^{**} \setminus X$. By Bishop-Phelps Theorem, we can choose such an x_0^{**} to be norm attaining. Let $x_0^* \in S(X^*)$ such that $x_0^{**}(x_0^*) = 1$. Let $r > 0$.

Since $x_0^{**} \in X^{**} \setminus X$, the subspace $\ker(x_0^{**})$ is norming, that is, $\overline{B(\ker(x_0^{**}))}^{w^*}$ contains some ball (say, $bB(X^*)$) centered at the origin. Fix $\delta > 0$ and let

$$G = (1+r)x_0^* + r_1 B(\ker(x_0^{**})), \qquad \text{where } r_1 = (1+r+\delta)/b.$$

It follows that $\overline{G}^{w^*} \supseteq \delta B(X^*)$.

Let $R > 0$ be so large that $RB(X^*) \supseteq G$. Since X is separable, the w*-topology is metrizable on any bounded subset. Let $d(x^*, y^*)$ be a metric on $RB(X^*)$ which gives the w*-topology on $RB(X^*)$. Let $\{u_i^*\}$ be w*-dense in $\delta B(X^*)$.

Coming to the inductive construction, take $z_1^* \in G \setminus \{(1+r)x_0^*\}$ and find $z_1 \in X$ with $z_1^*(z_1) = 1 > \sup z_1(B(X^*))$. Since the w*-open half-space $\{x^* \in X^* : x^*(z_1) < 1\}$ contains $\delta B(X^*)$ (in fact, $B(X^*)$) and since $\overline{G}^{w^*} \supseteq \delta B(X^*)$, it follows that $\overline{(G \cap \{x^* \in X^* : x^*(z_1) < 1\})}^{w^*} \supseteq \delta B(X^*)$. Hence there is an $z_2^* \in (G \cap \{x^* \in X^* : x^*(z_1) < 1\})$ with $d(u_2^*, z_2^*) < 1/2$. Put

$$L_2 = \{\sum_{i=1}^{2} \alpha_i z_i^* : \sum_{i=1}^{2} \alpha_i = 1\}.$$

It is clear that $L_2 \cap B(X^*) = \emptyset$ and hence, there is a $z_2 \in X$ with $z_2(L_2) = 1 > \sup z_2(B(X^*))$. As before, the w*-closure of $G \cap \{x^* \in X^* : x^*(z_1) < 1\} \cap \{x^* \in X^* : x^*(z_2) < 1\} \supseteq \delta B(X^*)$. Hence there is an $z_3^* \in (G \cap \{x^* \in X^* : x^*(z_1) < 1\} \cap \{x^* \in X^* : x^*(z_2) < 1\})$ with $d(u_3^*, z_3^*) < 1/3$.

Notice that from the construction of $\{z_i\}$ it follows that for each n, $\bigcap_{i=1}^{n} \{x^* \in X^* : x^*(z_i) < 1\} \supseteq B(X^*) \supseteq \delta B(X^*)$. This allows us to choose $z_i^* \in G \cap \bigcap_{k=1}^{i-1} \{x^* \in X^* : x^*(z_k) < 1\}$ with $d(z_i^*, u_i^*) < 1/i$, $i \geq 1$. The further construction is clear.

The rest of the proof runs along the lines of the proof of (a). □

Theorem 2.16. *Let X be an infinite dimensional Banach space. Then, for every $\varepsilon > 0$, there is an ε-equivalent norm and an unbounded nested sequence of balls $\{B_n = B(x_n, r_n)\}$ (in this norm) such that $B = \cup B_n$ is not a cone.*

Proof. Let $\{z_i\}$, $\{z_i^*\}$ and $|||\cdot|||$ be as given by Lemma 2.15 (a). Let $r_0 = 1$, $r_1 = 2$ and $z_0 = 0$. If $\{r_1, r_2, \ldots, r_n\}$ have been already constructed, put

$$r_{n+1} = \max \left\{ r_n + 1, \frac{\sum_{k=1}^{n} (r_k - r_{k-1}) z_{n+1}^*(z_{k-1}) - r_n z_{n+1}^*(z_n) + n + 1}{1 - z_{n+1}^*(z_n)} \right\}. \tag{3}$$

Put

$$x_n = \sum_{k=1}^{n} (r_k - r_{k-1}) z_{k-1}$$

for every $n \geq 1$. Since $|||x_{n+1} - x_n||| = r_{n+1} - r_n$, it is clear that the sequence of balls $\{B_n = B(x_n, r_n)\}$ is nested. It is also clear from the definition of r_n that the sequence is unbounded.

Let us estimate $\inf z_m^*(B)$ for $m \geq 1$.

$$\begin{aligned}
\inf z_m^*(B) &= \inf_{n\geq 1} \inf z_m^*(B_n) = \inf_{n\geq m+1} (z_m^*(x_n) - r_n) \\
&= \inf_{n\geq m+1} \left(\sum_{k=1}^{n} (r_k - r_{k-1}) z_m^*(z_{k-1}) - r_n \right) \\
&= \inf_{n\geq m+1} \left(\sum_{k=1}^{m} (r_k - r_{k-1}) z_m^*(z_{k-1}) + \sum_{k=m+1}^{n} (r_k - r_{k-1}) - r_n \right) \\
&= \inf_{n\geq m+1} \left(\sum_{k=1}^{m} (r_k - r_{k-1}) z_m^*(z_{k-1}) - r_m \right) \\
&= \sum_{k=1}^{m-1} (r_k - r_{k-1}) z_m^*(z_{k-1}) - r_{m-1} z_m^*(z_{m-1}) - (1 - z_m^*(z_{m-1})) r_m.
\end{aligned}$$

By (3), we get

$$\inf z_m^*(B) \leq -m, \quad m \geq 1,$$

which, by Lemma 2.8, proves that B is not a cone. □

Combining the above result with Proposition 2.7, we obtain

Corollary 2.17. *A Banach space X is finite dimensional if and only if for every equivalent norm, the union of each unbounded nested sequence of balls is a cone.*

Coming to the isometric question, recall that for $k \geq 1$, a Banach space X with $\dim(X) \geq k+1$ is k-rotund if given $(k+1)$ linearly independent points $x_1, x_2, \dots, x_{k+1} \in S(X)$, $\|x_1 + x_2 + \cdots + x_{k+1}\| < k+1$. This notion was introduced in [11] generalizing the notion of k-UR Banach spaces introduced by Sullivan in [14].

Corollary 2.18. *If X^* is k-rotund, then the union of any unbounded nested sequence of balls is a cone.*

Proof. Observe that X^* is k-rotund if and only if every face of $S(X^*)$ is the linear combination of at most k linearly independent points in $S(X^*)$.

Now, since for any unbounded nested sequence $\{B_n\}$ of balls in X, the set A is a face of $S(X^*)$, any maximal linearly independent subset of A contains at most k points. It then follows from the proof of Proposition 2.7 that B is a cone. □

And here is an example of an unbounded nested sequence of balls in c_0 (with its usual norm) whose union is not a cone.

Example 2.19. Let $X = c_0$. Let $\{e_n\}$ be the standard unit vector basis of c_0 and $\{e_n^*\}$ be the standard unit vector basis of ℓ_1.

Let $u_n = \sum_{k=1}^{n} e_k$ and $x_n = \sum_{k=1}^{n} u_k$. Then $\{u_n\} \subseteq S(X)$ and therefore the sequence $\{B_n = B(x_n, n)\}$ is nested. Let $B = \cup B_n$. Observe that every e_n^* is bounded below on B. Indeed,

$$\begin{aligned} \inf e_i^*(B_n) &= e_i^*(x_n) - n = \sum_{k=1}^{n} e_i^*(u_k) - n \;=\; \sum_{k=1}^{n}\sum_{j=1}^{k} e_i^*(e_j) - n \\ &= \begin{cases} \sum_{k=i}^{n} 1 - n & \text{if} \quad n \geq i \\ -n & \text{if} \quad n < i \end{cases} = \begin{cases} -i+1 & \text{if} \quad n \geq i \\ -n & \text{if} \quad n < i. \end{cases} \end{aligned}$$

It follows that

$$\inf e_i^*(B) = \inf_n \inf e_i^*(B_n) = -i+1,$$

and thus by Lemma 2.8, B is not a cone.

Similarly, $B^{**} = \cup B^{**}(x_n, n) \subseteq \ell_\infty$ also is not a cone.

Observe that $u_n \xrightarrow{\text{w}^*} \mathbf{1} = (1, 1, \ldots, 1, \ldots) \in \ell_\infty$, and therefore, $x_n/n \xrightarrow{\text{w}^*} \mathbf{1}$. It follows that $\{e_n^*\} \subseteq A \subseteq \{(a_n) \in S(\ell_1) : \sum_{n=1}^{\infty} a_n = 1\} = \{(a_n) \in S(\ell_1) : a_n \geq 0\}$. Therefore, $C = \{(\alpha_n) \in c_0 : \alpha_n > 0\} \neq \emptyset$. $\square$

Observe that if B is a cone, then $C \neq \emptyset$. On the other hand, in the above example, even though B is not a cone, $C \neq \emptyset$.

Indeed, for $X = c_0$ with its usual norm, given any $x_0^{**} \in S(\ell_\infty)$ restricting it to its first n coordinates, it is always possible to get a sequence $\{y_n\} \subseteq c_0$ such that $\|y_n - x_0^{**}\| \leq 1$ for all $n \geq 1$ and $\text{w}^*\text{-}\lim y_n = x_0^{**}$. This implies that $C \neq \emptyset$, as we can see from the following lemma.

Lemma 2.20. *Let $\{B_n = B(x_n, r_n)\}$ be an unbounded nested sequence of balls in X. Let B, A and C be as before. Let $x_0^{**} \in B(X^{**})$ is a w*-cluster point of the sequence $\{x_n/r_n\} \subseteq B(X)$. Then each of the following conditions is sufficient for C to be nonempty :*

- (*a*) $0 \notin \overline{A}^{\text{w}^*}$. *In particular, if A is w*-closed, then $C \neq \emptyset$.*
- (*b*) $d(x_0^{**}, X) < 1$. *In particular, if X is reflexive, then $C \neq \emptyset$.*
- (*c*) $\|x_n/r_n - x_0^{**}\| \leq 1$ *for all sufficiently large n.*
- (*d*) *there exists a sequence $\{y_n\} \subseteq X$ such that $\|y_n - x_0^{**}\| \leq 1$ for all sufficiently large $n \geq 1$ and x_0^{**} is a w*-cluster point of $\{y_n\}$. In particular, if X is an M-ideal in X^{**}, or more generally, if the canonical projection P from X^{***} to X^* satisfies $\|I - P\| \leq 1$, then $C \neq \emptyset$.*

Proof. Observe that $z \in C$ if and only if $A \subseteq \{x^* \in X^* : x^*(z) > 0\}$. So, if $0 \notin \overline{A}^{\text{w}^*}$, then $C \neq \emptyset$.

If $x_0^{**} \in B(X^{**})$ is a w*-cluster point of $\{x_n/r_n\}$, then $A \subseteq D(x_0^{**}) \cap X^*$. It follows that $B^{**}(x_0^{**}, 1) \cap X \subseteq C$.

Therefore, if $d(x_0^{**}, X) < 1$, then $C \neq \emptyset$. In particular, if $x_0^{**} \in X$, that is, if the sequence $\{x_n/r_n\} \subseteq B(X)$ has a weak cluster point $x_0^{**} \in B(X)$, then $C \neq \emptyset$. Observe that this happens if X is reflexive.

If $\|x_n/r_n - x_0^{**}\| \leq 1$ for sufficiently large n, we may assume it happens for all $n \geq 1$. Then for all $x^* \in A$,

$$x^*\left(\frac{x_n}{r_n}\right) \geq x^*(x_0^{**}) - \left\|\frac{x_n}{r_n} - x_0^{**}\right\| \geq 0.$$

Moreover, by (2), for all $x^* \in A$,

$$x^*\left(\frac{x_n}{r_n}\right) \geq 1 + \frac{c}{r_n} > \frac{1}{2}$$

for sufficiently large n (of course, n depends on x^*). Define

$$x_0 = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{x_n}{r_n}.$$

Now it is clear that $x^*(x_0) > 0$ for all $x^* \in A$. That is, $x_0 \in C$.

Observe that for the above argument to work, all we need is a sequence $\{y_n\} \subseteq X$ such that $\|y_n - x_0^{**}\| \leq 1$ for all $n \geq 1$ and x_0^{**} is an w*-cluster point of $\{y_n\}$. Then,

$$y_0 = \sum_{n=1}^{\infty} \frac{1}{2^n} y_n \in C.$$

Observe that if X is an M-ideal in its bidual, then by [10, Proposition III.1.9], the above condition is satisfied. And if the canonical projection P from X^{***} to X^* is such that $\|I - P\| \leq 1$, then also by [9, Proposition 2.3], the above condition is satisfied. □

Thus, for every unbounded nested sequence of balls in a reflexive space, C is non-empty. How about non-reflexive spaces? By arguments similar to Proposition 2.11, it follows that if the centers of the nested sequence of balls lie in a reflexive subspace, then also C is nonempty. However, this need not be generally the case. Indeed, we have

Theorem 2.21. *In every non-reflexive Banach space, there is an equivalent norm and an unbounded nested sequence of balls $\{B_n\}$ (in this norm) such that $C = \emptyset$.*

Proof. If X is separable, we use the construction of Theorem 2.16 with Lemma 2.15 (b), to obtain $\overline{A}^{w^*} \supseteq \delta B(X^*)$ for some $\delta > 0$. Then it is not difficult to see that $C = \emptyset$.

If X is not separable, let $Y \subseteq X$ be a separable non-reflexive subspace. As above, there is an equivalent norm on Y and an unbounded nested sequence $\{B_n = B(y_n, r_n)\}$ of balls in Y (in this norm) such that $\overline{A_Y}^{w^*} \supseteq \delta B(Y^*)$ for some $\delta > 0$. Get an equivalent norm on X that extends the new norm on Y. Consider the unbounded nested sequence $\{B(y_n, r_n)\}$ of balls in X.

Let $D \subseteq S(X^*)$ consist of a norm preserving extension of each $y^* \in A_Y$. Then by Lemma 2.10, $A_X = (D + Y^\perp) \cap B(X^*)$. Observe that, in fact, $A_X = (D + 2B(Y^\perp)) \cap B(X^*)$ [if $x^* \in A_X$, $x^* = x_1^* + x_2^*$, $x_1^* \in D$, $x_2^* \in Y^\perp$, then $\|x_2^*\| = \|x^* - x_1^*\| \leq \|x^*\| + \|x_1^*\| \leq 2$].

Let $i : Y \to X$ be the natural embedding, then $i^*(D) = A_Y$ and hence, $i^*(\overline{D}^{w^*}) \supseteq \overline{A_Y}^{w^*} \supseteq \delta B(Y^*)$. Thus for some $\eta \in (0,1)$, $\eta B(X^*) \subseteq i^{*-1}(\delta B(Y^*)) \subseteq \overline{D}^{w^*} + Y^\perp$. As before, $\eta B(X^*) \subseteq \overline{D}^{w^*} + 2B(Y^\perp)$. It follows that $\overline{A_X}^{w^*} = [\overline{D}^{w^*} + 2B(Y^\perp)] \cap B(X^*) \supseteq \eta B(X^*)$. As before, $C = \emptyset$. □

By combining this theorem and Lemma 2.20, we have

Corollary 2.22. *A Banach space is reflexive if and only if for every equivalent norm and for each unbounded nested sequence of balls, $C \neq \emptyset$.*

If an equivalent norm appears a bit artificial in the above result, here is an unbounded nested sequence of balls in ℓ_1 (with its usual norm) such that $C = \emptyset$.

Example 2.23. Let $X = \ell_1$, and $\{u_n\}$ be the standard unit vector basis of ℓ_1.

Let $x_n = \sum_{k=1}^{n} u_k$. Since $\{u_n\} \subseteq S(X)$, the sequence $\{B_n = B(x_n, n)\}$ is nested. Let $B = \cup B_n$, and $a^* = (a_n) \in S(\ell_\infty)$.

$$\inf a^*(B_n) = a^*(x_n) - n = \sum_{k=1}^{n} a^*(u_k) - n = -\sum_{k=1}^{n} (1 - a_k).$$

It follows that

$$A = \{(a_n) \in S(\ell_\infty) : \sum_{n=1}^{\infty} (1 - a_n) < \infty\}.$$

Claim : $C = \emptyset$.

Let $x = (\alpha_n) \in S(\ell_1)$. Choose $N \geq 1$ such that $\sum_{n=N+1}^{\infty} |\alpha_n| < 1/3$. Take

$$a^* = (-\text{sgn}(\alpha_1), -\text{sgn}(\alpha_2), \ldots, -\text{sgn}(\alpha_N), 1, \ldots .1, \ldots) \in A,$$

and observe that

$$a^*(x) = \sum_{n=1}^{\infty} a_n \alpha_n = -\sum_{n=1}^{N} |\alpha_n| + \sum_{n=N+1}^{\infty} \alpha_n = -1 + \sum_{n=N+1}^{\infty} (|\alpha_n| + \alpha_n) < -\frac{1}{3}.$$

Therefore, $x \notin C$. □

Definition 2.24. Let us say that a Banach space X has Property C_1 (resp. C_2) if for every unbounded nested sequence $\{B_n\}$ of balls in X, $B = \cup B_n$ is a cone (resp. $C \neq \emptyset$).

Then the main observations of this section can be summarized as :

Proposition 2.25. (*a*) *Finite dimensional Banach spaces and Banach spaces whose duals are k-rotund for some k, have Property C_1.*

(*b*) *Banach spaces with Property C_1, reflexive Banach spaces and Banach spaces that are M-ideal in their bidual have Property C_2.*

(*c*) *ℓ_1 with its usual norm fails Property C_2.*

(*d*) *Any infinite dimensional Banach space can be equivalently renormed to fail Property C_1.*

(*e*) *Any non-reflexive Banach space can be equivalently renormed to fail Property C_2.*

(*f*) *If every separable subspace of a Banach space X has Property C_1 (resp. C_2), then X has Property C_1 (resp. C_2).*

Remark 2.26. By the argument of Corollary 2.18, the condition "every face of $S(X^*)$ is the linear combination of finitely many linearly independent points in $S(X^*)$" is sufficient for X to have Property C_1. Is this also necessary? Is there a characterization of this property similar to k-rotundity? Observe that in the case of a straight nested sequence of balls in the direction of x_0, the union is a cone, but $A = D(x_0)$ may have infinitely many linearly independent elements. For example, take $x_0 = \mathbf{1} \in \ell_\infty$. Then $D(\mathbf{1})$ contains all $e_n \in \ell_1$. So for a given nested sequence of balls, this condition is not necessary for the union to be a cone.

3. A variant of k-rotundity

As we mentioned in the introduction, Vlasov [15] characterized the strict convexity of the dual of a Banach space in terms of the union of unbounded nested sequence of balls. Beauzamy and Maurey [6] obtained similar characterization of smoothness in terms of the union of straight unbounded nested sequence of balls.

Our aim in this section is to generalize these results. Concretely, we relate the fact that the union of every (resp. every straight) unbounded nested sequence of balls is the intersection of at most k half-spaces with the fact that every face (resp. certain faces) of the dual ball is the convex hull of at most k extreme points.

First we need some lemmas whose give us more information about the set A of functionals that are bounded below on the union. Let us fix a Banach space X, an unbounded nested sequence of balls $B_n = B(z_n, r_n)$, and let $B = \cup B_n$, and A be as above.

Lemma 3.1. *Let $A' \subseteq A$ be such that*

$$B = \bigcap_{x^* \in A'} \{x \in X : x^*(x) > \inf x^*(B)\}.$$

Then

$$\begin{aligned}\bigcap_{x^* \in A'} \ker x^* &= \{y \in X : x + \lambda y \in B, \text{ for all } \lambda \in \mathbb{R}, \text{ for all } x \in B\} \\ &= \{y \in X : x_0 + \lambda y \in B, \text{ for all } \lambda \in \mathbb{R}, \text{ for some } x_0 \in B\}.\end{aligned}$$

In particular,

$$\bigcap_{x^* \in A} \ker x^* = \bigcap_{x^* \in A'} \ker x^*.$$

The proof is straightforward.

Lemma 3.2. *Let $A' \subseteq A$ be a finite set such that*

$$B = \bigcap_{x^* \in A'} \{x \in X : x^*(x) > \inf x^*(B)\}.$$

Then $A = co(A')$.

Proof. Let $A' = \{x_1^*, \dots, x_n^*\}$. Let $x^* \in A$. By Lemma 3.1, $\ker x^* \supseteq \bigcap_{i=1}^n \ker x_i^*$, and therefore, there exist $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $x^* = \sum_{i=1}^n \lambda_i x_i^*$.

By Lemma 2.6, $\sum_{i=1}^n \lambda_i = 1$ and $\inf x^*(B) = \sum_{i=1}^n \lambda_i \inf x_i^*(B)$. We will show that $\lambda_i \geq 0$ for all $i = 1, \dots, n$.

Suppose not. For notational simplicity, suppose $\lambda_n < 0$.

If $x_n^* \in \mathrm{span}\{x_1^*, \dots, x_{n-1}^*\}$, then by Lemma 2.6, it is in the affine hull of $\{x_1^*, \dots, x_{n-1}^*\}$, and therefore, so is x^*. So we may assume that $x_n^* \notin \mathrm{span}\{x_1^*, \dots, x_{n-1}^*\}$.

Claim : There exists $x_0 \in X$ such that $x_i^*(x_0) = \inf x_i^*(B)$ for $i = 1, \dots, n-1$ and $x_n^*(x_0) > \inf x_n^*(B)$.

By Lemma 2.6, the map $x^* \to \inf x^*(B)$ restricted to A is linear. Thus, there exists $x_1 \in X$ such that $x_i^*(x_1) = \inf x_i^*(B)$ for $i = 1, \dots, n-1$. Moreover, by the above assumption, $\ker x_n^* \neq \bigcap_{i=1}^{n-1} \ker x_i^*$. So there exists $x_2 \in \bigcap_{i=1}^{n-1} \ker x_i^*$ such that $x_n^*(x_2) \neq 0$. Then for a suitable $\lambda \in \mathbb{R}$, $x_0 = x_1 + \lambda x_2$ satisfies the claimed conditions.

Since $\overline{B} = \bigcap_{i=1}^{n} \{x \in X : x_i^*(x) \geq \inf x^*(B)\}$, $x_0 \in \overline{B}$ and so, $x^*(x_0) \geq \inf x^*(B)$. But,

$$\begin{aligned} x^*(x_0) &= \sum_{i=1}^{n} \lambda_i x_i^*(x_0) = \sum_{i=1}^{n-1} \lambda_i \inf x_i^*(B) + \lambda_n x_n^*(x_0) \\ &< \sum_{i=1}^{n-1} \lambda_i \inf x_i^*(B) + \lambda_n \inf x_n^*(B) = \inf x^*(B). \end{aligned}$$

A contradiction ! □

Remark 3.3. Observe that since A' is finite, A is compact and A' contains the extreme points of A. And since A is a face of $B(X^*)$, these points are extreme points of $B(X^*)$ as well.

Now we can state the main results of this section.

Theorem 3.4. *For a Banach space X, and $k \geq 1$, the following are equivalent :*

(a) *For every $x_1^*, \dots, x_m^* \in S(X^*)$ with $\|x_1^* + \dots + x_m^*\| = m$, there exist $y_1^*, \dots, y_k^* \in S(X^*)$ such that $x_1^*, \dots, x_m^* \in \mathrm{co}(y_1^*, \dots, y_k^*)$.*

(b) *Every face (convex extremal set) F of $S(X^*)$ is the convex hull of at most k extreme points of $B(X^*)$.*

(c) *The union of every unbounded nested sequence of balls in X is the intersection of at most k half-spaces.*

Proof. $(a) \Rightarrow (b)$. Let F be a face of $S(X^*)$. By (a), any maximal affine independent subset of F has at most k points. So F is contained in the affine hull of k points. Then, by [7, Theorem 2.8], F is closed, in fact, compact and hence, is the closed convex hull of its extreme points. Now, we use (a) once more to deduce that F has at most k extreme points. And since F is a face of $B(X^*)$, these points are extreme points of $B(X^*)$ as well.

$(b) \Rightarrow (c)$. Consider an unbounded nested sequence of balls $\{B_n\}$ and define A as before. If $A = \emptyset$, $B = X$ and there is nothing to prove. And if $A \neq \emptyset$, by Lemma 2.3, A is a face of $B(X^*)$ and by (b), A is the convex hull of at most k extreme points of $B(X^*)$. Since A is a face, these extreme points are actually in A. If $A = \mathrm{co}\{x_1^*, \dots, x_m^*\}$ with $m \leq k$, then we have clearly

$$B = \bigcap_{i=1}^{m} \{x \in X : x_i^*(x) > \inf x_i^*(B)\}.$$

$(c) \Rightarrow (a)$. Let $x_1^*, x_2^*, \dots, x_m^* \in S(X^*)$ be such that $x^* = (x_1^* + x_2^* + \dots + x_m^*)/m \in S(X^*)$.

Choose a sequence $\{\delta_n\}$ such that $\delta_n > 0$ for all n and $\sum_{n=1}^{\infty} \delta_n < \infty$. Let $\{x_n\} \subseteq B(X)$ be such that $x^*(x_n) > 1 - \delta_n$.

Let $B_n = B\left(\sum_{j=1}^{n} x_j, n\right)$. Clearly $\{B_n\}$ is an unbounded nested sequence of balls. Define A as before. Observe that for any $n \geq 1$,

$$\inf x^*(B_n) = x^*\left(\sum_{j=1}^{n} x_j\right) - n = \sum_{j=1}^{n} (x^*(x_j) - 1) > -\sum_{j=1}^{n} \delta_j \geq -\sum_{j=1}^{\infty} \delta_j > -\infty.$$

That is, $x^* \in A$. Since A is a face, $x_1^*, x_2^*, \ldots, x_m^* \in A$.

Now, by (c), there are $y_1^*, \ldots, y_k^* \in A$ such that

$$B = \bigcap_{i=1}^{k} \{x \in X : y_i^*(x) > \inf y_i^*(B)\}.$$

By Lemma 3.2, $A = \mathrm{co}(y_1^*, \ldots, y_k^*)$. In particular, $x_1^*, x_2^*, \ldots, x_m^* \in co(y_1^*, \ldots, y_k^*)$, as desired. ☐

Remark 3.5. Condition (b) above clearly implies X^* is k-rotund, and therefore, the union of any unbounded nested sequence of balls is also a cone. It is not difficult to see that, in general, it is strictly stronger than k-rotundity of X^*.

We conclude this section by extending the result of Beauzamy and Maurey [6] characterizing smooth spaces in terms of the union of straight unbounded nested sequence of balls. The proof is almost same as the one above.

Corollary 3.6. *For a Banach space X, $x_0 \in S(X)$ and $k \geq 1$, the following are equivalent :*

- (a) *$D(x_0)$ is the convex hull of at most k extreme points of $B(X^*)$.*
- (b) *The union of every straight unbounded nested sequence of balls in the direction of x_0 is the intersection of at most k half-spaces.*

Proof. Observe that for every straight unbounded nested sequence of balls in the direction of x_0, the set A is either empty or equals either $D(x_0)$ or $-D(x_0)$.

For the other direction, observe that in this case, in the proof of $(c) \Rightarrow (a)$ above, it is possible to take $\delta_n = 0$ and $x_n = x_0$ for $n \geq 1$. ☐

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