SOME RECENT RESULTS ON THE NUMERICAL INDEX OF A BANACH SPACE

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1. Introduction

The numerical index of a Banach space is a constant relating the norm and the numerical range of operators on the space. The notion of numerical range was first introduced by O. Toeplitz in 1918 [16] for matrices, and it was extend in the sixties to operators on an arbitrary Banach space by F. Bauer [1] and G. Lumer [12]. Let us recall the relevant definitions. Given a real or complex Banach space X, we write B_X for the closed unit ball and S_X for the unit sphere of X. The dual space will be denoted by X^*, and $L(X)$ will be the Banach algebra of all bounded linear operators on X. The *numerical range* of an operator T in $L(X)$ is the subset $V(T)$ of the scalar field defined by

$$V(T) = \{x^*(Tx) \ : \ x \in S_X, \ x^* \in S_{X^*}, \ x^*(x) = 1\}.$$

The *numerical radius* of T is then given by

$$v(T) = \sup\{|\lambda| \ : \ \lambda \in V(T)\}.$$

It is clear that v is a seminorm on $L(X)$, and $v(T) \leq \|T\|$ for every $T \in L(X)$. Quite often, v is actually a norm and it is equivalent to the operator norm. Thus it is natural to consider the so called *numerical index* of the space X, namely the constant $n(X)$ defined by

$$n(X) = \inf\{v(T) \ : \ T \in S_{L(X)}\}.$$

Equivalently, $n(X)$ is the greatest constant $k \geq 0$ such that $k\|T\| \leq v(T)$ for every $T \in L(X)$. Note that $0 \leq n(X) \leq 1$, and $n(X) > 0$ if and only if v and $\|\cdot\|$ are equivalent norms (the numerical radius can be a non-equivalent norm on $L(X)$ –see [14, Example 3.b]–). Clearly, $n(X) = 1$ when the operator norm and the numerical radius agree on $L(X)$. This is the case of some classical spaces like $L_1(\mu)$ and $C(K)$ [5]. For general information and background we refer to [3, 4] and to the survey paper [13]. Recent results can be found in [6, 11, 14, 15] and the references therein.

The concept of numerical index was first suggested by G. Lumer in 1968. At that time, it was known that a Hilbert space of dimension

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greater than 1 has numerical index 1/2 in the complex case, and 0 in the real case. Two years later, J. Duncan, C. McGregor, J. Pryce, and A. White [5] determined the range of values of the numerical index. More precisely, for a real Banach space X, $n(X)$ can be any number in the interval $[0,1]$, while $\{n(X) : X \text{ complex Banach space}\} = [1/e, 1]$. The remarkable result that $n(X) \geq 1/e$ for every complex Banach space X goes back to H. Bohnenblust and S. Karlin [2] (see also [7])

Let us mention here some facts concerning the numerical index which will be relevant to our discussion. For instance, one has $v(T^*) = v(T)$ for every $T \in L(X)$, where T^* is the adjoint operator of T (see [3, §9]), and it clearly follows that $n(X^*) \leq n(X)$ for every Banach space X. The question if this is actually an equality seems to be open. We will prove a partial answer to this question in section 4.

The outline of the paper is as follows.

In §2 we expose some results of [14, 15] on the "stability" of the numerical index. We compute the numerical index of c_0-, l_1- and l_∞-sums of Banach spaces and we also compute the numerical index of some vector-valued function spaces.

Section 3 is devoted to comment some isomorphic results given in [6, 11]. First, On one hand, infinite-dimensional real reflexive spaces cannot be renormed to have numerical index 1. Second, "almost" every Banach space can be renormed to have any possible value of the numerical index but 1.

Finally, we prove in §4 that the dual of a real Banach space having RNP and numerical index 1, has also numerical index 1.

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2. "Stability" of the numerical index

The cited result of [5] that

$$n\big(C(K)\big) = n\big(L_1(\mu)\big) = 1$$

for every compact K and every positive measure μ, is generalized in [14, 15] to c_0-, l_1- and l_∞-sums of Banach spaces and to some spaces of vector-valued functions. In this section we expose these results.

Given an arbitrary family $\{X_\lambda : \lambda \in \Lambda\}$ of Banach spaces, let us denote by $[\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}$ (resp. $[\oplus_{\lambda\in\Lambda} X_\lambda]_{l_1}$, $[\oplus_{\lambda\in\Lambda} X_\lambda]_{l_\infty}$) the c_0-sum (resp. l_1-sum, l_∞-sum) of the family.

Proposition 1. ([14, Proposition 1]) *Let $\{X_\lambda :\ \lambda \in \Lambda\}$ be a family of Banach spaces. Then*

$$n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}\Big) = n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{l_1}\Big) = n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{l_\infty}\Big) = \inf_\lambda n(X_\lambda).$$

As an easy application of this proposition, one can exhibit an example of a real Banach space X such that the numerical radius is a norm on $L(X)$, but it is not equivalent to the operator norm, i.e. $n(X) = 0$ (see [14, Example 3.b].)

It is also possible to extend this result to spaces of vector-valued function spaces. Let us give the necessary definitions. Given a real or complex Banach space X, a compact Hausdorff space K, and a σ-finite measure μ, let $C(K, X)$ denotes the space of X-valued continuous functions on K, $L_1(\mu, X)$ denotes the space of X-valued μ-Bochner-integrable functions and finally, we write $L_\infty(\mu, X)$ for the space of X-valued μ-Bochner-measurable and essentially bounded functions.

Theorem 2. ([14, Theorems 5 and 8] and [15, Theorem 3]) *Let K be a compact Hausdorff space, and let μ be a σ-finite measure. Then*

$$n(C(K, X)) = n(L_1(\mu, X)) = n(L_\infty(\mu, X)) = n(X)$$

for every Banach space X.

Since $C(K, X) = C(K) \otimes_\varepsilon X$ and $L_1(\mu, X) = L_1(\mu) \otimes_\pi X$, one may wonder if the above result might be a special case of a general result giving $n(X \otimes_\varepsilon Y)$ and $n(X \otimes_\pi X)$ as a function of $n(X)$ and $n(Y)$. In [14, Example 10] it is proved that this is not the case. Indeed, using results by Å. Lima [10] we can prove that

$$n(l_1^4 \otimes_\pi l_1^4) = n(l_\infty^4 \otimes_\varepsilon l_\infty^4) = 1$$

and

$$n(l_1^4 \otimes_\varepsilon l_1^4) < 1, \quad n(l_\infty^4 \otimes_\pi l_\infty^4) < 1,$$

so it cannot exist such a general result for tensor products.

3. Isomorphic results

As the numerical index is strongly isometric, given a Banach space X, we consider the set of values of the numerical index of X when equipped with all equivalent norms. That is, we study the set

$$\mathcal{N}(X) = \{n(Y)\ :\ Y \simeq X\},$$

where $Y \simeq X$ means that X and Y are isomorphic.

In [6] appeared two "positive" results about this set. The first one applies to every Banach space.

Theorem 3. ([6, Theorem 9]) *Let X be a Banach space of dimension greater than one. Then $\mathcal{N}(X)$ is an interval containing $[0, 1/3[$ in the real case and $[e^{-1}, 1/2[$ in the complex case.*

The second result improves the first one, but it applies only to Banach spaces which admits a *long biorthogonal system* (v.g. separable or, more generally, WCG spaces).

Theorem 4. ([6, Theorem 10]) *Let X be a Banach space admitting a long biorthogonal system. Then* $\sup \mathcal{N}(X) = 1$. *Therefore,* $\mathcal{N}(X) \supset [0,1[$ *in the real case and* $\mathcal{N}(X) \supset [e^{-1},1[$ *in the complex case.*

With this on, the only interesting value of the numerical index (for an isomorphic point of view) is 1. Actually, it is proved in [11] that infinite-dimensional reflexive or quasi-reflexive real spaces cannot be renormed to have numerical index 1. Moreover,

Theorem 5. ([11, Corollary 5]) *Let X be an infinite-dimensional real Banach space with $n(X) = 1$. Then X^{**}/X is non-separable.*

Joining the two above results, we obtain

Corollary 6. ([6, Corollary 11]) *Let X be an infinite-dimensional real Banach space with X^{**}/X separable. Then $\mathcal{N}(X) = [0,1[$.*

4. Passing the numerical index to the dual

It is clear that $V(T) \subseteq V(T^*)$ for every bounded linear operator T on a Banach space X, where T^* is the adjoint of T. Moreover, it follows easily from a result by Lumer [12, Lemma 12] that

$$\overline{\text{co}}V(T) = \overline{\text{co}}V(T^*),$$

where $\overline{\text{co}}$ denotes closed convex hull, and therefore, $v(T) = v(T^*)$. So, we have:

Proposition 7. [5, Proposition 1.3] *If X is a Banach space, then $v(T^*) = v(T)$ for every $T \in L(X)$. Therefore, $n(X^*) \leq n(X)$.*

As we have mention in the introduction, the question if the above inequality is actually an equality is open. The aim of this section is to give a partial answer to the question. To this end, we require the notion of semi L-summand introduced by Å. Lima (see [8, §5] and [9, §3]). A closed subspace J of a Banach space X is a *semi L-summand* if for every $x \in X$ there exists a unique $y \in J$ such that $\|x-y\| = d(x,J)$, and moreover this y satisfies $\|x\| = \|y\| + \|x-y\|$. We will use a result of [9] which is only valid in the real case.

Proposition 8. ([9, Theorem 3.1]) *Let X be a real Banach space and let $x \in \mathrm{ex}\,(B_X)$. Then* $\mathrm{span}(x)$ *is a semi L-summand if and only if $|x^*(x)| = 1$ for all $x^* \in \mathrm{ex}\,(B_{X^*})$.*

Now, we can state the main result of the section.

Proposition 9. *Let X be a real Banach space satisfying the RNP. If $n(X) = 1$, then $n(X^*) = 1$.*

Proof. Fix a denting point $x \in B_X$. By [11, Lemma 1], we have that $|x^*(x)| = 1$ for every extreme point x^* of B_{X^*}. Then, $\operatorname{span}(x)$ is a semi L-summand of X by Proposition 8 and therefore, $\operatorname{span}(x)$ is a semi L-summand of X^{**} by [8, Theorem 6.14]. Now, we can use Proposition 8 again to get $|x^{***}(x)| = 1$ for every $x^{***} \in \operatorname{ex}(B_{X^{***}})$. Summarizing, we have

$$(*) \qquad |x^{***}(x)| = 1$$

for every $x^{***} \in \operatorname{ex}(B_{X^{***}})$ and every denting point x of B_X.

Let $T \in L(X^*)$ and let $\varepsilon > 0$. By using that the unit ball of X^{**} is the weak*-closed convex hull of the set of denting points of B_X, we can take such a denting point x so that

$$\|T^*x\| > \|T\| - \varepsilon.$$

Then, we can find $x^{***} \in \operatorname{ex}(B_{X^{***}})$ such that

$$|x^{***}(T^*x)| = \|T^*x\| > \|T\| - \varepsilon.$$

This fact, together with (*), imply that $\|T^*\| \leq v(T^*)$. We finally use Proposition 7 to get $\|T\| = v(T)$ and then, $n(X^*) = 1$. $\square$

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