A survey on the numerical index of a Banach space*

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1 Introduction

The numerical index of a Banach space is a constant relating the norm and the numerical range of operators on the space. The notion of numerical range was first introduced by O. Toeplitz in 1918 [32] for matrices, but his definition applies equally well to operators on infinite-dimensional Hilbert spaces. Let H denote a Hilbert space with inner product $(\cdot|\cdot)$, and let S_H denote the unit sphere of H. The *numerical range* of a bounded linear operator T on H is the subset $W(T)$ of the scalar field defined by

$$W(T) = \{(Tx|x) \,:\, x \in S_H\}.$$

Some properties of the Hilbert space numerical range are discussed in the classical book of P. Halmos [17, §17]. Further developments can be found in a recent book of K. Gustafson and D. Rao [16].

The concept of numerical range for operators on general Banach spaces had to wait until the sixties, when distinct (but somehow equivalent) definitions were independently introduced by G. Lumer [25] and F. Bauer [4]. Although Lumer's paper has been the most important in the development of the subject, Bauer gave the most convenient definition, which we will use here. Given a real or complex Banach space X, we write B_X for the closed unit ball and S_X for the unit sphere of X. The dual space will be denoted by X^* and $L(X)$ will be the Banach algebra of all bounded linear operators on X. The *numerical range* of an operator T in $L(X)$ is the subset $V(T)$ of the scalar field defined by

$$V(T) = \{x^*(Tx) \,:\, x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1\}.$$

The *numerical radius* of T is then given by

$$v(T) = \sup\{|\lambda| \,:\, \lambda \in V(T)\}.$$

*Research partially supported by Spanish D.G.E.S. project no. PB96-1406, and by P.F.P.D., Junta de Andalucía Grant.

AMS Subject Class. (1991): 46B20, 47A12.

It is clear that v is a seminorm on $L(X)$, and $v(T) \leq \|T\|$ for every $T \in L(X)$. Quite often, v is actually a norm and it is equivalent to the operator norm. Thus it is natural to consider the so called *numerical index* of the space X, namely the constant $n(X)$ defined by

$$n(X) = \inf\{v(T) \; : \; T \in S_{L(X)}\}.$$

Equivalently, $n(X)$ is the greatest constant $k \geq 0$ such that $k\|T\| \leq v(T)$ for every $T \in L(X)$. Note that $0 \leq n(X) \leq 1$, and $n(X) > 0$ if and only if v and $\|\cdot\|$ are equivalent norms.

A complete survey on numerical ranges and their relations to spectral theory of operators can be found in the books by F. Bonsall and J. Duncan [7, 8] and we refer the reader to these books for general information and background. These books contain some basic information on the numerical index. After their publication many interesting results have been found and some intriguing questions remain open. Our plan here is to make an account of these new developments.

The outline of this paper is as follows. In §2 we summarize some known properties of the numerical index, and compute it for some concrete spaces. We discuss the results of [9, 12, 15, 26] on the range of variation of the numerical index, and the differences between the real and complex cases. Also, we show some "stability properties" of the numerical index for operations like c_0-, l_1-, and l_∞-sums, and compute the numerical index of some vector-valued function spaces. These results appear in [27]. In §3 the facts known about spaces with numerical index 1 are exposed, specially those given in [28] and [24]. In §4 we discuss some geometrical properties of Banach spaces implying numerical index 1, and the relation between them. Finally, in §5 we list some remarks and open problems.

Acknowledgements: The author would like to express his gratitude to his supervisor, Dr. Rafael Payá, and to Dr. Ginés López for their day-to-day help and valuable suggestions. Also, it is his pleasure to thank the organizers of Jarandilla's meeting on Banach spaces for inviting him to write this survey.

2 The numerical index of a Banach space

The concept of numerical index was first suggested by G. Lumer in a lecture to the North British Functional Analysis Seminar in 1968. At that time, it was known that if H is a complex Hilbert space of dimension greater than 1, then $n(H) = 1/2$ (see [17, Page 114] for example.)

The real case is different. In a real Hilbert space H with dimension greater than 1 it is easy to build a norm-one operator T such that Tx is orthogonal to x for every $x \in S_H$, so $W(T) = \{0\}$ and $n(H) = 0$.

One of the most striking results on the numerical index is the fact that no complex Banach space may have numerical index 0. Lumer [25] proved that $\|T\| \leq 4v(T)$ for every bounded linear operator T on a complex Banach space X, so $n(X) \geq 1/4$. In 1970 B. Glickfeld [15] improved this estimate by just writing in terms of the numerical radius an inequality due to F. Bohnenblust and S. Karlin [5].

Theorem 1. [15, Theorem 1.4] *Let X be a complex Banach space. Then*

$$\|T\| \leq e\,v(T)$$

for all $T \in L(X)$. Equivalently, $n(X) \geq e^{-1}$.

Glickfeld also proves in [15] that e^{-1} is the best possible constant in a strong sense: there is a complex Banach space X and an operator $T \in L(X)$ such that $\|T\| = 1$ and $v(T) = e^{-1}$. Therefore, $n(X) = e^{-1}$ and the infimum defining $n(X)$ is attained. Finally, J. Duncan, C. McGregor, J. Pryce, and A. White [12] determined (also in 1970) the range of variation of the numerical index.

Theorem 2. [12, Theorems 3.5 and 3.6] *For every $t \in [0,1]$ (resp. $t \in [e^{-1},1]$), there is a real (resp. complex) Banach space X such that $n(X) = t$. Actually, X can be taken to be two-dimensional.*

The somehow surprising appearance of the number e in this world was due to the use of holomorphic techniques in the proof of the inequality by Bohnenblust and Karlin (see [5] for details).

Let us make one more remark on the difference between the real and complex cases. It is easy to show that the real space $X_{\mathbb{R}}$ underlying a complex Banach space X satisfies $n(X_{\mathbb{R}}) = 0$. Actually, $x \mapsto ix$ has numerical radius 0 when viewed as an operator on $X_{\mathbb{R}}$.

In 1972, G. Lumer [26] obtained a "universal" upper-bound for the norm of an operator in terms of the numerical radius of its powers. The result is interesting only in the real case.

Theorem 3. [26, Theorem 1] *There are constants c_1, c_2 such that for any Banach space X, one has*

$$\|T\| \leq c_1\,v(T) + c_2\,\left(v(T^2)\right)^{\frac{1}{2}} \quad \forall T \in L(X).$$

It follows from [9, Example 2.3] that one can take $c_1 = c_2 = 4$.

What about the numerical index of classical Banach spaces? Well, computing the numerical index of concrete spaces may be hard. For instance, the numerical index of l_p ($p \neq 1,2,\infty$) is yet unknown. However, there are some classical spaces whose numerical indexes have been calculated in the literature. In [12], the authors gave the first example of a Banach space such that the norm and the numerical radius coincide for all operators on it, that is, a space with numerical index 1: $C(K)$, the Banach space of continuous scalar-valued functions on a compact Hausdorff space K [12, Theorem 2.1]. To find new examples, we can look at the relation between the numerical indexes of a Banach space and its dual.

It is clear that $V(T) \subseteq V(T^*)$ for every bounded linear operator T on a Banach space X, where T^* is the adjoint of T. Moreover, it follows easily from a result by Lumer [25, Lemma 12] that

$$\overline{\text{co}}V(T) = \overline{\text{co}}V(T^*),$$

where $\overline{\text{co}}$ denotes closed convex hull, and therefore, $v(T) = v(T^*)$. Later on, by using a refinement of the Bishop-Phelps Theorem, B. Bollobás [6] proved that

$$V(T^*) \subseteq \overline{V(T)}.$$

We can now state:

Proposition 4. [12, Proposition 1.3] *If X is a Banach space, then*

$$n(X^*) \leq n(X).$$

The question if the above inequality is actually an equality seems to be open.

Back to the examples, [12, Theorem 2.2] gives us two families of Banach spaces with numerical index 1: L-spaces and M-spaces. Indeed, the dual of an L-space and the bidual of an M-space are isometric to a space of continuous functions on some compact Hausdorff space, and the above proposition applies.

It is natural to ask for the behaviour of the numerical index under some operations. It is shown in [27] that the numerical index of a c_0-, l_1-, or l_∞-sum of Banach spaces can be computed in the expected way. Given an arbitrary family $\{X_\lambda : \lambda \in \Lambda\}$ of Banach spaces, let us denote by $[\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}$ (resp. $[\oplus_{\lambda\in\Lambda} X_\lambda]_{l_1}$, $[\oplus_{\lambda\in\Lambda} X_\lambda]_{l_\infty}$) the c_0-sum (resp. l_1-sum, l_∞-sum) of the family.

Proposition 5. [27, Proposition 1] *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of Banach spaces. Then*

$$n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}\Big) = n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{l_1}\Big) = n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{l_\infty}\Big) = \inf_\lambda n(X_\lambda).$$

As an easy application of this proposition, one can exhibit an example of a real Banach space X such that the numerical radius is a norm on $L(X)$, but it is not equivalent to the operator norm, i.e. $n(X) = 0$ (see [27, Example 2.b].)

The numerical index of some vector-valued function spaces was also computed in [27]. Given a real or complex Banach space X, a compact Hausdorff space K, and a positive measure μ, let $C(K,X)$ (resp. $L_1(\mu,X)$) denote the space of X-valued continuous functions on K (resp. X-valued μ-Böchner-integrable functions).

Theorem 6. [27, Theorems 5 and 8] *Let K be a compact Hausdorff space, and let μ be a positive measure. Then*

$$n(C(K,X)) = n(L_1(\mu,X)) = n(X)$$

for every Banach space X.

Let us finally mention one further example of a Banach space with numerical index 1, namely the disk algebra (see [9, Theorem 3.3]). It actually follows from a recent result of D. Werner [33] that all function algebras have numerical index 1.

3 Banach spaces with numerical index 1

A Banach space X has numerical index 1 if and only if for every $T \in L(X)$ the norm of T can be evaluated by

$$\|T\| = \sup\{|x^*(Tx)| \,:\, x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1\}.$$

What are the consequences of this property on the geometry (or the topology) of a Banach space? For instance, is it possible to find an infinite-dimensional reflexive Banach space having numerical index 1? or, which infinite-dimensional Banach spaces have (or can be re-normed to have) the property? We will try to partially answer these questions here.

In the finite-dimensional context, a satisfactory characterization of spaces with numerical index 1 was given by C. McGregor in 1971. Denote by $\mathrm{ex}(B)$ the set of extreme points in the convex set B.

Theorem 7. [28, Theorem 3.1] *A finite-dimensional space X satisfies $n(X) = 1$ if and only if $|x^*(x)| = 1$ for every $x \in \mathrm{ex}(B_X)$ and $x^* \in \mathrm{ex}(B_{X^*})$.*

It is not clear how to carry this property to the infinite-dimensional context, because for infinite-dimensional X, $\mathrm{ex}(B_X)$ may be empty (e.g. c_0). One could reformulate McGregor's condition in a natural way: $|x^{**}(x^*)| = 1$ for every $x^* \in \mathrm{ex}(B_{X^*})$ and every $x^{**} \in \mathrm{ex}(B_{X^{**}})$. It is easy to show that this condition is sufficient to ensure $n(X) = 1$, but we do not know if it is also necessary.

In a recent paper [24], McGregor's result has been extended to the infinite-dimensional context by considering denting points instead of general extreme points. Recall that $x_0 \in B_X$ is said to be a *denting point* of B_X if it belongs to slices of B_X with arbitrarily small diameter. If X is a dual space and the slices can be taken to be defined by w^*-continuous functionals, then we say that x_0 is a *w^*-denting point*.

Lemma 8. [24, Lemma 1] *Let X be a Banach space with numerical index* 1. *Then:*

(i) $|x^{**}(x^*)| = 1$ *for every* $x^{**} \in \mathrm{ex}(B_{X^{**}})$ *and every* w^**-denting point* $x^* \in B_{X^*}$.

(ii) $|x^*(x)| = 1$ *for every* $x^* \in \mathrm{ex}(B_{X^*})$ *and every denting point* $x \in B_X$.

The above lemma can be combined with a useful sufficient condition for a *real* Banach space to contain a subspace isomorphic either to c_0 or to l_1, which follows easily from Rosenthal's l_1-Theorem [31] and Fonf's Theorem on containment of c_0 [13].

Proposition 9. [24, Proposition 2] *Let X be a real Banach space and assume that there is an infinite set $A \subset S_X$ such that $|x^*(a)| = 1$ for every $a \in A$ and all $x^* \in \mathrm{ex}(B_{X^*})$. Then X contains (an isomorphic copy of) either c_0 or l_1.*

The way to use Proposition 9 and Lemma 8 should be clear: take a real Banach space with numerical index 1 and infinitely many denting points (or w^*-denting points in its dual), and you obtain that the Banach space (or the dual) contains c_0 or l_1. A natural (isomorphic) assumption on an infinite-dimensional Banach space providing a lot of denting points is the Radon-Nikodým property (RNP for short). Actually the unit ball of a Banach space satisfying the RNP is the closed convex hull of its strongly exposed points, and strongly exposed points are denting. On the other hand, if X is an Asplund space (equivalently X^* has the RNP), then B_{X^*} is the w^*-closed convex hull of its w^*-strongly exposed (hence w^*-denting) points (see [29]). Therefore, we get:

Theorem 10. [24, Theorem 3] *Let X be an infinite-dimensional real Banach space with $n(X) = 1$. If X has the RNP, then X contains l_1. If X is an Asplund space, then X^* contains l_1.*

Note that the second part of the above theorem does not follow directly from the first one, because we require only $n(X) = 1$ and it is not known if this implies $n(X^*) = 1$.

Some interesting consequences of the above theorem are obtained by using the relationship between the RNP, containment of c_0 or l_1, reflexivity, etc. For instance, an Asplund space cannot contain l_1, so if X is a real Asplund space satisfying the RNP, and $n(X) = 1$, then X is finite-dimensional. As a special case, a reflexive or quasi-reflexive real Banach space with numerical index 1 must be finite-dimensional. Actually, if the quotient X^{**}/X is separable, it is known (see [11, page 219]) that X has the RNP and is an Asplund space. Therefore, if X is an infinite-dimensional real Banach space with $n(X) = 1$, then X^{**}/X is non-separable. All these results can be understood as necessary conditions for a Banach space to be re-normable with numerical index 1. We emphasize the following.

Corollary 11. *An infinite-dimensional real Asplund space with the RNP cannot be renormed to have numerical index* 1.

Unfortunately, we do not know how to extend the above results to the complex case. There, our knowledge of Banach spaces with numerical index 1 is too poor. As far as we know, an infinite-dimensional reflexive complex space might have numerical index 1.

4 CL-spaces and almost-CL-spaces

By using Theorem 7 and a result by Å. Lima [21, Corollary 3.7] we get that a finite-dimensional Banach space has numerical index 1 if and only if it is a CL-space. General CL-spaces were introduced by R. Fullerton in 1960 [14]. A Banach space is a *CL-space* if its unit ball is the absolutely convex hull of every maximal proper face. If the unit ball is the closed absolutely convex hull of every maximal proper face, we say, following Å. Lima [21], that the space is an *almost-CL-space*. $C(K)$ is a CL-space in the complex as well as in the real case. $L_1(\mu)$ is a CL-space in the real case but complex $L_1(\mu)$-spaces are only almost-CL-spaces. Actually, the complex space l_1 is an example of an almost-CL-space which is not a CL-space.

For the infinite-dimensional case, in 1990 M. D. Acosta proved that real CL-spaces have numerical index 1 (see [1] and [2, Teorema 5.5]). For the sake of completeness we include a simpler proof. Actually, we deal with real or complex almost-CL-spaces.

Proposition 12. *Let X be an almost-CL-space, then $n(X) = 1$.*

Proof. Every maximal proper face F of B_X is of the form

$$F = \{x \in B_X \ : \ x^*(x) = 1\}$$

for suitable $x^* \in S_{X^*}$. If $A \subseteq S_{X^*}$ is the set of those x^* generating maximal proper faces in the above sense, it is easy to see that B_{X^*} is the w^*-closed convex hull of A.

Now, given $T \in L(X)$ and $\varepsilon > 0$, let $x^* \in A$ be such that $\|T^*x^*\| > \|T^*\| - \varepsilon$, and find $x^{**} \in \mathrm{ex}(B_{X^{**}})$ satisfying

$$|x^{**}(T^*x^*)| = \|T^*x^*\| > \|T^*\| - \varepsilon.$$

We are left with only showing that $|x^{**}(x^*)| = 1$, for this will imply $v(T^*) > \|T^*\| - \varepsilon$. Since X is an almost-CL-space, B_X is the closed absolutely convex hull of the maximal proper face F defined by x^*, and it follows from Goldstine Theorem that $B_{X^{**}}$ is the w^*-closed absolutely convex hull of F. By the reversed Krein-Milman Theorem (see [10, Theorem 7.8], for example) every extreme point in $B_{X^{**}}$ belongs to the w^*-closure of F up to rotation, so $|x^{**}(x^*)| = 1$ as required. □

CL-spaces are related to an intersection property of balls introduced by J. Lindenstrauss in 1964 for real spaces [22], namely the *3.2. intersection property (3.2.I.P.)*.

A real Banach space has this property if every collection of three mutually intersecting closed balls has nonempty intersection. Å. Lima showed in 1977 [20] that every real Banach space with the 3.2.I.P. is a CL-space, but the converse is false even in the finite-dimensional case (see [18, Remark 3.6] and [22, Page 47]). It follows from Proposition 12 that every real Banach space with the 3.2.I.P. has numerical index 1. Moreover, the second assertion in Theorem 10 can be established for this kind of space without the Asplund assumption. Indeed, if X is an infinite-dimensional real Banach space with the 3.2.I.P., then X^* satisfies the 3.2.I.P. as well [20, Corollary 3.3], and we can apply a result by J. Lindenstrauss [22, Theorem 4.7] to get that $|x^{**}(x^*)| = 1$ for every $x^* \in \mathrm{ex}(B_{X^*})$ and every $x^{**} \in \mathrm{ex}(B_{X^{**}})$. Since $\mathrm{ex}(B_{X^*})$ is infinite, Proposition 9 shows that X^* contains either c_0 or l_1, but a dual space contains l_∞ (hence also l_1) as soon as it contains c_0. So we have proved

Proposition 13. [24, Corollary 7] *Let X be an infinite-dimensional real Banach space with the 3.2.I.P. Then X^* contains l_1.*

The fact that an infinite-dimensional real Banach space with the 3.2.I.P. cannot be reflexive was known to J. Lindenstrauss and R. Phelps in 1968 [23, Corollary 2.4].

To finish this section, we cite a result obtained by S. Reisner in 1991 [30], which emphasizes the difference between spaces with the 3.2.I.P. and CL-spaces. In 1981, A. B. Hannsen and Å. Lima had given a structure theorem for real finite-dimensional spaces with the 3.2.I.P. [19]: any such space is obtained from the real line by repeated l_1- and l_∞-sums. That is, it can be constructed in a finite sequence of steps, using only one type of "brick", which is the real line, and two "construction tools", l_1- and l_∞-sums. In [30], Reisner proved that nothing similar can be expected for CL-spaces. He showed that it does not exist a finite set of "bricks" which is sufficient to construct all finite-dimensional real CL-spaces by l_1 and l_∞ sums (see [30, Section 3] for details).

For more information and background on CL-spaces and 3.2.I.P. we refer the interested reader to [3], and to the already mentioned [14], [19], [20], [21], [22], and [30].

5 Remarks and problems

(a) *Power inequality and numerical index:* We say that the power inequality holds for a Banach space X if
$$v(T^n) \leq v(T)^n$$
for every $T \in L(X)$. This obviously happens if the numerical index of a Banach space is 1, and it is known (see [17] and [7]) that it also happens for every complex Hilbert space (in the real case, there exists an operator T so that $v(T) = 0$ and $v(T^2) > 0$ –see §2–). Is it then the case that the power inequality holds for spaces

whose numerical index is sufficiently close to 1? In [9, Section 3], this question was answered in the negative by proving that for every $\alpha \in [e^{-1}, 1[$ there is a complex Banach space X_α with $n(X_\alpha) = \alpha$ failing the power inequality. We do not know of any spaces satisfying the power inequality, other than complex Hilbert spaces and spaces with numerical index 1.

(b) It would be of interest to compute the numerical index for various classical Banach spaces such as l_p (or in general $L_p(S, \Sigma, \mu)$) for $1 < p < \infty$ ($p \neq 2$), and the Hardy spaces. In a next step, it would also be desirable to compute the numerical index of l_p-sums of Banach spaces and spaces $L_p(\mu, X)$.

(c) It is proved in [7, Theorem 2.6 and Theorem 10.1] that for a complex Banach space X,

$$\rho(T) \leq v(T) (\leq \|T\|) \qquad (T \in L(X)),$$

where $\rho(T)$ is the spectral radius of T. In §3, we study one of the extreme case of this inequality, $v(T) = \|T\|$ for all operator T, and it might seem of interesting to study the other extreme case, that is, what's the matter if $\rho(T) = v(T)$ for every bounded linear operator T of $L(X)$. But this is only possible in a trivial case. If a complex Banach space X satisfies that the spectral and the numerical radius coincide, then the spectral radius is an equivalent norm in $L(X)$, and it is a classical result in the theory of Banach algebras that in such a case $L(X)$ is commutative, so $X = \mathbb{C}$. See [7, §3 and §4] for a more detailed account.

(d) In the infinite-dimensional context the relation between CL-spaces, almost-CL-spaces, and spaces with numerical index 1 is not clear enough. We already mentioned that $l_1(\mathbb{C})$ is an almost-CL-space which is not a CL-space. We do not know what's the matter in the real case, and also, we do not know if every real or complex Banach space with numerical index 1 is an almost-CL-space.

(e) We conjecture that a Banach space with numerical index 1 must contain either c_0 or l_1, but we do not know yet how to attack this conjecture. In the complex case we even do not know if the results in [24] are valid.

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