Real Banach spaces with numerical index 1*

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Abstract

We show that an infinite-dimensional real Banach space with numerical index 1 and satisfying the Radon-Nikodỳm property contains ℓ_1. It follows that a reflexive or quasi-reflexive real Banach space cannot be renormed to have numerical index 1, unless it is finite-dimensional.

1 Introduction.

A Banach space has numerical index 1 if the norm of every bounded linear operator on it agrees with the numerical radius. Let us recall the relevant definitions. Given a real or complex Banach space X, we write B_X for the closed unit ball and S_X for the unit sphere of X. The dual space will be denoted by X^* and $L(X)$ will be the Banach algebra of all bounded linear operators on X. The *numerical range* of such an operator T is the subset $V(T)$ of the scalar field defined by

$$V(T) = \{x^*(Tx) \ : \ x \in S_X, \ x^* \in S_{X^*}, \ x^*(x) = 1\}.$$

The *numerical radius* of T is then given by

$$v(T) = \sup\{|\lambda| \ : \ \lambda \in V(T)\}.$$

It is clear that v is a seminorm on $L(X)$, and $v(T) \leq \|T\|$ for every $T \in L(X)$. Quite often, v is actually a norm and it is equivalent to the operator norm $\|\cdot\|$. Thus it is natural to consider the so called *numerical index* of the space X, namely the constant $n(X)$ defined by

$$n(X) = \inf\{v(T) \ : \ T \in S_{L(X)}\}.$$

Equivalently, $n(X)$ is the greatest constant $k \geq 0$ such that $k\|T\| \leq v(T)$ for every $T \in L(X)$. Note that $0 \leq n(X) \leq 1$, and $n(X) > 0$ if and only if v and $\|\cdot\|$ are equivalent norms.

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A complete survey on numerical ranges and their relations to spectral theory of operators can be found in the books by F. Bonsall and J. Duncan [1, 2] and we refer the reader to these books for general information and background. Let us mention here a couple of facts concerning the numerical index wich are relevant to our discussion. First, one has $v(T^*) = v(T)$ for every $T \in L(X)$, where T^* is the adjoint operator of T (see [1, §9]) and it clearly follows that $n(X^*) \leq n(X)$ for every Banach space X. The question if this is actually an equality seems to be open. Second, real and complex Banach spaces behave in a very diferent way with regard to the numerical index, as summarized in the following equalities (see [6]):

$$\{n(X) \ : \ X \text{ complex Banach space } \} = [e^{-1}, 1]$$
$$\{n(X) \ : \ X \text{ real Banach space } \} = [0, 1]$$

For instance, it is easy to show that the real space $X_{\mathbb{R}}$ underlying a complex Banach space X satisfies $n(X_{\mathbb{R}}) = 0$ in spite of the fact that $n(X) \geq e^{-1}$.

We are interested in the study of Banach spaces with numerical index 1. In the finite-dimensional context a satisfactory characterization of this kind of spaces was given by C. McGregor [11]. With regard to the infinite-dimensional case, a large family of classical spaces with numerical index 1 was exhibited in the already cited paper of J. Duncan, C. McGregor, J. Pryce, and A. White [6], namely $L_1(\mu)$ spaces and their preduals. The disk algebra is another interesting example (see [2, Theorem 32.9]).

In this paper we try to investigate the isomorphic properties of Banach spaces with numerical index 1. To say it in another way, we get necessary conditions for an infinite-dimensional Banach space to admit an equivalent norm with numerical index 1. As far as we know, this is an unexplored line. For instance, it is known that a Hilbert space H ($\dim(H) \geq 2$) has numerical index 0 or 1/2 depending whether it is real or complex, but the question if an infinite-dimensional Banach space with numerical index 1 can be isomorphic to a Hilbert space seems to be open.

To summarize our results, let us fix an infinite-dimensional real Banach space X with $n(X) = 1$. We show that if X satisfies the Radon-Nikodỳm property (RNP for short), then X contains (a subspace isomorphic to) ℓ_1 while if X is an Asplund space (i.e. X^* satisfies RNP) then X^* contains ℓ_1 (note that we do not require $n(X^*) = 1$). It follows that the quotient space X^{**}/X cannot be separable. Therefore, a reflexive or quasi-reflexive real Banach space cannot be equivalently renormed to have numerical index 1, unless it is finite-dimensional. Unfortunately, we could not get analogous results for complex Banach spaces. For background on RNP and Asplund spaces we refer to [3], [5], or [12].

The arguments we use can be applied to a more restrictive class of real Banach spaces introduced by J. Lindenstrauss [9], to obtain slightly sharper results. Recall that a real Banach space is said to satisfy the *3.2 Intersection Property (3.2.I.P.)* if each family of three mutually intersecting closed balls has nonempty intersection. It is not difficult to check that $n(X) = 1$ whenever X satisfies the 3.2.I.P., but the converse is not true. We prove that the dual of an infinite-dimensional real Banach space with the 3.2.I.P. always contains ℓ_1. The fact that such an space cannot be reflexive was known to J. Lindenstrauss and R. Phelps [10].

2 Main result.

Our starting point is McGregor's characterization of finite-dimensional spaces with numerical index 1 [11, Theorem 3.1]: *A finite-dimensional space X satisfies $n(X) = 1$ if and only if $|x^*(x)| = 1$ for all extreme points $x \in B_X$ and $x^* \in B_{X^*}$.* We will denote by $\mathrm{ex}(B)$ the set of extreme points in the convex set B. For infinite-dimensional X, $\mathrm{ex}(B_X)$ may be empty (e.g. c_0), so the right statement of McGregor's condition in this case should read $|x^{**}(x^*)| = 1$ for every $x^* \in \mathrm{ex}(B_{X^*})$ and every $x^{**} \in \mathrm{ex}(B_{X^{**}})$. One can easily show that this condition is sufficient to ensure $n(X) = 1$ but we do not know if the condition is also necessary. Nevertheless, by considering denting points instead of general extreme points we will get a (weaker) necessary condition. Recall that $x_0 \in B_X$ is said to be a *denting point* of B_X if it belongs to slices of B_X with arbitrarily small diameter. More precisely, for each $\varepsilon > 0$ one can find a functional $x^* \in S_{X^*}$ and a positive number α such that the slice $\{x \in B_X \ : \ \mathrm{Re}\ x^*(x) > 1 - \alpha\}$ is contained in the closed ball centered at x_0 with radius ε. If X is a dual space and the functionals x^* can be taken to be w^*-continuous, then we say that x_0 is a *w^*-denting point.*

Lemma 1. *Let X be a Banach space with numerical index 1. Then:*

(i) $|x^{**}(x^*)| = 1$ *for every* $x^{**} \in \mathrm{ex}(B_{X^{**}})$ *and every w^*-denting point* $x^* \in B_{X^*}$.

(ii) $|x^*(x)| = 1$ *for every* $x^* \in \mathrm{ex}(B_{X^*})$ *and every denting point* $x \in B_X$.

Proof. We only give the proof of (i); the other part is analogous.

Let us fix $x_0^{**} \in \mathrm{ex}(B_{X^{**}})$, a w^*-denting point $x_0^* \in B_{X^*}$, and $0 < \varepsilon < 1$. Consider the w^*-closed set $F = \{x^{**} \in B_{X^{**}} \ : \ |(x^{**} - x_0^{**})(x_0^*)| \geq \varepsilon\}$, and let K be the w^*-closed convex hull of F. Note that $x_0^{**} \notin K$, since otherwise x_0^{**} would be an extreme point in K and the "reversed" Krein-Milman Theorem (see [4, Theorem 7.8], for example) would give $x_0^{**} \in F$, a contradiction. We can now use the Hahn-Banach separation theorem to find $y^* \in S_{X^*}$ and $\alpha > 0$ such that

$$\mathrm{Re}\ x_0^{**}(y^*) > 1 - \alpha \geq \mathrm{Re}\ x^{**}(y^*)$$

for every $x^{**} \in F$. It follows that $|(x^{**} - x_0^{**})(x_0^*)| < \varepsilon$ whenever $x^{**} \in B_{X^{**}}$ satisfies $\mathrm{Re}\ x^{**}(y^*) > 1 - \alpha$. On the other hand, since x_0^* is a w^*-denting point, we can find $y \in S_X$ and $\beta > 0$ such that $\|x^* - x_0^*\| < \varepsilon$ whenever $x^* \in B_{X^*}$ satisfies $\mathrm{Re}\ x^*(y) > 1 - \beta$.

Consider the rank-one operator $T \in L(X)$ defined by $Tx = y^*(x)y$ for every $x \in X$. Since $n(X) = 1$, we have $v(T) = \|T\| = 1$ and the definition of the numerical radius provides us with $x \in S_X$ and $x^* \in S_{X^*}$, such that $x^*(x) = 1$ and $|x^*(Tx)| = |y^*(x)||x^*(y)| > 1 - \delta$, where we take $\delta = \min\{\alpha, \beta\}$. By choosing suitable modulus one scalars s and t we have

$$\begin{cases} \text{Re } y^*(sx) = |y^*(x)| > 1 - \delta \geq 1 - \alpha \\ \text{Re } tx^*(y) = |x^*(y)| > 1 - \delta \geq 1 - \beta. \end{cases}$$

It follows that $|x_0^*(sx) - x_0^{**}(x_0^*)| < \varepsilon$ and $\|tx^* - x_0^*\| < \varepsilon$, so

$$\begin{aligned} 1 - |x_0^{**}(x_0^*)| &\leq |tx^*(sx) - x_0^{**}(x_0^*)| \leq \\ &\leq |tx^*(sx) - x_0^*(sx)| + |x_0^*(sx) - x_0^{**}(x_0^*)| < 2\varepsilon \end{aligned}$$

and we let $\varepsilon \downarrow 0$. □

Our next result is a useful sufficient condition for a real Banach space to contain a subspace isomorphic either to c_0 or to ℓ_1, which follows easily from Rosenthal's ℓ_1-Theorem [13] and a result by Fonf [7]. The above lemma shows that this sufficient condition will be fulfilled by a Banach space with numerical index 1 (or by its dual), provided that the unit ball has infinitely many denting (or w^*-denting) points. From now on we only work in the real case.

Proposition 2. *Let X be a real Banach space and assume that there is an infinite set $A \subset S_X$ such that $|x^*(a)| = 1$ for every $a \in A$ and all $x^* \in \text{ex}(B_{X^*})$. Then X contains (an isomorphic copy of) either c_0 or ℓ_1.*

Proof. Suppose that X does not contain ℓ_1. Then, by Rosenthal's ℓ_1-Theorem [13], every bounded sequence in X has a weakly Cauchy subsequence, so there is a weakly Cauchy sequence $\{a_n\}$ of distinct members of A. Let Y be the closed subspace generated by this sequence. The assumption on A clearly gives $\|a_n - a_m\| = 2$ for $n \neq m$, so Y is infinite-dimensional. The proof will be finished by showing that Y contains c_0, and this will follow from Fonf's Theorem [7] if we are able to prove that $\text{ex}(B_{Y^*})$ is countable.

By a well-known application of the Hahn-Banach and Krein-Milman theorems, every $y^* \in \text{ex}(B_{Y^*})$ is the restriction to Y of some extreme point in B_{X^*}, so $|y^*(a_n)| = 1$ for every n. Since $\{a_n\}$ is weakly Cauchy, the sequence $\{y^*(a_n)\}$ must be eventually 1 or -1. This shows that

$$\text{ex}(B_{Y^*}) = \bigcup_{k=1}^{\infty} (E_k \cup -E_k)$$

where $E_k = \{y^* \in \text{ex}(B_{Y^*}) \ : \ y^*(a_n) = 1 \ \text{ for } n \geq k\}$. Since the sequence $\{a_n\}$ separates the points of Y^*, each set E_k is finite and we are done. □

A natural (isomorphic) assumption on an infinite-dimensional Banach space providing a lot of denting points is RNP. Actually the unit ball of a Banach space satisfying RNP is the closed convex hull of its strongly exposed points, and strongly exposed points are denting. On the other hand, if X is an Asplund space, then B_{X^*} is the w^*-closed convex hull of its w^*-strongly exposed (hence w^*-denting) points (see [12]). Therefore, we have

Theorem 3. *Let X be an infinite-dimensional real Banach space with $n(X) = 1$. If X has RNP, then X contains ℓ_1. If X is an Asplund space, then X^* contains ℓ_1.* □

We do not know if the second part of the above theorem follows directly from the first. Note that X^* satisfies RNP if X is Asplund, but we only require $n(X) = 1$ and we do not know if $n(X^*) = 1$.

An Asplund space cannot contain ℓ_1, so Theorem 3 has the following consequence.

Corollary 4. *Let X be a real Asplund space satisfying RNP. If $n(X) = 1$, then X is finite-dimensional.* □

As a special case of the above corollary a reflexive real Banach space with numerical index 1 must be finite-dimensional. In fact, we have:

Corollary 5. *Let X be an infinite-dimensional real Banach space with $n(X) = 1$. Then X^{**}/X is non-separable.*

Proof. It is known (see [5, page 219]) that X and X^* have RNP if X^{**}/X is separable. □

Remark 6. It is worth mentioning that in the proof of Lemma 1 only rank-one operators were involved. Thus, the lemma remains true if we only assume that $v(T) = \|T\|$ for every rank-one operator $T \in L(X)$. However this observation does not lead to an improvement of our main results. In fact, a Banach space satisfying RNP or an Asplund space has numerical index one as soon as rank-one operators on the space have numerical radius equal to the norm. We show this for an Asplund space, the RNP case being similar. Take an arbitrary operator $T \in L(X)$, fix $\varepsilon > 0$, and use that B_{X^*} is the w^*-closed convex hull of its w^*-denting points to get $\|T^*(x^*)\| > \|T^*\| - \varepsilon$ for some w^*-denting point x^*. Now choose $x^{**} \in \mathrm{ex}(B_{X^{**}})$ such that $|x^{**}(T^*x^*)| > \|T^*\| - \varepsilon$. With the above observation in mind, Lemma 1 tells us that, up to rotation, we may arrange $x^{**}(x^*) = 1$, so $v(T^*) \geq \|T^*\| - \varepsilon$. Now let $\varepsilon \downarrow 0$ and use that $v(T) = v(T^*)$ (see [1, §9, Corollary 6]).

As we announced in the introduction, one of the assertions in Theorem 3 can be improved for spaces with the 3.2.I.P. Indeed, A. Lima proved that the 3.2.I.P. is a self-dual property [8, Corollary 3.3]. Now, if X is an infinite-dimensional real Banach space with the 3.2.I.P. we may apply to X^* a result by Lindenstrauss [9, Theorem 4.7] to obtain that $|x^{**}(x^*)| = 1$ for every $x^{**} \in \mathrm{ex}(B_{X^{**}})$ and every $x^* \in \mathrm{ex}(B_{X^*})$. Since $\mathrm{ex}(B_{X^*})$ is infinite, Proposition 2 shows that X^* contains either c_0 or ℓ_1, but a dual space contains l_∞ (hence also ℓ_1) as soon as it contains c_0. Therefore, we have shown the following:

Corollary 7. *Let X be an infinite-dimensional real Banach space with the 3.2.I.P. Then X^* contains ℓ_1.* □

The fact that an infinite-dimensional real Banach space with the 3.2.I.P. cannot be reflexive was proved in [10, Corollary 2.4].

We have seen above that every real Banach space with the 3.2.I.P. satisfies

$$|x^{**}(x^*)| = 1 \ \forall x^* \in \text{ex}(B_{X^*}) \ \forall x^{**} \in \text{ex}(B_{X^{**}}). \tag{$*$}$$

Recall that $(*)$ implies $n(X) = 1$ and we do not know if the converse is true. On the other hand, it is known that property $(*)$ does not imply the 3.2.I.P., even in the finite-dimensional case (see [9, pp. 47]).

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