Generating operators between Banach spaces

Miguel Martín

UNIVERSIDAD DE GRANADA

IMAG
INSTITUTO DE MATEMÁTICAS
Universidad de Granada

joint work with V. Kadets, J. Merí, and A. Quero

Colloquium
Departamento de Análisis Matemático y Matemática Aplicada
Universidad Complutense de Madrid

MINISTERIO DE CIENCIA E INNOVACIÓN
UNIÓN EUROPEA
FONDO EUROPEO DE DESARROLLO REGIONAL
"Una manera de hacer Europa"
AGENCIA ESTATAL DE INVESTIGACIÓN

PID2021-122126NB-C31

Some notation

X, Y real or complex Banach spaces

$\mathbb{K}$	scalar field $\mathbb{R}$ or $\mathbb{C}$
$\mathbb{T}$	modulus one scalars
B_X	closed unit ball
S_X	unit sphere
X^*	topological dual
$\mathcal{L}(X,Y)$	bounded linear operators from X to Y
$\operatorname{conv}(\cdot)$	convex hull
$\overline{\operatorname{conv}}(\cdot)$	closed convex hull
$\operatorname{ext}(C)$	extreme points of a (closed convex bounded) set

Introduction. Spear operators

Spear vector (Ardalani, 2014)

X Banach space, $u \in S_X$. We say that u is a *spear vector* if

$$\max_{\theta\in\mathbb{T}} \|u+\theta x\| = 1 + \|x\| \quad \text{for every } x \in X.$$

Examples of spear vectors:

- $\text{Spear}(\mathbb{K}) = \mathbb{T}$
- $\text{Spear}(\ell_1) = \mathbb{T}\{e_n : n \in \mathbb{N}\}$
- $\text{Spear}(\ell_\infty) = \mathbb{T}^{\mathbb{N}}$
- $\text{Spear}(L_\infty(\mu)) = \{f \in L_\infty(\mu) : |f(t)| = 1 \ \mu\text{-a.e.}\}$

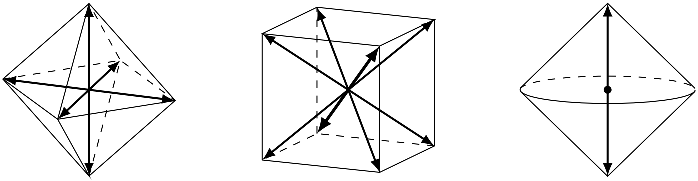

Figure: Spear vectors in the real spaces ℓ_1^3, ℓ_∞^3 and $\ell_2^2 \oplus_1 \mathbb{R}$, respectively.

Spear vector (Ardalani, 2014)

X Banach space, $u \in S_X$. We say that u is a *spear vector* if

$$\max_{\theta\in\mathbb{T}} \|u + \theta x\| = 1 + \|x\| \quad \text{for every } x \in X.$$

Some properties of spear vectors:

- $z \in \operatorname{Spear}(X)$ iff $|x^*(z)| = 1$ for every $x^* \in \operatorname{ext}(B_{X^*})$
- $\operatorname{Spear}(X) \subseteq \operatorname{ext}(B_X)$ (actually, they are strongly extreme)
- $J_X(\operatorname{Spear}(X)) \subseteq \operatorname{Spear}(X^{**})$
- $z^* \in \operatorname{Spear}(X^*)$ then z^* is norm attaining. Actually,

$$B_X = \overline{\operatorname{conv}}\big\{x \in B_X : |z^*(x)| = 1\big\}$$

- X real, $\#\operatorname{Spear}(X) = \infty$, then $X \supset c_0$ or $X \supset \ell_1$
- X strictly convex, $\dim(X) \geqslant 2$, then $\operatorname{Spear}(X) = \emptyset = \operatorname{Spear}(X^*)$

Spear operator (Ardalani, 2014)

X, Y Banach spaces, $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$.

$$G \text{ *spear operator* } \overset{\text{def}}{\iff} \max_{\theta\in\mathbb{T}} \|G + \theta T\| = 1 + \|T\| \text{ for every } T \in \mathcal{L}(X,Y)$$

Examples of spear operators (Kadets–M.–Merí–Pérez, 2018)

- The identity operator on $C(K)$, $L_1(\mu)$, c_0, ℓ_1, ℓ_∞...
- The inclusion $A(\mathbb{D}) \hookrightarrow C(\mathbb{T})$.
- The Fourier transforms
  (for instance, $\mathcal{F}\colon L_1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$ or $\mathcal{F}\colon L_1(\mathbb{T}) \longrightarrow c_0(\mathbb{Z})$).
- $G = x_0^* \otimes y_0$ spear $\iff$ $x_0^* \in \mathrm{Spear}(X^*)$ and $y_0 \in \mathrm{Spear}(Y)$.

Some properties **(Kadets–M.–Merí–Pérez, 2018)**

- $G\colon \ell_1 \longrightarrow Y$ spear $\iff$ $G(e_n) \in \operatorname{Spear}(Y)\ \forall n \in \mathbb{N}$
- $G\colon X \longrightarrow c_0$ spear $\iff$ $G^*(e_n) \in \operatorname{Spear}(X^*)\ \forall n \in \mathbb{N}$
- If $\dim(X) < \infty$, $G\colon X \longrightarrow Y$ spear $\iff$ $G(x) \in \operatorname{Spear}(Y)\ \forall x \in \operatorname{ext}(B_X)$
- If $\dim(Y) < \infty$, $G\colon X \longrightarrow Y$ spear $\iff$ $G^*(y^*) \in \operatorname{Spear}(X^*)\ \forall y^* \in \operatorname{ext}(B_{Y^*})$

Isomorphic and isometric consequences (Kadets–M.–Merí–Pérez, 2018)

X, Y Banach spaces, $G \in \mathcal{L}(X,Y)$ spear operator.

- If X is real and $\dim G(X) = \infty$, then $X^* \supset \ell_1$.
- If X^* is strictly convex or smooth, then $X = \mathbb{K}$.
- If Y^* is strictly convex, then $Y = \mathbb{K}$.
- If X contains strongly exposed points, Y smooth, then $Y = \mathbb{K}$. (M.-Merí-Quero-Sain-Roy, 2023)

Spear operators. III

Numerical radius with respect to G

X, Y Banach spaces, $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$, $T \in \mathcal{L}(X,Y)$

$$v_G(T) := \inf_{\delta>0} \sup\Big\{|y^*(Tx)| \colon y^* \in S_{Y^*},\, x \in S_X,\, \operatorname{Re} y^*(Gx) > 1-\delta\Big\}.$$

- $v_G(\cdot)$ is a seminorm on $\mathcal{L}(X,Y)$.
- $v_G(T) \leqslant \|T\|$ for every $T \in \mathcal{L}(X,Y)$.

G spear operator $\iff$ $v_G(T) = \|T\|$ for every $T \in \mathcal{L}(X,Y)$.

Leading idea of the manuscript (and the talk)

Introduce a new seminorm on $\mathcal{L}(X,Y)$ between $v_G(\cdot)$ and the usual operator norm to get a weaker property than being spear:

$$v_G(T) = \inf_{\delta>0} \sup\{|y^*(Tx)| \colon y^* \in S_{Y^*},\, x \in S_X,\, \operatorname{Re} y^*(Gx) > 1-\delta\}.$$

$$\|T\| = \inf_{\delta>0} \sup\{|y^*(Tx)| \colon y^* \in S_{Y^*},\, x \in S_X,\, \cancel{\operatorname{Re} y^*(Gx) > 1-\delta}\}.$$

$$\begin{aligned}\|T\|_G &:= \inf_{\delta>0} \sup\{|y^*(Tx)| \colon y^* \in S_{Y^*},\, x \in S_X,\, \|Gx\| > 1-\delta\} \\ &= \inf_{\delta>0} \sup\{\|Tx\| \colon y^* \in S_{Y^*},\, x \in S_X,\, \|Gx\| > 1-\delta\}.\end{aligned}$$

Generating operators

Norm relative to an operator

X, Y Banach spaces, $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$, $T \in \mathcal{L}(X,Y)$.

$$\|T\|_G := \inf_{\delta>0} \sup\{\|Tx\| : x \in S_X,\ \|Gx\| > 1-\delta\}.$$

Relative norm (Kadets–M.–Merí–Quero, 2023)

X, Y, Z Banach spaces and $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$. The *(semi-)norm of* $T \in \mathcal{L}(X,Z)$ *relative to* G is

$$\|T\|_G := \inf_{\delta>0} \sup\left\{\|Tx\| : x \in \operatorname{att}(G,\delta)\right\},$$

where, for each $\delta > 0$, $\operatorname{att}(G,\delta) := \{x \in S_X : \|Gx\| > 1-\delta\}$.

- $\|\cdot\|_G$ is a seminorm on $\mathcal{L}(X,Z)$.
- If $Z = Y$, then for every $T \in \mathcal{L}(X,Y)$,

$$v_G(T) \leqslant \|T\|_G \leqslant \|T\|.$$

Generating operator (Kadets–M.–Merí–Quero, 2023)

Let X, Y be Banach spaces and let $G \in \mathcal{L}(X, Y)$ be a norm-one operator. We say that G is *generating* if $\|T\|_G = \|T\|$ for every $T \in \mathcal{L}(X, Y)$.

Proposition

X, Y Banach spaces and $G \in \mathcal{L}(X, Y)$ with $\|G\| = 1$. Then, TFAE:

1. G is generating.
2. $\|T\|_G = \|T\|$ for every $T \in \mathcal{L}(X, Z)$ and every Banach space Z.
3. There is a (non null) Banach space Z such that $\|T\|_G = \|T\|$ for every *rank-one* operator $T \in \mathcal{L}(X, Z)$.
4. $B_X = \overline{\text{conv}}(\text{att}(G, \delta))$ for every $\delta > 0$.

If $\dim(X) < \infty$, G generating $\iff$ $B_X = \text{conv}(\{x \in S_X : \|Gx\| = 1\})$.

Examples:

- The identity operator on every Banach space and every isometric embedding.
- Spear operators.
- The natural inclusion $G\colon \ell_1 \hookrightarrow c_0$ is generating since

$$\mathbb{T}\{e_n\colon n\in\mathbb{N}\}\subset \operatorname{att}(G,\delta)\ \forall\delta>0 \implies B_{\ell_1}=\overline{\operatorname{conv}}(\operatorname{att}(G,\delta))\ \forall\delta>0.$$

- The natural inclusion $G\colon L_\infty[0,1]\hookrightarrow L_1[0,1]$ is generating since

$$\{f\in L_\infty[0,1]\colon |f(t)|=1 \text{ a.e.}\}\subset \operatorname{att}(G,\delta)\ \forall\delta>0$$
$$\implies B_{L_\infty[0,1]}=\overline{\operatorname{conv}}(\operatorname{att}(G,\delta))\ \forall\delta>0.$$

Lemma

X, Y Banach spaces and $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$. If G is generating, then $\|Gx\| = 1$ for every denting point x of B_X.

Proposition

X, Y Banach spaces and $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$. Suppose that X has the RNP. Then, G is generating if and only if $\|Gx\| = 1$ for every denting point x of B_X.

Particular cases: X, Y Banach spaces

- $G\colon \ell_1 \longrightarrow Y$ generating $\iff$ $\|Ge_n\| = 1$ for every $n \in \mathbb{N}$.
- If $\dim(X) < \infty$, $G\colon X \longrightarrow Y$ generating $\iff$ $\|Gx\| = 1$ for every $x \in \operatorname{ext}(B_X)$.

Spear sets (Kadets–M.–Merí–Pérez, 2018)

Let X be a Banach space. We say that $F \subset B_X$ is a *spear set* if

$$\max_{\theta\in\mathbb{T}} \sup_{z\in F} \|z+\theta x\| = 1 + \|x\| \qquad \text{for every } x \in X.$$

- $F = B_X$ and $F = S_X$ are spear sets
- $F = \{z\}$ is a spear set $\iff$ z is a spear vector
- $F \subset B_{L_\infty[0,1]}$ spear set $\iff$ $\forall A \subset [0,1]$ with $|A| > 0$ and $\forall \varepsilon > 0$ $\exists B \subset A$ with $|B| > 0$ and $f \in F$ such that $|f(t)| > 1-\varepsilon$ $\forall t \in B$.
- $F \subset B_{X^*}$ spear set iff $\forall \varepsilon > 0$,

$$B_X = \overline{\operatorname{conv}} \left\{ x \in B_X : \sup_{x^*\in F} \operatorname{Re} x^*(x) > 1-\varepsilon \right\}.$$

Theorem

X, Y Banach spaces and $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$.
Then, G is generating if and only if $G^*(B_{Y^*})$ is a spear set of X^*.

Spear sets (Kadets–M.–Merí–Pérez, 2018)

Let X be a Banach space. We say that $F \subset B_X$ is a *spear set* if

$$\max_{\theta\in\mathbb{T}} \sup_{z\in F} \|z + \theta x\| = 1 + \|x\| \qquad \text{for every } x \in X.$$

Theorem

X, Y Banach spaces and $G \in \mathcal{L}(X, Y)$ with $\|G\| = 1$.
Then, G is generating if and only if $G^*(B_{Y^*})$ is a spear set of X^*.

Corollary

X, Y Banach spaces, $x_0^* \in S_{X^*}$, and $y_0 \in S_Y$.
$G = x_0^* \otimes y_0$ is generating if and only if x_0^* is a spear vector of X^*.

- X has denting points, $G \in \mathcal{L}(X,Y)$ generating $\Longrightarrow$ G attains its norm.
- Every spear $x^* \in X^*$ attains its norm
  $\Longrightarrow$ G rank-one and generating attains its norm.

Question: Does every generating operator attain its norm?
No, there are counterexamples even of rank two.

Example: Consider $g\colon [0,1] \longrightarrow \ell_2^2$ given by $g(t) = (\cos t, \sin t)$ and $G\colon L_1[0,1] \longrightarrow \ell_2^2$ defined by:

$$G(x) = \int_0^1 x(t)g(t)\,dt \qquad (x \in L_1[0,1]).$$

Then, G is generating but does not attain its norm.

- G generating $\iff$ $\{y^* \circ g\colon y^* \in B_{Y^*}\}$ is a spear set of $B_{L_\infty[0,1]}$
  $\iff$ $\|g(t)\| = 1$ a.e. ✓
- Suppose that there is a non-zero $x \in L_1[0,1]$ such that $\|Gx\| = \|x\|$.

$$\left|\int_0^1 x(t)g(t)dt\right| = \left\|\int_0^1 x(t)g(t)dt\right\| = \int_0^1 |x(t)|dt = \int_0^1 |x(t)g(t)|dt.$$

$\Longrightarrow$ $\exists \lambda \in \mathbb{T}$ such that $xg = \lambda|xg| = \lambda|x|$ a.e. which is a contradiction

Theorem

Let X be a Banach space. Then, TFAE:

1. There exists a Banach space Y and a norm-one operator $G \in \mathcal{L}(X,Y)$ such that G is generating but does not attain its norm.
2. There exists a spear set $\mathcal{B} \subset B_{X^*}$ such that $\sup_{x^*\in\mathcal{B}} |x^*(x)| < 1$ for every $x \in S_X$.

(1)⇒(2) Take $\mathcal{B} = G^*(B_{Y^*})$ spear set of B_{X^*} and

$$\sup_{x^*\in\mathcal{B}} |x^*(x)| = \sup_{y^*\in B_{Y^*}} |[G^*y^*](x)| = \sup_{y^*\in B_{Y^*}} |y^*(Gx)| = \|Gx\| < 1 \quad \forall x \in S_X.$$

(2)⇒(1) Define $G\colon X \longrightarrow \ell_\infty(\mathcal{B})$ by

$$[Gx](x^*) = x^*(x) \qquad (x^* \in \mathcal{B},\ x \in X).$$

- $\|G\| = 1$ but $\|Gx\| = \sup_{x^*\in\mathcal{B}} |[Gx](x^*)| = \sup_{x^*\in\mathcal{B}} |x^*(x)| < 1$ for every $x \in S_X$.
- $\mathcal{B} \subset G^*(B_{\ell_1(\mathcal{B})}) \subset G^*(B_{\ell_\infty(\mathcal{B})^*}) \implies G^*(B_{\ell_\infty(\mathcal{B})^*})$ is a spear set. □

The set of all generating operators

X, Y Banach spaces

$$\mathrm{Gen}(X,Y) := \{G \in \mathcal{L}(X,Y) \colon \|G\| = 1,\ G \text{ is generating}\}.$$

Questions:

- Are there Banach spaces Y such that $\mathrm{Gen}(X,Y) \neq \emptyset$ for every Banach space X?
- Are there Banach spaces X such that $\mathrm{Gen}(X,Y) \neq \emptyset$ for every Banach space Y?

Range spaces:

Proposition

For every Banach space Y there is a Banach space X such that $\mathrm{Gen}(X,Y) = \emptyset$.

Proposition

For every **separable** X, $\mathrm{Gen}(X, C[0,1]) \neq \emptyset$.

The set of all generating operators. II

X, Y Banach spaces

$$\operatorname{Gen}(X,Y) := \{G \in \mathcal{L}(X,Y)\colon \|G\| = 1,\ G \text{ is generating}\}.$$

Questions:

- Are there Banach spaces Y such that $\operatorname{Gen}(X,Y) \neq \emptyset$ for every Banach space X?
- Are there Banach spaces X such that $\operatorname{Gen}(X,Y) \neq \emptyset$ for every Banach space Y?

Domain space:

Proposition

X Banach space. Then,

$$\operatorname{Gen}(X,Y) \neq \emptyset \text{ for every Banach space } Y \iff \operatorname{Spear}(X^*) \neq \emptyset.$$

The set of all generating operators. III

Questions: X Banach space

- When $\mathrm{Gen}(X,Y) = S_{\mathcal{L}(X,Y)}$ for all Banach spaces Y?

Proposition

X Banach space. Then,

$$\exists\, Y \text{ Banach space such that } \mathrm{Gen}(X,Y) = S_{\mathcal{L}(X,Y)} \implies X = \mathbb{K}.$$

In this case, $\mathrm{Gen}(\mathbb{K},Y) = S_{\mathcal{L}(\mathbb{K},Y)}$ for every Banach space Y.

- When $B_{\mathcal{L}(X,Y)} = \overline{\mathrm{conv}}(\mathrm{Gen}(X,Y))$ for all Banach spaces Y?

Theorem

(Ω,Σ,μ) finite measure space, Y Banach space. Then,

$$\{T \in \mathcal{L}(L_1(\mu),Y)\colon \|T\| \leqslant 1,\ T \text{ is representable}\} \subset \overline{\mathrm{conv}}\left(\mathrm{Gen}(L_1(\mu),Y)\right).$$

As a consequence, if Y has the RNP with respect to μ, then

$$B_{\mathcal{L}(L_1(\mu),Y)} = \overline{\mathrm{conv}}\left(\mathrm{Gen}(L_1(\mu),Y)\right).$$

The set of all generating operators. IV

Corollary

- $B_{\mathcal{L}(\ell_1,Y)} = \overline{\operatorname{conv}}\,(\operatorname{Gen}(\ell_1,Y))$ for every Banach space Y.
- $B_{\mathcal{L}(\ell_1^n,Y)} = \operatorname{conv}\,(\operatorname{Gen}(\ell_1^n,Y))$ for every Banach space Y and every $n \in \mathbb{N}$.

Theorem

X *real* Banach space, $\dim(X) = n$.

$$B_{\mathcal{L}(X,Y)} = \overline{\operatorname{conv}}(\operatorname{Gen}(X,Y)) \text{ for every Banach space } Y \implies X = \ell_1^n.$$

The set of all generating operators. V

Theorem

X *real* Banach space, $\dim(X) = n$.

$$B_{\mathcal{L}(X,Y)} = \overline{\operatorname{conv}}(\operatorname{Gen}(X,Y)) \text{ for every Banach space } Y \implies X = \ell_1^n.$$

Sketch of the proof.

- $B_{\mathcal{L}(X,\mathbb{K})} = \overline{\operatorname{conv}}(\operatorname{Gen}(X,\mathbb{K})) \implies X^*$ is almost CL-space and so $n(X) = 1$.
- **[McGregor, 1971]**: X real finite-dimensional, $n(X) = 1 \implies \operatorname{ext}(B_X)$ is finite.
- The set $\operatorname{ext}(B_X)$ has exactly $2n$ elements and so $X = \ell_1^n$. Indeed, if $\operatorname{ext}(B_X)$ has more than $2n$ elements, we can take $\{e_1, \ldots, e_n\} \subset \operatorname{ext}(B_X)$ linearly independent and $\mathrm{e}_{n+1} \in \operatorname{ext}(B_X) \setminus \{\pm e_j : j = 1, \ldots, n\}$.
  Consider Y Banach space with
  $$B_Y = \operatorname{conv}\left(\operatorname{ext}(B_X) \cup \{\pm(1+\varepsilon)e_{n+1}\}\right)$$
  so that $e_1, \ldots, e_n \operatorname{ext}(B_Y)$.
  The natural identity $G\colon X \longrightarrow Y$ is not generating as $\|Ge_{n+1}\|_Y < 1$ and $G \notin \operatorname{conv}(\operatorname{Gen}(X,Y))$. □

References

M. Ardalani
Numerical index with respect to an operator
Studia Math. **225** (2014), 165–171.

V. Kadets, M. Martín, J. Merí, and A. Pérez
Spear operators between Banach spaces
Lecture Notes in Mathematics, **2205**, Springer, 2018.

V. Kadets, M. Martín, J. Merí, A. Pérez, and A. Quero
On the numerical index with respect to an operator
Diss. Math. **547** (2020), 1–58.

V. Kadets, M. Martín, J. Merí, and A. Quero
Generating operators between Banach spaces
Preprint available at arXiv:2306.02645 (2023)