When the typical operator is norm attaining?

Miguel Martín
Universidad de Granada / IMAG

UNIVERSIDAD DE GRANADA
Instituto de Matemáticas
Universidad de Granada

2023 ILJU School of Mathematics

PMI
POSTECH Mathematics Institute
KIAS
KOREA INSTITUTE FOR ADVANCED STUDY
태광그룹 일주학술문화재단
ILJU

MINISTERIO DE CIENCIA E INNOVACIÓN
UNIÓN EUROPEA
Junta de Andalucía

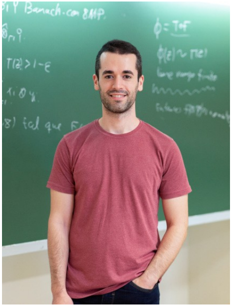
Preliminaries

Section 1

The minicourse is mainly based on the paper

M. Jung, M. Martín, and A. Rueda Zoca.
Residuality in the set of norm attaining operators between Banach spaces.
J. Funct. Anal. 284 (2023), 109746, 46pp.

Mingu Jung (KIAS, Korea)



Abraham Rueda Zoca (Granada, Spain)

Preliminaries

Section 1

Notation

X, Y real or complex Banach spaces

- $\mathbb{K}$ base field $\mathbb{R}$ or $\mathbb{C}$,
- $\mathbb{T}$ modulus one scalars,
- $B_X = \{x \in X : \|x\| \leqslant 1\}$ closed unit ball of X,
- $S_X = \{x \in X : \|x\| = 1\}$ unit sphere of X,
- $\overline{\text{conv}}(C)$ closed convex hull of C,
- $\mathcal{L}(X, Y)$ bounded linear operators from X to Y,
    - $\|T\| = \sup\{\|T(x)\| : x \in S_X\}$,
- $\mathcal{K}(X, Y)$ compact linear operators from X to Y,
- $\mathcal{F}(X, Y)$ bounded linear operators from X to Y with finite rank,
- $X^* = \mathcal{L}(X, \mathbb{K})$ topological dual of X.

Preliminaries

Section 1

Norm attaining functionals

Norm attaining functionals

$x^* \in X^*$ attains its norm when

$$\exists\, x \in S_X \ : \ |x^*(x)| = \|x^*\|$$

★ $\mathrm{NA}(X,\mathbb{K}) := \{x^* \in X^* : x^* \text{ attains its norm}\}$

Examples and comments

- $\dim(X) < \infty \implies \mathrm{NA}(X,\mathbb{K}) = \mathcal{L}(X,\mathbb{K})$ (Heine-Borel).
- X reflexive $\iff \mathrm{NA}(X,\mathbb{K}) = \mathcal{L}(X,\mathbb{K})$ (Hahn-Banach, James).
- $\mathrm{NA}(c_0,\mathbb{K}) = c_{00} \leqslant \ell_1$,
- $\mathrm{NA}(\ell_1,\mathbb{K}) = \left\{ x \in \ell_\infty : \|x\|_\infty = \max_n\{|x(n)|\} \right\} \subseteq \ell_\infty$, residual, contains c_0,
- $\mathrm{NA}(X,\mathbb{K})$ may be "wild", for instance:
  - it may contain NO two-dimensional subspaces (Read, 2017; Rmoutil, 2017),
  - it can be NOT norm Borel (Kaufman, 1991).
- (Petunin–Plichko 1974; Godefroy 1987): X separable, $Z \leqslant X^*$ closed, separating for X, $Z \subseteq \mathrm{NA}(X,\mathbb{K}) \implies Z$ is an isometric predual of X.

Norm attaining operators

Norm attaining operators

$T \in \mathcal{L}(X,Y)$ attains its norm when

$$\exists\, x \in S_X \ : \ \|T(x)\| = \|T\|$$

★ $\mathrm{NA}(X,Y) := \{T \in \mathcal{L}(X,Y)\colon T \text{ attains its norm}\}$

Some examples and comments

- $\dim(X) < \infty \implies \mathrm{NA}(X,Y) = \mathcal{L}(X,Y)$ for every Y (Heine-Borel),
- $\dim(X) = \infty \implies \mathrm{NA}(X,c_0) \neq \mathcal{L}(X,c_0)$ (see M.-Merí-Payá, 2006).
- X reflexive $\iff \mathcal{K}(X,Y) \subseteq \mathrm{NA}(X,Y)$ for every Y (James).
- $\mathcal{L}(X,\ell_\infty) = \left[\bigoplus_{n\in\mathbb{N}} \mathcal{L}(X,\mathbb{K})\right]_{\ell_\infty} = \ell_\infty(X^*)$.
  $\mathrm{NA}(X,\ell_\infty) = \left\{(x_n^*) \in \ell_\infty(X^*)\colon \exists k \in \mathbb{N},\ \|x_k^*\| = \|(x_n^*)\|_\infty,\ x_k^* \in \mathrm{NA}(X,\mathbb{K})\right\}$.
- $\mathcal{L}(\ell_1,Y) = \left[\bigoplus_{n\in\mathbb{N}} \mathcal{L}(\mathbb{K},Y)\right]_{\ell_\infty} = \ell_\infty(Y)$.
  $\mathrm{NA}(\ell_1,Y) = \left\{(y_n) \in \ell_\infty(Y)\colon \exists k \in \mathbb{N},\ \|y_k\| = \|(y_n)\|_\infty\right\}$.
- $\mathrm{NA}(L_1[0,1], L_\infty[0,1])$???

The problem of denseness of norm attaining functionals

Problem

Is $\mathrm{NA}(X, \mathbb{K})$ always dense in X^***?**

Theorem (E. Bishop & R. Phelps, 1961)

The set of norm attaining functionals is dense in X^* (for the norm topology).

Problem

Is $\mathrm{NA}(X, Y)$ always dense in $\mathcal{L}(X, Y)$**?**

The answer is **No**, and this is the origin of the study of norm attaining operators.

Modified problem

When is $\mathrm{NA}(X, Y)$ dense in $\mathcal{L}(X, Y)$**?**

The study of this problem was initiated by J. Lindenstrauss in 1963,
who provided the first negative and positive examples.

An overview on “classical” results on norm attaining operators

Section 2

Bibliography for this overview

M. D. Acosta
Denseness of norm attaining mappings
RACSAM (2006)

A. Capel
Norm-attaining operators
Master thesis. Universidad Autónoma de Madrid, Spain. 2015
http://hdl.handle.net/10486/682502

M. Martín
The version for compact operators of Lindenstrauss properties A and B
RACSAM (2016)

An overview on “classical” results on norm attaining operators

Section 2

Lindenstrauss' seminal paper of 1963

Negative answer

$$\mathrm{NA}(X,Y) \text{ is NOT always dense}$$

Lemma

Y LUR, $T\colon X \longrightarrow Y$ bounded from below (monomorphism).
If T attains its norm, then it does at a strongly exposed point.

Example

X separable without strongly exposed points (e.g. c_0, $C[0,1]$, $L_1[0,1]$), Y LUR renorming of X. Then, $\mathrm{NA}(X,Y)$ is not dense in $\mathcal{L}(X,Y)$.

Lemma

If Y is strictly convex, then $\mathrm{NA}(c_0,Y) \subseteq \mathcal{F}(c_0,Y)$.

Example

Y strictly convex, $Y \supset c_0$. Then, $\mathrm{NA}(X,Y)$ is not dense in $\mathcal{L}(X,Y)$.

Lindenstrauss properties A and B

Observation

- The question then is for which X and Y the density holds.
- As this problem is too general, Lindenstrauss introduced two properties.

Definition

X, Y Banach spaces,

- X has (Lindenstrauss) property A iff $\overline{\mathrm{NA}(X,Z)} = \mathcal{L}(X,Z) \quad \forall Z$
- Y has (Lindenstrauss) property B iff $\overline{\mathrm{NA}(Z,Y)} = \mathcal{L}(Z,Y) \quad \forall Z$

First examples

- If X is finite-dimensional, then X has property A,
- $\mathbb{K}$ has property B (Bishop-Phelps theorem),
- c_0, $C[0,1]$, $L_1[0,1]$ fail property A,
- if Y is strictly convex, $Y \supset c_0$, then Y fails property B.

Positive results I

Theorem (Lindenstrauss, 1963)

X, Y Banach spaces. Then

$$\big\{T \in \mathcal{L}(X,Y):\ \ T^{**}\colon X^{**} \longrightarrow Y^{**} \text{ attains its norm}\big\}$$

is dense in $\mathcal{L}(X,Y)$.

Consequence

If X is reflexive, then X has property A.

An improvement (Zizler, 1973)

X, Y Banach spaces. Then

$$\big\{T \in \mathcal{L}(X,Y):\ \ T^{*}\colon Y^{*} \longrightarrow X^{*} \text{ attains its norm}\big\}$$

is dense in $\mathcal{L}(X,Y)$.

Positive results II

Definitions (Lindenstrauss, Schachermayer)

Let Z be a Banach space. Consider for two sets $\{z_i : i \in I\} \subset S_Z$, $\{z_i^* : i \in I\} \subset S_{X^*}$ and a constant $0 \leqslant \rho < 1$, the following four conditions:

1. $z_i^*(z_i) = 1, \forall i \in I$;
2. $|z_i^*(z_j)| \leqslant \rho < 1$ if $i, j \in I, i \neq j$;
3. B_Z is the absolutely closed convex hull of $\{z_i : i \in I\}$ (i.e. $\|z^*\| = \sup\{|z^*(z_i)| : i \in I\}$ for ever $z^* \in Z^*$);
4. B_{Z^*} is the absolutely weakly*-closed convex hull of $\{z_i^* : i \in I\}$ (i.e. $\|z\| = \sup\{|z_i^*(z)| : i \in I\}$ for every $z \in Z$).

- Z has property α if 1, 2, and 3 are satisfied (e.g. ℓ_1).
- Z has property β if 1, 2, and 4 are satisfied (e.g. c_0, ℓ_∞).

Theorem (Lindenstrauss, 1963)

- Property α implies property A.
- Property β implies property B.

Positive results III

Examples

- The following spaces have property α:
  - ℓ_1,
  - finite-dimensional spaces whose unit ball has finitely many extreme points (up to rotation).
- The following spaces have property β:
  - every Y such that $c_0 \subset Y \subset \ell_\infty$,
  - finite-dimensional spaces such that the dual unit ball has finitely many extreme points (up to rotation).

Examples

- The following spaces have property A: ℓ_1 and **all** finite-dimensional spaces.
- The following spaces have property B: every Y such that $c_0 \subset Y \subset \ell_\infty$, finite-dimensional spaces such that the dual unit ball has finitely many extreme points (up to rotation).
- Every finite-dimensional space has property A, but the only known (in the 1960's) finite-dimensional real spaces with property B were the polyhedral ones. Only a little bit more is known nowadays...

An overview on “classical” results on norm attaining operators

---

Section 2

The Radon-Nikodým property

Definitions

X Banach space.

- X has the Radon-Nikodým property (RNP) if the Radon-Nikodým theorem is valid for X-valued vector measures (with respect to every finite positive measure).
- $C \subset X$ is dentable if it contains slices of arbitrarily small diameter.
- $C \subset X$ is subset-dentable if every subset of C is dentable.

Theorem (Rieffel, Maynard, Huff, David, Phelps, 1970's)

X RNP $\iff$ every bounded $C \subset X$ is dentable $\iff$ B_X subset-dentable.

Remark

In the book

J. Diestel and J. J. Uhl
Vector Measures
Math. Surveys **15**, AMS, Providence 1977.

there are more than 30 different reformulations of the RNP.

The RNP and property A: positive results

Theorem (Bourgain, 1977)

X Banach space, $C \subset X$ absolutely convex closed bounded subset-dentable, Y Banach space. Then

$$\{T \in \mathcal{L}(X,Y) \colon \text{the norm of } T \text{ attains its supremum on } C\}$$

is dense in $\mathcal{L}(X,Y)$.

★ In particular, RNP $\Longrightarrow$ property A.

Remark

It is actually shown that for every bounded linear operator there are arbitrary closed **compact** perturbations of it attaining the norm.

Non-linear Bourgain-Stegall variational principle (Stegall, 1978)

X, Y Banach spaces, $C \subset X$ bounded subset-dentable, $\varphi : C \longrightarrow Y$ uniformly bounded such that $x \longmapsto \|\varphi(x)\|$ is upper semicontinuous.
Then for every $\delta > 0$, there exists $x_0^* \in X^*$ with $\|x_0^*\| < \delta$ and $y_0 \in S_Y$ such that the function $x \longmapsto \|\varphi(x) + x^*(x)y_0\|$ attains its supremum on C.

The RNP and property A: negative results

Theorem (Bourgain, 1977)

$C \subset X$ separable, bounded, closed and convex,
$\{T \in \mathcal{L}(X,Y)$: the norm of T attains its supremum on $C\}$ dense in $\mathcal{L}(X,Y)$.
$\Longrightarrow$ C is dentable.

★ In particular, if X is separable and has property A $\Longrightarrow$ B_X is dentable.

Remark

Lindenstrauss actually showed that if X is separable and has property A
$\Longrightarrow$ B_X is the closed convex hull of its strongly exposed points.

A refinement (Huff, 1980)

X Banach space failing the RNP.

Then there exist X_1 and X_2 equivalent renorming of X such that

$$\mathrm{NA}(X_1, X_2) \text{ is NOT dense in } \mathcal{L}(X,Y).$$

The RNP and property A: isomorphic characterization

Main consequence

Every renorming of X has property A $\iff$ X has the RNP.

Example

ℓ_1 has property A in every equivalent norm.

Another consequence

Every renorming of X has property B $\implies$ X has the RNP.

Observations

1. The converse of the implication above is NOT TRUE (Gowers, 1990)
2. To get an equivalence, a weaker property is needed, quasi norm attainment:

G. Choi, Y.-S. Choi, M. Jung, M. M.
On quasi norm attaining operators between Banach spaces
RACSAM (2022)

An overview on "classical" results on norm attaining operators

Section 2

Counterexamples for property B

Observation

It was an open question in the 1980's whether RNP $\implies$ property B

Counterexamples

- (Gowers, 1990): ℓ_p does not have property B for any $1 < p < \infty$.
- (Acosta, 1999): No infinite-dimensional strictly convex space has property B.
- (Acosta, 1999): ℓ_1 fails property B.

Consequence

Y separable, every renorming of Y has property B $\implies$ Y is finite-dimensional

The main open problem

★ Do all finite-dimensional spaces have property B**?**
Equivalently, does $\mathcal{F}(X,Y) \subset \overline{\mathrm{NA}(X,Y)}$ for all Banach spaces X and Y**?**

An overview on "classical" results on norm attaining operators

Section 2

Some classical spaces: positive results

Example (Johnson-Wolfe, 1979)

In the real case, $\mathrm{NA}(C(K_1), C(K_2))$ is dense in $\mathcal{L}(C(K_1), C(K_2))$.

Example (Iwanik, 1979)

$\mathrm{NA}(L_1(\mu), L_1(\nu))$ is dense in $\mathcal{L}(L_1(\mu), L_1(\nu))$.

Theorem (Schachermayer, 1983)

Every weakly compact operator from $C(K)$ can be approximated by (weakly compact) norm attaining operators.

Consequence (Schachermayer, 1983)

$\mathrm{NA}(C(K), L_p(\mu))$ is dense in $\mathcal{L}(C(K), L_p(\mu))$ for $1 \leqslant p < \infty$.

Example (Finet-Payá, 1998)

$\mathrm{NA}(L_1[0,1], L_\infty[0,1])$ is dense in $\mathcal{L}(L_1[0,1], L_\infty[0,1])$.

Some classical spaces: negative results

Example (Schachermayer, 1983)

$$\mathrm{NA}(L_1[0,1], C[0,1]) \text{ is NOT dense in } \mathcal{L}(L_1[0,1], C[0,1]).$$

Consequence

$C[0,1]$ does not have property B and it was the first "classical" example.

Example (Aron-Choi-Kim-Lee-M., 2015; M., 2014)

$$\left.\begin{array}{c} Z = C[0,1] \oplus_1 L_1[0,1] \\ \text{or} \\ Z = C[0,1] \oplus_\infty L_1[0,1] \end{array}\right\} \implies \mathrm{NA}(Z,Z) \text{ not dense in } \mathcal{L}(Z).$$

An overview on “classical” results on norm attaining operators

Section 2

Direct sums

Lemma (Payá-Saleh, 2000)

$\{X_i\}$, $\{Y_j\}$ families of Banach spaces,
$X = \left[\bigoplus X_i\right]_{\ell_1}$, $Y = \left[\bigoplus Y_j\right]_{c_0}$ or $Y = \left[\bigoplus Y_j\right]_{\ell_\infty}$.
TFAE:

- $\mathrm{NA}(X, Y)$ dense in $\mathcal{L}(X, Y)$,
- $\mathrm{NA}(X_i, Y_j)$ dense in $\mathcal{L}(X_i, Y_j)$ for all i, j.

Consequence

- Property A is stable by ℓ_1-sums;
- Property B is stable by c_0-sums and by ℓ_∞-sums.

Observation

- Property α is NOT stable by ℓ_1-sums;
- Property β is NOT stable by c_0-sums.

Properties quasi-α and quasi-β

Two new properties

- Acosta-Aguirre-Payá, 1996: introduced property quasi-β which is a weakening of property β.
- Choi-Song, 2008: introduced property quasi-α which is a weakening of property α.

Proposition

- Property quasi-α implies property A.
- Property quasi-β implies property B.

Observation

- Property quasi-β is stable by c_0-sums.

Example

- Property quasi-β provides examples of finite-dimensional real spaces with property B which are not polyhedral. . .
- but these new examples still have a "discrete" (maybe not finite) set of extreme points.

More on direct sums

Proposition (Aron-Choi-Kim-Lee-M., 2015; M., 2014)

$$Z = X \oplus_1 Y \text{ or } Z = X \oplus_\infty Y.$$

$$\mathrm{NA}(Z,Z) \text{ dense in } \mathcal{L}(Z) \implies \mathrm{NA}(X,Y) \text{ dense in } \mathcal{L}(X,Y).$$

This provides with many examples of Z such that $\mathrm{NA}(Z,Z)$ is not dense in $\mathcal{L}(Z,Z)$.

An interesting example

G domain space in Gowers' example (i.e. $\mathrm{NA}(G,\ell_2)$ not dense in $\mathcal{L}(G,\ell_2)$).

- G has property quasi-β (but not β) and so G has property B (Acosta-Aguirre-Payá, 1996).
- ℓ_2 has property A.
- But for $Z = G \oplus_\infty \ell_2$, $\mathrm{NA}(Z,Z)$ is not dense in $\mathcal{L}(Z)$.

An overview on "classical" results on norm attaining operators

Section 2

The question of norm attainment for compact operators

Question (open from 1970's till 2014)

Can every compact operator be approximated by norm-attaining operators?

Observations

- In all the negative examples of the previous sections, the authors constructed NON COMPACT operators which cannot be approximated by norm attaining operators.
- Actually, the idea of the proofs is to use that the operator which is not going to be approximated is not compact or, even, it is an isomorphism.
- In most examples, it was even known that compact operators attaining the norm are dense.

Positive results on norm attaining compact operators

Question (open from 1970's till 2014)

Can every compact operator be approximated by norm-attaining operators?

Positive results

- If X is reflexive, then ALL compact operators from X into Y are norm attaining. (Indeed, compact operators carry weak convergent sequences to norm convergent sequences.)
- Some classical spaces (Johnson-Wolfe, 1979):
  - $X = C_0(L)$ or $X = L_1(\mu)$, Y arbitrary;
  - X arbitrary, $Y = L_1(\mu)$ (only real case) or $Y^* \equiv L_1(\mu)$;
  - X arbitrary, $Y \leqslant c_0$ with AP.
- More recent results:
  - (Cascales-Guirao-Kadets, 2013) X arbitrary, Y uniform algebra;
  - (M. 2014) $X^* = \ell_1$, Y arbitrary.

The solution

Negative answer (M. 2014)

There are compact operators which cannot be approximated by norm attaining operators

Some examples

- When $X \leqslant c_0$ with X^* failing AP, exists Y such that...
- Exists $X \leqslant c_0$ with AP and Y such that...
- When Y is strictly convex without AP, exists X such that...

$\mathcal{K}(X,Y)$ is not contained in $\overline{\mathrm{NA}(X,Y)}$.

The remaining open problem

Can every **finite-rank** operator be approximated by norm-attaining operators?

Note

We do not even know whether it may exists a Banach space X such that the elements in $\mathrm{NA}(X,\ell_2)$ are of rank one.

What is residuality and why can it be interesting here?

Section 3

What is residuality or typicality?

Residual set

C subset of a complete metric space M is residual if $M \setminus C$ is of the first Baire category. Equivalently, C contains a G_δ dense subset. The elements of C are called typical.

Equivalent reformulation

M complete metric space, $C \subset M$. TFAE:

- C is residual
- $C = \bigcap_{n=1}^{\infty} C_n$ and $\operatorname{int}(C_n)$ dense for all n
- $C \supseteq \bigcap_{n=1}^{\infty} O_n$ and O_n open dense for all n
  (i.e. C contains a subset which is G_δ in M and dense in M)

Main property (Baire's Category Theorem)

The countable intersection of residual sets is residual, hence dense.

Why residuality can be interesting here?

Consequences of the residuality of norm attaining operators

X, Y Banach spaces, suppose $\mathrm{NA}(X,Y)$ is residual in $\mathcal{L}(X,Y)$. Then:

- $\mathcal{L}(X,Y) = \mathrm{NA}(X,Y) - \mathrm{NA}(X,Y)$,
- Given $\{S_n\} \subset \mathcal{L}(X,Y)$ (maybe unbounded), the set

$$\big\{T \in \mathcal{L}(X,Y)\colon S_n + T \in \mathrm{NA}(X,Y)\big\}$$

  is residual (in particular, dense) in $\mathcal{L}(X,Y)$.
  ★ In particular, given $\varepsilon > 0$ there is $T \in \mathcal{L}(X,Y)$ with $\|T\| < \varepsilon$ such that $S_n + T \in \mathrm{NA}(X,Y)$ for all $n \in \mathbb{N}$.

Example

$\mathrm{NA}(c_0,\mathbb{K}) = \ell_1 \cap c_{00} \subsetneq \ell_1$, so it is not residual. Besides:

- $\mathrm{NA}(c_0,\mathbb{K}) - \mathrm{NA}(c_0,\mathbb{K}) = \ell_1 \cap c_{00} \neq \ell_1$,
- Given $x_1^* = 0$ and $x_2^* \in \ell_1 \setminus c_{00}$, there is NO $x^* \in \mathcal{L}(c_0,\mathbb{K})$ such that $x_1^* + x^* \in \mathrm{NA}(c_0,\mathbb{K})$ and $x_2^* + x^* \in \mathrm{NA}(c_0,\mathbb{K})$.

How frequent is residuality of norm-attaining operators?

Theorem (Bourgain, 1977)

X RNP $\implies \mathrm{NA}(X,Y)$ is residual for every Y.

Remark

The converse result is not true: ALL known sufficient conditions for property A on X actually gives residuality of $\mathrm{NA}(X,Y)$ for all Y's.

Example where there is no residuality

$\mathrm{NA}(c_0,\mathbb{K})$ is dense and $X^*\setminus \mathrm{NA}(c_0,\mathbb{K})$ is dense and second category.

Even stronger properties

In some (unfrequent) cases, $\mathrm{NA}(X,Y)$ contains an OPEN dense subset:

- $X=\ell_1$, Y arbitrary.
- $X=\mathcal{F}(M)$, Y arbitrary, for some metric concrete compact spaces M such that $\mathcal{F}(M)$ DO NOT have RNP.

Recalling Lindenstrauss' and Bourgain's results on property A

Section 4

Recalling Lindenstrauss' and Bourgain's results on property A

Section 4

Definitions

Definition 1: strong exposition

$C \subset X$ bounded. $x_0 \in C$ is strongly exposed if there is $x^* \in X^*$ such that whenever $\{x_n\} \subset C$ satisfies $\operatorname{Re} x^*(x_n) \longrightarrow \sup \operatorname{Re} x^*(C)$, then $\{x_n\} \longrightarrow x_0$.
Equivalently, the slices of C defined by x^* contain x_0 and are arbitrarily small.

- In this case, we say that x^* strongly exposes C (at x_0).
- $\text{str-exp}(C)$ set of strongly exposed points of C.
- $\mathrm{SE}(C)$ functionals which strongly expose C at some (strongly exposed) point.
- $\mathrm{SE}(C)$ is a G_δ subset of X^*.

For the case $C = B_X \ldots$

- If $\mathrm{SE}(B_X)$ is dense, $\mathrm{NA}(X, \mathbb{K})$ is residual.
- Šmulyan's test:

$$x^* \in \mathrm{SE}(B_X) \iff \text{the norm of } X^* \text{ is Fréchet-differentiable at } x^*$$

Lindenstrauss's and Bourgain's necessary conditions on property A

Lindenstrauss, 1963

If X admits a LUR renorming and has property A
$\implies$ B_X is the closed convex hull of $\text{str-exp}(B_X)$.

Bourgain, 1977

$C \subseteq X$ *separable bounded closed convex* such that for every Y the set

$$\left\{T \in \mathcal{L}(X,Y) \colon \exists \max_{x \in C} \|Tx\|\right\}$$

is dense in $\mathcal{L}(X,Y)$ (C has the Bishop-Phelps property in Bourgain's terminology)
$\implies$ C is dentable (i.e. C contains slices of arbitrarily small diameter).

Our result (to be shown in session 2)

X admitting a LUR renorming, $C \subseteq X$ bounded with the Bishop-Phelps property
$\implies$ $\mathrm{SE}(C)$ dense in X^* (hence $C = \overline{\mathrm{conv}}\big(\text{str-exp}(B_X)\big)$).
★ In particular, X admitting a LUR renorming, X with property A
$\implies$ $\mathrm{SE}(B_X)$ is dense in X^*, hence $\mathrm{NA}(X,\mathbb{K})$ is residual.

Recalling Lindenstrauss' and Bourgain's results on property A

Section 4

Lindenstrauss: uniform strong exposition

Set of uniformly strongly exposed points

X Banach space, $A \subset S_X$ is a set of uniformly strongly exposed points if for every $a \in A$ there is $a^* \in S_{X^*}$ with $\operatorname{Re} a^*(a) = 1$ satisfying that for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$x \in B_X, \quad \operatorname{Re} a^*(x) > 1 - \delta \quad \Longrightarrow \quad \|x - a\| < \varepsilon.$$

(That is, the elements in B are strongly exposed in a uniform way).

Lindenstrauss, 1963

If $B_X = \overline{\operatorname{conv}}(A)$ with A uniformly strongly exposed, then X has property A.

Examples

- Obviously, if X is uniformly convex.
- Property α.

This not cover...

Reflexivity (also Lindenstrauss), Property quasi-α (Choi-Song), RNP (Bourgain).

The RNP and absolutely strongly exposing operators

Definition (Bourgain, 1977)

$T \in \mathcal{L}(X,Y)$ is absolutely strongly exposing ($T \in \mathrm{ASE}(X,Y)$) iff there exists $x_0 \in S_X$ such that whenever $\{x_n\} \subset B_X$ satisfies $\|T(x_n)\| \longrightarrow \|T\|$ then $\exists \{\theta_n\} \subset \mathbb{T}$ for which $\{\theta_n x_n\} \longrightarrow x_0$.

★ $\mathrm{ASE}(X,Y)$ is a G_δ-set. Therefore, if $\mathrm{ASE}(X,Y)$ is dense, $\mathrm{NA}(X,Y)$ is residual.

Theorem (Bourgain, 1977)

X RNP, Y arbitrary $\implies$ $\mathrm{ASE}(X,Y)$ is dense.

★ It is just used in the proof that B_X is subset-dentable.

Much more: the non-linear Bourgain-Stegall variational principle (Stegall, 1978)

$C \subset X$ bounded RNP set, $\varphi\colon C \longrightarrow \mathbb{R}$ bounded upper semicontinuous.
Then, the set

$$\left\{x^* \in X^* : \varphi + \operatorname{Re} x^* \text{ strongly exposed } C\right\}$$

is residual in X^*.

What more? (I)

Observation (Chiclana–GarcíaLirola–M.–RuedaZoca, 2021)

ALL known sufficient conditions for property A actually imply that absolutely strongly exposing operators are dense:

- RNP,
- properties α and quasi-α,
- $B_X = \overline{\mathrm{conv}}(A)$, A uniformly strongly exposed.

Open problem 1 (still open)

Does the property A of X imply that $\mathrm{ASE}(X, Y)$ is dense for every Y**?**

Observation

If $\mathrm{ASE}(X, Y)$ is dense for some $Y \implies \mathrm{SE}(X)$ is dense.

Open problem 2 (still open)

Does the denseness of $\mathrm{SE}(X)$ imply that $\mathrm{ASE}(X, Y)$ is dense for every Y**?**

What more? (II)

Less ambitious question

If $\mathrm{SE}(X)$ is dense, for which Ys is $\mathrm{ASE}(X,Y)$ dense**?**

Examples of when $\mathrm{SE}(X)$ is dense

- If X has RNP,
- If X has property A and admits a LUR renorming,
- If X is LUR (a property which is not known to imply property A),
- If $\text{str-exp}(B_X) = S_X$ (a property which is not known to imply property A),
- $X = JT^*$ (the dual of the James-tree space, not known if it has property A).

The main objective (for session 3)

To find spaces Y (better if they are not known to have property B) such that $\mathrm{ASE}(X,Y)$ is dense whenever $\mathrm{SE}(X)$ is dense.