# Residuality in the set of norm attaining operators

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# *Preliminaries*

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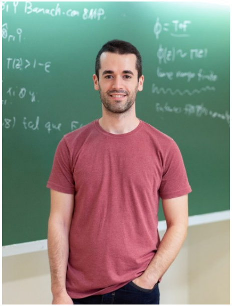
## Section 1

# The talk is based on the preprint

M. Jung, M. Martín, and A. Rueda Zoca.
Residuality in the set of norm attaining operators between Banach spaces.
*Preprint* (2022). Arxiv code: 2203.04023

Mingu Jung (KIAS, Korea)



Abraham Rueda Zoca (Murcia, Spain)

# Notation

$X$, $Y$ real or complex Banach spaces

- $\mathbb{K}$ base field $\mathbb{R}$ or $\mathbb{C}$,
- $\mathbb{T}$ modulus one scalars,
- $B_X = \{x \in X \colon \|x\| \leqslant 1\}$ closed unit ball of $X$,
- $S_X = \{x \in X \colon \|x\| = 1\}$ unit sphere of $X$,
- $\overline{\operatorname{conv}}(C)$ closed convex hull of $C$,
- $\mathcal{L}(X, Y)$ bounded linear operators from $X$ to $Y$,
  - $\|T\| = \sup\{\|T(x)\| \colon x \in S_X\}$,
- $\mathcal{K}(X, Y)$ compact linear operators from $X$ to $Y$,
- $\mathcal{F}(X, Y)$ bounded linear operators from $X$ to $Y$ with finite rank,
- $X^* = \mathcal{L}(X, \mathbb{K})$ topological dual of $X$.

# Norm attaining functionals

## Norm attaining functionals

$x^* \in X^*$ attains its norm when

$$\exists\, x \in S_X \;:\; |x^*(x)| = \|x^*\|$$

★ $\mathrm{NA}(X, \mathbb{K}) := \{x^* \in X^* : x^* \text{ attains its norm}\}$

## Examples and comments

- $\dim(X) < \infty \implies \mathrm{NA}(X, \mathbb{K}) = \mathcal{L}(X, \mathbb{K})$ (Heine-Borel).
- $X$ reflexive $\iff \mathrm{NA}(X, \mathbb{K}) = \mathcal{L}(X, \mathbb{K})$ (Hahn-Banach, James).
- $\mathrm{NA}(c_0, \mathbb{K}) = c_{00} \leqslant \ell_1$,
- $\mathrm{NA}(\ell_1, \mathbb{K}) = \Big\{x \in \ell_\infty : \|x\|_\infty = \max_n\{|x(n)|\}\Big\} \subseteq \ell_\infty$, residual, contains $c_0$,
- $\mathrm{NA}(X, \mathbb{K})$ may be "wild", for instance:
  - it may contain NO two-dimensional subspaces (Read, 2017; Rmoutil, 2017),
  - it can be NOT norm Borel (Kaufman, 1991).
- (Petunin–Plichko 1974; Godefroy 1987): $X$ separable, $Z \leqslant X^*$ closed, separating for $X$, $Z \subseteq \mathrm{NA}(X, \mathbb{K}) \implies Z$ is an isometric predual of $X$.

# Norm attaining operators

## Norm attaining operators

$T \in \mathcal{L}(X,Y)$ attains its norm when

$$\exists\, x \in S_X \ : \ \|T(x)\| = \|T\|$$

★ $\mathrm{NA}(X,Y) := \{T \in \mathcal{L}(X,Y) \colon T$ attains its norm$\}$

## Some examples and comments

- $\dim(X) < \infty \implies \mathrm{NA}(X,Y) = \mathcal{L}(X,Y)$ for every $Y$ (Heine-Borel),
- $\dim(X) = \infty \implies \mathrm{NA}(X,c_0) \neq \mathcal{L}(X,c_0)$ (see M.-Merí-Payá, 2006).
- $X$ reflexive $\iff \mathcal{K}(X,Y) \subseteq \mathrm{NA}(X,Y)$ for every $Y$ (James).
- $\mathcal{L}(X,\ell_\infty) = \left[\bigoplus_{n\in\mathbb{N}} \mathcal{L}(X,\mathbb{K})\right]_{\ell_\infty} = \ell_\infty(X^*)$.
  $\mathrm{NA}(X,\ell_\infty) = \left\{(x_n^*) \in \ell_\infty(X^*) \colon \exists k \in \mathbb{N},\ \|x_k^*\| = \|(x_n^*)\|_\infty,\ x_k^* \in \mathrm{NA}(X,\mathbb{K})\right\}$.
- $\mathcal{L}(\ell_1,Y) = \left[\bigoplus_{n\in\mathbb{N}} \mathcal{L}(\mathbb{K},Y)\right]_{\ell_\infty} = \ell_\infty(Y)$.
  $\mathrm{NA}(\ell_1,Y) = \left\{(y_n) \in \ell_\infty(Y) \colon \exists k \in \mathbb{N},\ \|y_k\| = \|(y_n)\|_\infty\right\}$.
- $\mathrm{NA}(L_1[0,1], L_\infty[0,1])$???

# The problem of denseness of norm attaining functionals

**Problem**

Is $\mathrm{NA}(X, \mathbb{K})$ always dense in $X^*$**?**

**Theorem (E. Bishop & R. Phelps, 1961)**

The set of norm attaining functionals is dense in $X^*$ (for the norm topology).

**Problem**

Is $\mathrm{NA}(X, Y)$ always dense in $\mathcal{L}(X, Y)$**?**

The answer is **No**, and this is the origin of the study of norm attaining operators.

**Modified problem**

When is $\mathrm{NA}(X, Y)$ dense in $\mathcal{L}(X, Y)$**?**

The study of this problem was initiated by J. Lindenstrauss in 1963, who provided the first negative and positive examples.

# Lindenstrauss' seminal paper of 1963

## Negative answer

There are bounded linear operators which cannot be approximated by norm attaining operators

## Idea:

$Y$ LUR, $T\colon X \longrightarrow Y$ bounded from below (monomorphism).
If $T$ attains its norm, then it does at a strongly exposed point.

## Examples

Take $X$ separable without strongly exposed points (e.g. $c_0$, $C[0,1]$, $L_1[0,1]$), let $Y$ be a LUR renorming of $X$. Then, $\mathrm{NA}(X,Y)$ is not dense in $\mathcal{L}(X,Y)$.

## Observation

- The question then is for which $X$ and $Y$ the density holds.
- As this problem is too general, Lindenstrauss introduced two properties.

# Lindenstrauss properties A and B

## Definition

$X$, $Y$ Banach spaces,

- $X$ has (Lindenstrauss) property A iff $\overline{\mathrm{NA}(X,Z)} = \mathcal{L}(X,Z) \quad \forall Z$
- $Y$ has (Lindenstrauss) property B iff $\overline{\mathrm{NA}(Z,Y)} = \mathcal{L}(Z,Y) \quad \forall Z$

## Examples

- Reflexive spaces have property A,
- $\ell_1$ has property A (property $\alpha$),
- $c_0$, $C[0,1]$, $L_1[0,1]$ fail property A,

- $\mathbb{K}$ has property B (Bishop-Phelps theorem),
- every $Y$ such that $c_0 \subset Y \subset \ell_\infty$ (canonical copies) has property B (property $\beta$),
- every LUR renorming of $c_0$ fails property B,
- finite-dimensional polyhedral spaces have property B (property $\beta$),
- there are SOME non-polyhedral finite-dimensional spaces which are known to have property B (property quasi-$\beta$, Acosta-Aguirre-Payá, 1996).

# The relationship with the Radon-Nikodým property

**Theorem (Bourgain, 1977)**

Radon Nikodým Property (RNP) $\Longrightarrow$ property A.

**Much more: the non-linear Bourgain-Stegall variational principle (Stegall, 1978)**

$X$, $Y$ Banach spaces, $C \subset X$ bounded RNP set, $\varphi\colon C \longrightarrow Y$ uniformly bounded such that $x \longmapsto \|\varphi(x)\|$ is upper semicontinuous.
Then for every $\delta > 0$, there exists $x_0^* \in X^*$ with $\|x_0^*\| < \delta$ and $y_0 \in S_Y$ such that the function $x \longmapsto \|\varphi(x) + x_0^*(x)y_0\|$ strongly attains its supremum on $C$.

**Theorem (Bourgain, 1977)**

$X$ separable with property A $\Longrightarrow$ $B_X$ is dentable.

**Consequence (uses a refinement by Huff, 1980)**

Every renorming of $X$ has property A $\Longleftrightarrow$ $X$ has the RNP.
Every renorming of $Y$ has property B $\Longrightarrow$ $X$ has the RNP.

# Counterexamples for property B

## Observation

It was an open question in the 1980's whether RNP $\implies$ property B

## Counterexamples

- (Gowers, 1990): $\ell_p$ does not have property B for any $1 < p < \infty$.
- (Acosta, 1999): No infinite-dimensional strictly convex space has property B.

## Consequence

$Y$ separable, every renorming of $Y$ has property B $\implies$ $Y$ is finite-dimensional

## The main open problem

★ Do all finite-dimensional spaces have property B**?**
Equivalently, does $\mathcal{F}(X,Y) \subset \overline{\mathrm{NA}(X,Y)}$ for all Banach spaces $X$ and $Y$**?**

## We cannot go further... (M. 2014)

There are **compact** operators which cannot be approximated by norm attaining ones.

## Some pairs of classical spaces

**Example (Johnson-Wolfe, 1979)**

In the real case, $\mathrm{NA}(C(K_1), C(K_2))$ is dense in $\mathcal{L}(C(K_1), C(K_2))$.

**Example (Iwanik, 1979)**

$\mathrm{NA}(L_1(\mu), L_1(\nu))$ is dense in $\mathcal{L}(L_1(\mu), L_1(\nu))$.

**Examples (Schachermayer, 1983)**

$\mathrm{NA}(C(K), L_p(\mu))$ is dense in $\mathcal{L}(C(K), L_p(\mu))$ for $1 \leqslant p < \infty$.

**Example (Finet-Payá, 1998)**

$\mathrm{NA}(L_1[0,1], L_\infty[0,1])$ is dense in $\mathcal{L}(L_1[0,1], L_\infty[0,1])$.

**Example (Schachermayer, 1983)**

$\mathrm{NA}(L_1[0,1], C[0,1])$ is NOT dense in $\mathcal{L}(L_1[0,1], C[0,1])$.

## Why residuality is interesting here?

### Residual set

A subset $C$ of a complete metric space $M$ is residual if $M \setminus C$ is of the first Baire category. Equivalently, if $C$ contains a $G_\delta$ dense subset.

### Residuality of norm attaining operators

$X$, $Y$ Banach spaces, suppose $\mathrm{NA}(X,Y)$ is residual in $\mathcal{L}(X,Y)$. Then:

- $\mathcal{L}(X,Y) = \mathrm{NA}(X,Y) - \mathrm{NA}(X,Y)$,
- Given $\{S_n\} \subset \mathcal{L}(X,Y)$, the set

$$\{T \in \mathcal{L}(X,Y) \colon S_n + T \in \mathrm{NA}(X,Y)\}$$

  is residual (in particular, dense) in $\mathcal{L}(X,Y)$.

### Example

$\mathrm{NA}(c_0,\mathbb{K}) = \ell_1 \cap c_{00} \subsetneq \ell_1$, so it is not residual. Besides:

- $\mathrm{NA}(c_0,\mathbb{K}) - \mathrm{NA}(c_0,\mathbb{K}) = \ell_1 \cap c_{00} \neq \ell_1$,
- Given $x_1^* = 0$ and $x_2^* \in \ell_1 \setminus c_{00}$, there is NO $x^* \in \mathcal{L}(c_0,\mathbb{K})$ such that $x_1^* + x^* \in \mathrm{NA}(c_0,\mathbb{K})$ and $x_2^* + x^* \in \mathrm{NA}(c_0,\mathbb{K})$.

# Roadmap of the talk

# *Necessary conditions for the denseness of norm attaining operators*

Section 2

# A bit of notation

**Definition 1**

$C \subset X$ bounded. $x_0 \in C$ is strongly exposed if there is $x^* \in X^*$ such that whenever $\{x_n\} \subset C$ satisfies $\operatorname{Re} x^*(x_n) \longrightarrow \sup \operatorname{Re} x^*(C)$, then $\{x_n\} \longrightarrow x_0$.
Equivalently, the slices of $C$ defined by $x^*$ contain $x_0$ and are arbitrarily small.

★ In this case, we say that $x^*$ strongly exposes $C$ (at $x_0$).

**For the case $C = B_X$...**

- The set of strongly exposed points of $B_X$ is denoted by $\text{str-exp}(B_X)$.
- We write $\mathrm{SE}(X)$ for the set of functionals strongly exposing $B_X$ which is a $G_\delta$-set.
- Hence, if $\mathrm{SE}(X)$ is dense, $\mathrm{NA}(X, \mathbb{K})$ is residual.
- $x^* \in \mathrm{SE}(X) \iff$ the norm of $X^*$ is Fréchet-differentiable at $x^*$.

# The previous results vs the new result

## Previous results

- (Lindenstrauss, 1963) $X$ admitting a LUR renorming (e.g. separable), $X$ having property A
  $\implies B_X = \overline{\text{conv}}(\text{str-exp}(B_X))$.
- (Bourgain, 1977) $C \subseteq X$ *separable bounded closed convex* such that for every $Y$ the set
  $$\left\{T \in \mathcal{L}(X,Y) \colon \exists \max_{x \in C} \|Tx\|\right\}$$
  is dense in $\mathcal{L}(X,Y)$ ($C$ has the Bishop-Phelps property in Bourgain's terminology)
  $\implies C$ is dentable (i.e. $C$ contains slices of arbitrarily small diameter).

## Our result

$X$ admitting a LUR renorming, $C \subseteq X$ bounded with the Bishop-Phelps property
$\implies$ the functionals in $X^*$ which strongly expose $C$ are dense in $X^*$.

★ In particular, $X$ admitting a LUR renorming, $X$ with property A
$\implies \text{SE}(X)$ is dense in $X^*$, hence $\text{NA}(X, \mathbb{K})$ is residual.

# Sketch of the proof of particular case

**Our result (particular case)**

$X$ admitting a LUR renorming, $X$ with property A $\Longrightarrow$ $\mathrm{SE}(X)$ is dense in $X^*$.

**Lemma**

$S\colon X \longrightarrow Y$ bounded below, $Y$ LUR, $x_0 \in S_X$ such that $\|S\| = \|Sx_0\|$. Then, $x_0$ is strongly exposed by $S^* y^*$ for every $y^* \in S_{Y^*}$ with $\operatorname{Re} y^*(Sx_0) = \|S\|$.

- Consider a LUR norm $|\!|\!| \cdot |\!|\!|$ on $X$ and let $Y = (X, |\!|\!| \cdot |\!|\!|) \oplus_2 \mathbb{K}$ which is LUR.
- For $x^* \in S_{X^*}$, define $T_n \in \mathcal{L}(X,Y)$ by $T_n(x) = (n^{-1}x, x^*(x))$, which are monomorphisms, and $S \in \mathcal{L}(X,Y)$ by $S(x) = (0, x^*(x))$. Observe $\{T_n\} \longrightarrow S$.
- We may find **monomorphisms** $S_n \in \mathrm{NA}(X,Y)$, $\|S_n\| = 1$, such that $\{S_n\} \longrightarrow S$.
- By the lemma, there are $y_n^* = (x_n^*, \lambda_n) \in Y^* = X^* \oplus_2 \mathbb{K}$ such that $S_n^* y_n^* \in \mathrm{SE}(X)$ and $\|S_n^* y_n^*\| = \|S_n\| = 1$.
- Suppose $\lambda_n \longrightarrow \lambda_0$ and observe

$$\|\lambda_0 x^* - S_n^* y_n^*\| = \|\lambda_0 x^* - (\lambda_n x^* - S^* y_n^*) - S_n^* y_n^*\| \leqslant |\lambda_0 - \lambda_n| + \|S^* - S_n^*\| \longrightarrow 0.$$

- As $\lambda_0 \neq 0$, $x^* = \lambda_0^{-1}(\lambda_0 x^*) \in \overline{\lambda_0^{-1}\,\mathrm{SE}(X)} = \overline{\mathrm{SE}(X)}$.

# An interesting example

## Example

The Lipschitz-free space on the two-dimensional Euclidean space, $\mathcal{F}(\mathbb{T})$, satisfies:

- $\mathrm{SE}\big(\mathcal{F}(\mathbb{T})\big)$ is not dense in $\mathcal{F}(\mathbb{T})^* \equiv \mathrm{Lip}_0(\mathbb{T}, \mathbb{R})$ (using known results),
- hence, by the new result, $\mathcal{F}(\mathbb{T})$ fails Lindenstrauss property A.
- On the other hand, $B_{\mathcal{F}(\mathbb{T})} = \overline{\mathrm{conv}}\big(\text{str-exp}(B_{\mathcal{F}(\mathbb{T})})\big)$ (so it satisfies Lindenstrauss necessary condition for property A).
- Besides, the set of those elements in $\mathcal{F}(\mathbb{T})^*$ which attain their norms at extreme points of $B_{\mathcal{F}(\mathbb{T})}$ is not dense in $\mathcal{F}(\mathbb{T})^*$.

## Compare with...

- (Lindenstrauss): If $B_X = \overline{\mathrm{conv}}(C)$ and the elements of $C$ are *uniformly* strongly exposed, then $X$ has property A.
- (Bourgain): $X$ has RNP $\iff$ $\text{str-exp}(B_Z) \neq \emptyset \ \forall Z \simeq X$
  $\iff$ $B_Z = \overline{\mathrm{conv}}(\text{str-exp}(B_Z)) \ \forall Z \simeq X$
  $\iff$ $\mathrm{SE}(Z)$ dense $\forall Z \simeq X$.

# *From residuality of norm attaining functionals to residuality of norm attaining operators*

Section 3

# Absolutely strongly exposing operators

**Definition (Bourgain, 1977)**

$T \in \mathcal{L}(X,Y)$ is absolutely strongly exposing ($T \in \mathrm{ASE}(X,Y)$) iff there exists $x_0 \in S_X$ such that whenever $\{x_n\} \subset B_X$ satisfies $\|T(x_n)\| \longrightarrow \|T\|$ then $\exists \{\theta_n\} \subset \mathbb{T}$ for which $\{\theta_n x_n\} \longrightarrow x_0$.

★ $\mathrm{ASE}(X,Y)$ is a $G_\delta$-set. Therefore, if $\mathrm{ASE}(X,Y)$ is dense, $\mathrm{NA}(X,Y)$ is residual.

**Theorem (Bourgain, 1977)**

$X$ RNP, $Y$ arbitrary $\implies$ $\mathrm{ASE}(X,Y)$ is dense.

**Observation (Chiclana–GarcíaLirola–M.–RuedaZoca, 2021)**

ALL known sufficient conditions for property A actually imply that absolutely strongly exposing operators are dense:

- RNP,
- properties $\alpha$ and quasi-$\alpha$,
- $B_X = \overline{\mathrm{conv}}(C)$, $C$ uniformly strongly exposed.

# The proposed questions. I

**Observation (Chiclana–GarcíaLirola–M.–RuedaZoca, 2021)**

ALL known sufficient conditions for property A actually imply that absolutely strongly exposing operators are dense.

**Open problem 1**

Does the property A of $X$ imply that $\mathrm{ASE}(X,Y)$ is dense for every $Y$**?**

**Observation**

If $\mathrm{ASE}(X,Y)$ is dense for some $Y \implies \mathrm{SE}(X)$ is dense.

**Open problem 2**

Does the denseness of $\mathrm{SE}(X)$ imply that $\mathrm{ASE}(X,Y)$ is dense for every $Y$**?**

# The proposed questions. II

## Less ambitious question

If $\mathrm{SE}(X)$ is dense, for which $Y$s is $\mathrm{ASE}(X, Y)$ dense**?**

## Examples of when $\mathrm{SE}(X)$ is dense

- If $X$ has RNP,
- If $X$ has property A and admits a LUR renorming,
- If $X$ is LUR (a property which is not known to imply property A),
- If $\text{str-exp}(B_X) = S_X$ (a property which is not known to imply property A).

## Our main objective

To find spaces $Y$ (better if they are not known to have property B) such that $\mathrm{ASE}(X, Y)$ is dense whenever $\mathrm{SE}(X)$ is dense.

# When the denseness of norm attaining operators was already known

## Main result in this case

If $\mathrm{SE}(X)$ is dense and $Y$ has any of the known sufficient conditions for property B ($\beta$ or quasi-$\beta$ properties), then $\mathrm{ASE}(X, Y)$ is dense.

## This includes...

- Property $\beta$: e.g. $Y$'s such that $c_0 \leqslant Y \leqslant \ell_\infty$ (canonical copies) or $Y$ finite-dimensional polyhedral.
- Property quasi-$\beta$: some more examples including $Y$ finite-dimensional non-polyhedral in dimension greater than or equal to $3$.
- Restricting to compact operators, there are more examples like isometric preduals of $L_1(\mu)$, uniform algebras,...

## A by-product of our study

Every closed subspace of $c_0$ has property quasi-$\beta$ and hence, Lindenstrauss property B.

## A first family of new examples. The general result

**Theorem**

$X$, $Y$ Banach spaces, $\mathcal{I}(X,Y) \leqslant \mathcal{L}(X,Y)$ containing rank-one operators. Suppose:

- $\mathrm{SE}(X)$ is dense,
- there is $\{y_n^*\} \subset S_{Y^*}$ such that the set $\mathcal{A} = \{T \in \mathcal{I}(X,Y) \colon \|T\| = \|T^* y_n^*\| \text{ for some } n \in \mathbb{N}\}$ is residual in $\mathcal{I}(X,Y)$.

Then, $\mathrm{ASE}(X,Y) \cap \mathcal{I}(X,Y)$ is dense in $\mathcal{I}(X,Y)$.

Idea of the proof:

**Lemma**

$T \in \mathcal{L}(X,Y)$, $y^* \in S_{Y^*}$ with $T^* y^* \in \mathrm{SE}(X)$, $\|T^* y^*\| = \|T\|$, then there is $x_0 \in \text{str-exp}(B_X)$ such that $\|T x_0\| = \|T\|$, and this implies that $T \in \overline{\mathrm{ASE}(X,Y)}$.

- The set $\mathcal{B} = \{T \in \mathcal{I}(X,Y) \colon T^* y_n^* \in \mathrm{SE}(X)\ \forall n \in \mathbb{N}\}$ is residual,
- $\mathcal{A} \cap \mathcal{B}$ is residual and contained in $\overline{\mathrm{ASE}(X,Y) \cap \mathcal{I}(X,Y)}$.

# A first family of new examples. Consequences I

## Consequence 1

$\mathrm{SE}(X)$ dense, $Y^*$ RNP with $\text{str-exp}(B_{Y^*})$ countable up to rotations. Then:

$$\mathrm{ASE}(X,Y) \text{ dense in } \mathcal{L}(X,Y), \quad \mathrm{ASE}(X,Y)\cap\mathcal{K}(X,Y) \text{ dense in } \mathcal{K}(X,Y).$$

## This result applies to...

- $Y$ being a predual of $\ell_1$,
- $Y$ being finite-dimensional such that $\operatorname{ext}(B_{Y^*})$ is countable (up to rotation),
- $Y = \mathrm{lip}_0(M)$ when $M$ is a countable compact metric space.

## Consequence 2

$\mathrm{SE}(X)$ dense, $Y$ RNP with $\text{str-exp}(B_Y)$ countable up to rotations. Then:

$$\mathrm{ASE}(X,Y^*) \text{ dense in } \mathcal{L}(X,Y^*), \quad \mathrm{ASE}(X,Y^*)\cap\mathcal{K}(X,Y^*) \text{ dense in } \mathcal{K}(X,Y^*).$$

## This result applies to...

- $Y = \mathcal{F}(M)$ (so $Y^* = \mathrm{Lip}_0(M)$) when $M$ is a countable proper metric space.

# A first family of new examples. Consequences II

## Consequence 3

$\mathrm{SE}(X)$ dense, $Y$ such that every separable subspace of $Y$ admits a countable James boundary. Then:

$$\mathrm{ASE}(X,Y) \cap \mathcal{K}(X,Y) \text{ dense in } \mathcal{K}(X,Y).$$

## This result applies to...

- $Y$ polyhedral real Banach space,
- $Y$ closed subspace of (the real or complex space) $C(K)$ where $K$ is a Hausdorff scattered compact space.

(no separability condition is needed!)

# A second family of new examples. The general result

**Theorem**

$X$, $Y$ Banach spaces, $\mathcal{I}(X,Y) \leqslant \mathcal{L}(X,Y)$ containing rank-one operators. Suppose:

- $\mathrm{SE}(X)$ is dense,
- $Y$ has the RNP and $\text{str-exp}(B_Y)$ is discrete up to rotations (i.e. for every sequence $\{y_n\}$ of elements of $\text{str-exp}(B_Y)$ converging to an element $y_0 \in \text{str-exp}(B_Y)$, there is a sequence $\{\theta_n\} \subset \mathbb{T}$ such that $y_n = \theta_n y_0$ for large $n$).

Then, $\mathrm{ASE}(X,Y^*) \cap \mathcal{I}(X,Y^*)$ is dense in $\mathcal{I}(X,Y^*)$.

Idea of the proof:

- We use Stegall variational principle in $\mathcal{L}(Y,X^*) \equiv \mathcal{L}(X,Y^*)$.
- We use Bourgain's ideas, the discreteness hypothesis, and the residuality of $\mathrm{SE}(X)$, to get operators $T\colon Y \longrightarrow X^*$ and norm-one elements $y$ such that $\|Ty\| = \|T\|$ and $Ty \in \mathrm{SE}(X)$.
- The (pre)adjoints of these operators attains their norms at strongly exposed points of $B_X$. Hence, they belong to $\overline{\mathrm{ASE}(X,Y^*)}$.

## A second family of new examples. Consequence

### Consequence 4

$\mathrm{SE}(X)$ dense, $Y$ RNP with $\text{str-exp}(B_Y)$ discrete up to rotations. Then:

$$\mathrm{ASE}(X, Y^*) \text{ dense in } \mathcal{L}(X, Y^*), \quad \mathrm{ASE}(X, Y^*) \cap \mathcal{K}(X, Y^*) \text{ dense in } \mathcal{K}(X, Y^*).$$

### This result applies to...

- $Y = \mathcal{F}(M)$ (hence $Y^* = \mathrm{Lip}_0(M)$) when $M$ is a discrete metric space.

# *Some open problems*

Section 4

# Some open problems

## Open problem 1

Does the denseness of $\mathrm{SE}(X)$ imply Lindenstrauss property A**?**

## Remarks

- A positive answer would give an isometric characterization of property A for (say) separable spaces.
- Otherwise, maybe property A implies something more than $\mathrm{SE}(X)$ dense...

## Open problem 2

Find more conditions on $Y$ to get denseness of $\mathrm{ASE}(X, Y)$ from the one of $\mathrm{SE}(X)$:

- finite-dimensional $Y$'s**?**
- Asplundness of $Y$**?**
- $Y = Z^*$ for $Z$ RNP**?**

## Remark

In our proofs we use the RNP of $Y^*$ or $Y_*$ and also some discreteness assumptions on the extremal structure.