When the typical operator is norm attaining?

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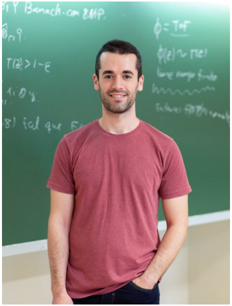
Preliminaries

Section 1

The talk is based on the preprint

M. Jung, M. Martín, and A. Rueda Zoca.
Residuality in the set of norm attaining operators between Banach spaces.
Preprint (2022). Arxiv code: 2203.04023

Mingu Jung (KIAS, Korea)



Abraham Rueda Zoca (Murcia, Spain)

Notation

X, Y real or complex Banach spaces

- $\mathbb{K}$ base field $\mathbb{R}$ or $\mathbb{C}$,
- $\mathbb{T}$ modulus one scalars,
- $B_X = \{x \in X \colon \|x\| \leqslant 1\}$ closed unit ball of X,
- $S_X = \{x \in X \colon \|x\| = 1\}$ unit sphere of X,
- $\overline{\mathrm{conv}}(C)$ closed convex hull of C,
- $\mathcal{L}(X, Y)$ bounded linear operators from X to Y,
  - $\|T\| = \sup\{\|T(x)\| \colon x \in S_X\}$,
- $\mathcal{K}(X, Y)$ compact linear operators from X to Y,
- $\mathcal{F}(X, Y)$ bounded linear operators from X to Y with finite rank,
- $X^* = \mathcal{L}(X, \mathbb{K})$ topological dual of X.

Norm attaining functionals

Norm attaining functionals

$x^* \in X^*$ attains its norm when

$$\exists\, x \in S_X \;:\; |x^*(x)| = \|x^*\|$$

★ $\mathrm{NA}(X, \mathbb{K}) := \{x^* \in X^* : x^* \text{ attains its norm}\}$

Examples and comments

- $\dim(X) < \infty \implies \mathrm{NA}(X, \mathbb{K}) = \mathcal{L}(X, \mathbb{K})$ (Heine-Borel).
- X reflexive $\iff \mathrm{NA}(X, \mathbb{K}) = \mathcal{L}(X, \mathbb{K})$ (Hahn-Banach, James).
- $\mathrm{NA}(c_0, \mathbb{K}) = c_{00} \leqslant \ell_1$,
- $\mathrm{NA}(\ell_1, \mathbb{K}) = \left\{x \in \ell_\infty : \|x\|_\infty = \max_n\{|x(n)|\}\right\} \subseteq \ell_\infty$, residual, contains c_0,
- $\mathrm{NA}(X, \mathbb{K})$ may be "wild", for instance:
  - it may contain NO two-dimensional subspaces (Read, 2017; Rmoutil, 2017),
  - it can be NOT norm Borel (Kaufman, 1991).
- (Petunin–Plichko 1974; Godefroy 1987): X separable, $Z \leqslant X^*$ closed, separating for X, $Z \subseteq \mathrm{NA}(X, \mathbb{K}) \implies Z$ is an isometric predual of X.

Norm attaining operators

Norm attaining operators

$T \in \mathcal{L}(X,Y)$ attains its norm when

$$\exists\, x \in S_X \ : \ \|T(x)\| = \|T\|$$

★ $\mathrm{NA}(X,Y) := \{T \in \mathcal{L}(X,Y) \colon T$ attains its norm$\}$

Some examples and comments

- $\dim(X) < \infty \implies \mathrm{NA}(X,Y) = \mathcal{L}(X,Y)$ for every Y (Heine-Borel),
- $\dim(X) = \infty \implies \mathrm{NA}(X,c_0) \neq \mathcal{L}(X,c_0)$ (see M.-Merí-Payá, 2006).
- X reflexive $\iff \mathcal{K}(X,Y) \subseteq \mathrm{NA}(X,Y)$ for every Y (James).
- $\mathcal{L}(X,\ell_\infty) = \left[\bigoplus_{n\in\mathbb{N}} \mathcal{L}(X,\mathbb{K})\right]_{\ell_\infty} = \ell_\infty(X^*)$.
  $\mathrm{NA}(X,\ell_\infty) = \left\{(x_n^*) \in \ell_\infty(X^*) \colon \exists k \in \mathbb{N},\ \|x_k^*\| = \|(x_n^*)\|_\infty,\ x_k^* \in \mathrm{NA}(X,\mathbb{K})\right\}$.
- $\mathcal{L}(\ell_1,Y) = \left[\bigoplus_{n\in\mathbb{N}} \mathcal{L}(\mathbb{K},Y)\right]_{\ell_\infty} = \ell_\infty(Y)$.
  $\mathrm{NA}(\ell_1,Y) = \left\{(y_n) \in \ell_\infty(Y) \colon \exists k \in \mathbb{N},\ \|y_k\| = \|(y_n)\|_\infty\right\}$.
- $\mathrm{NA}(L_1[0,1], L_\infty[0,1])$???

The problem of denseness of norm attaining functionals

Problem

Is $\mathrm{NA}(X, \mathbb{K})$ always dense in X^***?**

Theorem (E. Bishop & R. Phelps, 1961)

The set of norm attaining functionals is dense in X^* (for the norm topology).

Problem

Is $\mathrm{NA}(X, Y)$ always dense in $\mathcal{L}(X, Y)$**?**

The answer is **No**, and this is the origin of the study of norm attaining operators.

Modified problem

When is $\mathrm{NA}(X, Y)$ dense in $\mathcal{L}(X, Y)$**?**

The study of this problem was initiated by J. Lindenstrauss in 1963,
who provided the first negative and positive examples.

Lindenstrauss' seminal paper of 1963

Negative answer

There are bounded linear operators which cannot be approximated by norm attaining operators

Idea:

Y LUR, $T\colon X \longrightarrow Y$ bounded from below (monomorphism).
If T attains its norm, then it does at a strongly exposed point.

Examples

Take X separable without strongly exposed points (e.g. c_0, $C[0,1]$, $L_1[0,1]$), let Y be a LUR renorming of X. Then, $\mathrm{NA}(X,Y)$ is not dense in $\mathcal{L}(X,Y)$.

Observation

- The question then is for which X and Y the density holds.
- As this problem is too general, Lindenstrauss introduced two properties.

Lindenstrauss properties A and B

Definition

X, Y Banach spaces,

- X has (Lindenstrauss) property A iff $\overline{\mathrm{NA}(X,Z)} = \mathcal{L}(X,Z) \quad \forall\, Z$
- Y has (Lindenstrauss) property B iff $\overline{\mathrm{NA}(Z,Y)} = \mathcal{L}(Z,Y) \quad \forall\, Z$

Examples

- Reflexive spaces have property A,
- ℓ_1 has property A (property α),
- c_0, $C[0,1]$, $L_1[0,1]$ fail property A,

- $\mathbb{K}$ has property B (Bishop-Phelps theorem),
- every Y such that $c_0 \subset Y \subset \ell_\infty$ (canonical copies) has property B (property β),
- every LUR renorming of c_0 fails property B,
- finite-dimensional polyhedral spaces have property B (property β),
- there are SOME non-polyhedral finite-dimensional spaces which are known to have property B (property quasi-β, Acosta-Aguirre-Payá, 1996).

The relation with the Radon-Nikodým property

Theorem (Bourgain, 1977)

Radon Nikodým Property (RNP) $\Longrightarrow$ property A.

Much more: the non-linear Bourgain-Stegall variational principle (Stegall, 1978)

X, Y Banach spaces, $C \subset X$ bounded RNP set, $\varphi\colon C \longrightarrow Y$ uniformly bounded such that $x \longmapsto \|\varphi(x)\|$ is upper semicontinuous.
Then for every $\delta > 0$, there exists $x_0^* \in X^*$ with $\|x_0^*\| < \delta$ and $y_0 \in S_Y$ such that the function $x \longmapsto \|\varphi(x) + x_0^*(x)y_0\|$ strongly attains its supremum on C.

Theorem (Bourgain, 1977)

X separable with property A $\Longrightarrow$ B_X is dentable.

Consequence (uses a refinement by Huff, 1980)

Every renorming of X has property A $\Longleftrightarrow$ X has the RNP.

Every renorming of Y has property B $\Longrightarrow$ X has the RNP.

Counterexamples for property B

Observation

It was an open question in the 1980's whether RNP $\implies$ property B

Counterexamples

- (Gowers, 1990): ℓ_p does not have property B for any $1 < p < \infty$.
- (Acosta, 1999): No infinite-dimensional strictly convex space has property B.

Consequence

Y separable, every renorming of Y has property B $\implies$ Y is finite-dimensional

The main open problem

★ Do all finite-dimensional spaces have property B**?**
Equivalently, does $\mathcal{F}(X,Y) \subset \overline{\mathrm{NA}(X,Y)}$ for all Banach spaces X and Y**?**

We cannot go further... (M. 2014)

There are **compact** operators which cannot be approximated by norm attaining ones.

Some pairs of classical spaces

Example (Johnson-Wolfe, 1979)

In the real case, $\mathrm{NA}(C(K_1), C(K_2))$ is dense in $\mathcal{L}(C(K_1), C(K_2))$.

Example (Iwanik, 1979)

$\mathrm{NA}(L_1(\mu), L_1(\nu))$ is dense in $\mathcal{L}(L_1(\mu), L_1(\nu))$.

Examples (Schachermayer, 1983)

$\mathrm{NA}(C(K), L_p(\mu))$ is dense in $\mathcal{L}(C(K), L_p(\mu))$ for $1 \leqslant p < \infty$.

Example (Finet-Payá, 1998)

$\mathrm{NA}(L_1[0,1], L_\infty[0,1])$ is dense in $\mathcal{L}(L_1[0,1], L_\infty[0,1])$.

Example (Schachermayer, 1983)

$\mathrm{NA}(L_1[0,1], C[0,1])$ is NOT dense in $\mathcal{L}(L_1[0,1], C[0,1])$.

Why residuality is interesting here?

Residual set

C subset of a complete metric space M is residual if $M \setminus C$ is of the first Baire category. Equivalently, C contains a G_δ dense subset. The elements of C are called typical.

Residuality of norm attaining operators

X, Y Banach spaces, suppose $\mathrm{NA}(X,Y)$ is residual in $\mathcal{L}(X,Y)$. Then:

- $\mathcal{L}(X,Y) = \mathrm{NA}(X,Y) - \mathrm{NA}(X,Y)$,
- Given $\{S_n\} \subset \mathcal{L}(X,Y)$, the set

$$\{T \in \mathcal{L}(X,Y) \colon S_n + T \in \mathrm{NA}(X,Y)\}$$

  is residual (in particular, dense) in $\mathcal{L}(X,Y)$.

Example

$\mathrm{NA}(c_0,\mathbb{K}) = \ell_1 \cap c_{00} \subsetneq \ell_1$, so it is not residual. Besides:

- $\mathrm{NA}(c_0,\mathbb{K}) - \mathrm{NA}(c_0,\mathbb{K}) = \ell_1 \cap c_{00} \neq \ell_1$,
- Given $x_1^* = 0$ and $x_2^* \in \ell_1 \setminus c_{00}$, there is NO $x^* \in \mathcal{L}(c_0,\mathbb{K})$ such that $x_1^* + x^* \in \mathrm{NA}(c_0,\mathbb{K})$ and $x_2^* + x^* \in \mathrm{NA}(c_0,\mathbb{K})$.

Roadmap of the talk

Necessary conditions for the denseness of norm attaining operators

Section 2

A bit of notation

Definition 1

$C \subset X$ bounded. $x_0 \in C$ is strongly exposed if there is $x^* \in X^*$ such that whenever $\{x_n\} \subset C$ satisfies $\operatorname{Re} x^*(x_n) \longrightarrow \sup \operatorname{Re} x^*(C)$, then $\{x_n\} \longrightarrow x_0$.
Equivalently, the slices of C defined by x^* contain x_0 and are arbitrarily small.

★ In this case, we say that x^* strongly exposes C (at x_0).

For the case $C = B_X$...

- The set of strongly exposed points of B_X is denoted by $\text{str-exp}(B_X)$.
- We write $\mathrm{SE}(X)$ for the set of functionals strongly exposing B_X which is a G_δ-set.
- Hence, if $\mathrm{SE}(X)$ is dense, $\mathrm{NA}(X, \mathbb{K})$ is residual.
- $x^* \in \mathrm{SE}(X) \iff$ the norm of X^* is Fréchet-differentiable at x^*.

The previous results vs the new result

Previous results

- (Lindenstrauss, 1963) X admitting a LUR renorming (e.g. separable), X having property A
  $\implies B_X = \overline{\mathrm{conv}}(\text{str-exp}(B_X))$.
- (Bourgain, 1977) $C \subseteq X$ *separable bounded closed convex* such that for every Y the set
  $$\left\{T \in \mathcal{L}(X,Y) \colon \exists \max_{x \in C} \|Tx\|\right\}$$
  is dense in $\mathcal{L}(X,Y)$ (C has the Bishop-Phelps property in Bourgain's terminology)
  $\implies C$ is dentable (i.e. C contains slices of arbitrarily small diameter).

Our result

X admitting a LUR renorming, $C \subseteq X$ bounded with the Bishop-Phelps property
$\implies$ the functionals in X^* which strongly expose C are dense in X^*.

★ In particular, X admitting a LUR renorming, X with property A
$\implies \mathrm{SE}(X)$ is dense in X^*, hence $\mathrm{NA}(X, \mathbb{K})$ is residual.

Sketch of the proof of particular case

Our result (particular case)

X admitting a LUR renorming, X with property A $\implies$ $\mathrm{SE}(X)$ is dense in X^*.

Lemma

$S\colon X \longrightarrow Y$ bounded below, Y LUR, $x_0 \in S_X$ such that $\|S\| = \|Sx_0\|$. Then, x_0 is strongly exposed by S^*y^* for every $y^* \in S_{Y^*}$ with $\operatorname{Re} y^*(Sx_0) = \|S\|$.

- Consider a LUR norm $|\!|\!| \cdot |\!|\!|$ on X and let $Y = (X, |\!|\!| \cdot |\!|\!|) \oplus_2 \mathbb{K}$ which is LUR.
- For $x^* \in S_{X^*}$, define $T_n \in \mathcal{L}(X,Y)$ by $T_n(x) = (n^{-1}x, x^*(x))$, which are monomorphisms, and $S \in \mathcal{L}(X,Y)$ by $S(x) = (0, x^*(x))$. Observe $\{T_n\} \longrightarrow S$.
- We may find **monomorphisms** $S_n \in \mathrm{NA}(X,Y)$, $\|S_n\| = 1$, such that $\{S_n\} \longrightarrow S$.
- By the lemma, there are $y_n^* = (x_n^*, \lambda_n) \in Y^* = X^* \oplus_2 \mathbb{K}$ such that $S_n^* y_n^* \in \mathrm{SE}(X)$ and $\|S_n^* y_n^*\| = \|S_n\| = 1$.
- Suppose $\lambda_n \longrightarrow \lambda_0$ and observe

$$\|\lambda_0 x^* - S_n^* y_n^*\| = \|\lambda_0 x^* - (\lambda_n x^* - S^* y_n^*) - S_n^* y_n^*\| \leqslant |\lambda_0 - \lambda_n| + \|S^* - S_n^*\| \longrightarrow 0.$$

- As $\lambda_0 \neq 0$, $x^* = \lambda_0^{-1}(\lambda_0 x^*) \in \overline{\lambda_0^{-1}\,\mathrm{SE}(X)} = \overline{\mathrm{SE}(X)}$.

An interesting example

Example

The Lipschitz-free space on the Euclidean unit circle, $\mathcal{F}(\mathbb{T})$, satisfies:

- $\mathrm{SE}\big(\mathcal{F}(\mathbb{T})\big)$ is not dense in $\mathcal{F}(\mathbb{T})^* \equiv \mathrm{Lip}_0(\mathbb{T}, \mathbb{R})$ (C-GL-M-RZ, 2021),
- hence, by the new result, $\mathcal{F}(\mathbb{T})$ fails Lindenstrauss property A.
- On the other hand, $B_{\mathcal{F}(\mathbb{T})} = \overline{\mathrm{conv}}\big(\text{str-exp}(B_{\mathcal{F}(\mathbb{T})})\big)$ (C-GL-M-RZ, 2021) (so it satisfies Lindenstrauss necessary condition for property A).
- Besides, the set of those elements in $\mathcal{F}(\mathbb{T})^*$ which attain their norms at extreme points of $B_{\mathcal{F}(\mathbb{T})}$ is not dense in $\mathcal{F}(\mathbb{T})^*$.

Compare with...

- (Lindenstrauss): If $B_X = \overline{\mathrm{conv}}(C)$ and the elements of C are *uniformly* strongly exposed, then X has property A.
- (Bourgain): X has RNP $\iff$ $\text{str-exp}(B_Z) \neq \emptyset \ \forall Z \simeq X$
  $\iff$ $B_Z = \overline{\mathrm{conv}}(\text{str-exp}(B_Z)) \ \forall Z \simeq X$
  $\iff$ $\mathrm{SE}(Z)$ dense $\forall Z \simeq X$.

From residuality of norm attaining functionals to residuality of norm attaining operators

Section 3

Absolutely strongly exposing operators

Definition (Bourgain, 1977)

$T \in \mathcal{L}(X,Y)$ is absolutely strongly exposing ($T \in \mathrm{ASE}(X,Y)$) iff there exists $x_0 \in S_X$ such that whenever $\{x_n\} \subset B_X$ satisfies $\|T(x_n)\| \longrightarrow \|T\|$ then $\exists\{\theta_n\} \subset \mathbb{T}$ for which $\{\theta_n x_n\} \longrightarrow x_0$.

★ $\mathrm{ASE}(X,Y)$ is a G_δ-set. Therefore, if $\mathrm{ASE}(X,Y)$ is dense, $\mathrm{NA}(X,Y)$ is residual.

Theorem (Bourgain, 1977)

X RNP, Y arbitrary $\implies \mathrm{ASE}(X,Y)$ is dense.

Observation (Chiclana–GarcíaLirola–M.–RuedaZoca, 2021)

ALL known sufficient conditions for property A actually imply that absolutely strongly exposing operators are dense:

- RNP,
- properties α and quasi-α,
- $B_X = \overline{\mathrm{conv}}(C)$, C uniformly strongly exposed.

The proposed questions. I

Observation (Chiclana–GarcíaLirola–M.–RuedaZoca, 2021)

ALL known sufficient conditions for property A actually imply that absolutely strongly exposing operators are dense.

Open problem 1

Does the property A of X imply that $\mathrm{ASE}(X,Y)$ is dense for every Y**?**

Observation

If $\mathrm{ASE}(X,Y)$ is dense for some $Y \implies \mathrm{SE}(X)$ is dense.

Open problem 2

Does the denseness of $\mathrm{SE}(X)$ imply that $\mathrm{ASE}(X,Y)$ is dense for every Y**?**

The proposed questions. II

Less ambitious question

If $\mathrm{SE}(X)$ is dense, for which Ys is $\mathrm{ASE}(X,Y)$ dense**?**

Examples of when $\mathrm{SE}(X)$ is dense

- If X has RNP,
- If X has property A and admits a LUR renorming,
- If X is LUR (a property which is not known to imply property A),
- If $\text{str-exp}(B_X) = S_X$ (a property which is not known to imply property A),
- $X = JT^*$ (the dual of the James-tree space, not known if it has property A).

Our main objective

To find spaces Y (better if they are not known to have property B) such that $\mathrm{ASE}(X,Y)$ is dense whenever $\mathrm{SE}(X)$ is dense.

When the denseness of norm attaining operators was already known

Main result in this case

If $\mathrm{SE}(X)$ is dense and Y has any of the known sufficient conditions for property B (β or quasi-β properties), then $\mathrm{ASE}(X, Y)$ is dense.

This includes...

- Property β: e.g. Y's such that $c_0 \leqslant Y \leqslant \ell_\infty$ (canonical copies) or Y finite-dimensional polyhedral.
- Property quasi-β: some more examples including Y finite-dimensional non-polyhedral in dimension greater than or equal to 3.
- Restricting to compact operators, there are more examples like isometric preduals of $L_1(\mu)$, uniform algebras,...

A by-product of our study

Every closed subspace of c_0 has property quasi-β and hence, Lindenstrauss property B.

A first family of new examples. The general result

Theorem

X, Y Banach spaces, $\mathcal{I}(X,Y) \leqslant \mathcal{L}(X,Y)$ containing rank-one operators. Suppose:

- $\mathrm{SE}(X)$ is dense,
- there is $\{y_n^*\} \subset S_{Y^*}$ such that the set $\mathcal{A} = \{T \in \mathcal{I}(X,Y) \colon \|T\| = \|T^* y_n^*\| \text{ for some } n \in \mathbb{N}\}$ is residual in $\mathcal{I}(X,Y)$.

Then, $\mathrm{ASE}(X,Y) \cap \mathcal{I}(X,Y)$ is dense in $\mathcal{I}(X,Y)$.

Idea of the proof:

Lemma

$T \in \mathcal{L}(X,Y)$, $y^* \in S_{Y^*}$ with $T^* y^* \in \mathrm{SE}(X)$, $\|T^* y^*\| = \|T\|$, then there is $x_0 \in \text{str-exp}(B_X)$ such that $\|T x_0\| = \|T\|$, and this implies that $T \in \overline{\mathrm{ASE}(X,Y)}$.

- The set $\mathcal{B} = \{T \in \mathcal{I}(X,Y) \colon T^* y_n^* \in \mathrm{SE}(X)\ \forall n \in \mathbb{N}\}$ is residual,
- $\mathcal{A} \cap \mathcal{B}$ is residual and contained in $\overline{\mathrm{ASE}(X,Y) \cap \mathcal{I}(X,Y)}$.

A first family of new examples. Consequences I

Consequence 1

$\mathrm{SE}(X)$ dense, Y^* RNP with $\text{str-exp}(B_{Y^*})$ countable up to rotations. Then:

$$\mathrm{ASE}(X,Y) \text{ dense in } \mathcal{L}(X,Y), \quad \mathrm{ASE}(X,Y)\cap\mathcal{K}(X,Y) \text{ dense in } \mathcal{K}(X,Y).$$

This result applies to...

- Y being a predual of ℓ_1,
- Y being finite-dimensional such that $\mathrm{ext}\,(B_{Y^*})$ is countable (up to rotation),
- $Y = \mathrm{lip}_0(M)$ when M is a countable compact metric space.

Consequence 2

$\mathrm{SE}(X)$ dense, Y RNP with $\text{str-exp}(B_Y)$ countable up to rotations. Then:

$$\mathrm{ASE}(X,Y^*) \text{ dense in } \mathcal{L}(X,Y^*), \quad \mathrm{ASE}(X,Y^*)\cap\mathcal{K}(X,Y^*) \text{ dense in } \mathcal{K}(X,Y^*).$$

This result applies to...

- $Y = \mathcal{F}(M)$ (so $Y^* = \mathrm{Lip}_0(M)$) when M is a countable proper metric space.

A first family of new examples. Consequences II

Consequence 3

$\mathrm{SE}(X)$ dense, Y such that every separable subspace of Y admits a countable James boundary. Then:

$$\mathrm{ASE}(X,Y) \cap \mathcal{K}(X,Y) \text{ dense in } \mathcal{K}(X,Y).$$

This result applies to...

- Y polyhedral real Banach space,
- Y closed subspace of (the real or complex space) $C(K)$ where K is a Hausdorff scattered compact space.

(no separability condition is needed!)

A second family of new examples. The general result

Theorem

X, Y Banach spaces, $\mathcal{I}(X,Y) \leqslant \mathcal{L}(X,Y)$ containing rank-one operators. Suppose:

- $\mathrm{SE}(X)$ is dense,
- Y has the RNP and $\text{str-exp}(B_Y)$ is discrete up to rotations (i.e. for every sequence $\{y_n\}$ of elements of $\text{str-exp}(B_Y)$ converging to an element $y_0 \in \text{str-exp}(B_Y)$, there is a sequence $\{\theta_n\} \subset \mathbb{T}$ such that $y_n = \theta_n y_0$ for large n).

Then, $\mathrm{ASE}(X,Y^*) \cap \mathcal{I}(X,Y^*)$ is dense in $\mathcal{I}(X,Y^*)$.

Idea of the proof:

- We use Stegall variational principle in $\mathcal{L}(Y,X^*) \equiv \mathcal{L}(X,Y^*)$.
- We use Bourgain's ideas, the discreteness hypothesis, and the residuality of $\mathrm{SE}(X)$, to get operators $T\colon Y \longrightarrow X^*$ and norm-one elements y such that $\|Ty\| = \|T\|$ and $Ty \in \mathrm{SE}(X)$.
- The (pre)adjoints of these operators attains their norms at strongly exposed points of B_X. Hence, they belong to $\overline{\mathrm{ASE}(X,Y^*)}$.

A second family of new examples. Consequence

Consequence 4

$\mathrm{SE}(X)$ dense, Y RNP with $\text{str-exp}(B_Y)$ discrete up to rotations. Then:

$$\mathrm{ASE}(X, Y^*) \text{ dense in } \mathcal{L}(X, Y^*), \quad \mathrm{ASE}(X, Y^*) \cap \mathcal{K}(X, Y^*) \text{ dense in } \mathcal{K}(X, Y^*).$$

This result applies to...

- $Y = \mathcal{F}(M)$ (hence $Y^* = \mathrm{Lip}_0(M)$) when M is a discrete metric space.

Some open problems

Section 4

Some open problems

Open problem 1

Does the denseness of $\mathrm{SE}(X)$ imply Lindenstrauss property A**?**

Remarks

- A positive answer would give an isometric characterization of property A for (say) separable spaces.
- Otherwise, maybe property A implies something more than $\mathrm{SE}(X)$ dense...

Open problem 2

Find more conditions on Y to get denseness of $\mathrm{ASE}(X, Y)$ from the one of $\mathrm{SE}(X)$:

- finite-dimensional Y's**?**
- Asplundness of Y**?**
- $Y = Z^*$ for Z RNP**?**

Remark

In our proofs we use the RNP of Y^* or Y_* and also some discreteness assumptions on the extremal structure.