Slicely Countably Determined sets, spaces, and operators

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Roadmap of the talk

Introduction

Section 1

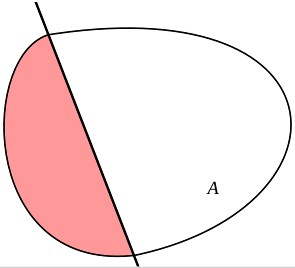
Basic notation

Basic notation

X real or complex Banach space.

- S_X unit sphere, B_X closed unit ball, $\mathbb{T}$ modulus-one scalars.
- X^* dual space, $L(X)$ bounded linear operators from X to X.
- $\operatorname{conv}(\cdot)$ convex hull, $\overline{\operatorname{conv}}(\cdot)$ closed convex hull,
- A slice of $A \subset X$ is a (nonempty) subset of the form

$$S(A, x^*, \alpha) := \{x \in A \colon \operatorname{Re} x^*(x) > \sup \operatorname{Re} x^*(A) - \alpha\} \quad (x^* \in X^*,\ \alpha > 0)$$



Two classical concepts: Radon-Nikodým property and Asplund spaces

The Radon-Nikodým property or RNP (1930's)

- X has the RNP iff the Radon-Nikodým theorem is valid for X-valued meassures;
- Equivalently [1960's–1970's], every bcc subset contains a denting point (i.e. a point belonging to slices of arbitrarily small diameter).

X Asplund $\Longleftrightarrow$ X^* RNP

Reflexive (say) $\Longrightarrow$ (RNP and Asplund)

(RNP or Asplund) $\Longrightarrow$ **???**

Asplund spaces (1960's)

- X is an Asplund space if every continuous convex real-valued function defined on an open subset of X is Fréchet differentiable on a dense subset;
- Equivalently [1970's], every separable subspace has separable dual.

The road map of the talk

The property

We introduce an isomorphic property for (separable) Banach spaces, the so-called

slicely countably determination (SCD)

such that

- it is satisfied by RNP spaces
  (actually, by strongly regular spaces – CPCP in particular–);
- it is satisfied by Asplund spaces
  (actually, by spaces not containing ℓ_1).

We also present examples and stability properties.

The applications

- We apply SCD to get results for the Daugavet property, the alternative Daugavet property and spaces with numerical index 1.
- We present SCD operators and applications.

Slicely Countably Determined sets and spaces

Section 2

SCD sets: Definitions and preliminary remarks

X Banach space, $A \subset X$ **bounded** and **convex**.

Determining sequence

A sequence $\{V_n : n \in \mathbb{N}\}$ of subsets of A is determining for A if one of the following equivalent conditions holds:

- if $B \subseteq A$ satisfies $B \cap V_n \neq \emptyset\ \forall n$, then $A \subseteq \overline{\mathrm{conv}}(B)$,
- given $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in V_n\ \forall n \in \mathbb{N}$, $A \subseteq \overline{\mathrm{conv}}\big(\{x_n : n \in \mathbb{N}\}\big)$,
- every slice of A contains one of the V_n's,

SCD sets

A is Slicely Countably Determined if A has a determining sequence of slices.

Remarks

- A is SCD iff $\overline{A}$ is SCD.
- If A is SCD, then it is separable.

SCD sets: Elementary positive examples I

Example

A separable and $A = \overline{\mathrm{conv}}(\mathrm{dent}(A)) \Longrightarrow A$ is SCD.

Proof.

- Take $\{a_n : n \in \mathbb{N}\}$ denting points with $A = \overline{\mathrm{conv}}\big(\{a_n : n \in \mathbb{N}\}\big)$.
- For every $n, m \in \mathbb{N}$, take a slice $S_{n,m}$ containing a_n and of diameter $1/m$.
- If $B \cap S_{n,m} \neq \emptyset \Longrightarrow a_n \in \overline{B}$.
- Therefore, $A = \overline{\mathrm{conv}}\big(\{a_n : n \in \mathbb{N}\}\big) \subseteq \overline{\mathrm{conv}}(\overline{B}) = \overline{\mathrm{conv}}(B)$.

Example

In particular, A RNP separable $\Longrightarrow A$ SCD.

Corollary

- If X is separable LUR $\Longrightarrow B_X$ is SCD.
- So, every separable space can be renormed such that $B_{(X,|\cdot|)}$ is SCD.

SCD sets: Elementary positive examples II

Example

If X^* is separable $\Longrightarrow$ A is SCD.

Proof.

- Take $\{x_n^* \colon n \in \mathbb{N}\}$ dense in S_{X^*}.
- For every $n, m \in \mathbb{N}$, consider $S_{n,m} = S(A, x_n^*, 1/m)$.
- It is easy to show that any slice of A contains one of the $S_{n,m}$

Example

Actually, it is enough that $(X, \rho_A)^*$ is separable, where ρ_A is the Minkowski functional associated to A.

Negative examples

Example

$B_{C[0,1]}$ and $B_{L_1[0,1]}$ are not SCD.

More general example

If X has the Daugavet property, then B_X is not SCD.

Actually the following anti-SCD phenomenon happens:

- For every sequence $\{S_n\}$ of slices of B_X and every $x \in S_X$, there is a sequence $\{x_n\}$ with $x_n \in S_n$ for $n \in \mathbb{N}$ such that $x \notin \overline{\text{lin}}\{x_n : n \in \mathbb{N}\}$.

A consequence

A subset of an SCD set is not necessarily SCD:

Renorm $C[0,1]$ to be LUR and let A be the new ball, which is SCD but contains a multiple of $B_{C[0,1]}$ which is not.

SCD sets: Further examples I (extending the RNP case)

Convex combination of slices

$W = \sum_{k=1}^{m} \lambda_k S_k \subset A$ where $\lambda_k \geqslant 0$, $\sum \lambda_k = 1$, S_k slices.

Proposition

In the definition of SCD we can use a sequence $\{W_n : n \in \mathbb{N}\}$ of convex combination of slices.

Small combinations of slices

A has small combinations of slices iff every slice of A contains convex combinations of slices of A with arbitrary small diameter.

Example

If A has small combinations of slices + separable $\Longrightarrow A$ is SCD.

Particular case

A strongly regular (in particular, CPCP) + separable $\Longrightarrow A$ is SCD.

SCD sets: Further examples II

Bourgain's lemma

Every relative weak open subset of A contains a convex combination of slices.

Corollary

In the definition of SCD we can use a sequence of relative weak open subsets:
a set A is SCD iff there is a sequence $\{V_n : n \in \mathbb{N}\}$ of relative weak open subsets of A such that every slice of A contains one of the V_n's.

π-bases

A π-base of the weak topology of A is a family $\{V_i : i \in I\}$ of weak open sets of A such that every weak open subset of A contains one of the V_i's.

Proposition

If $(A, \sigma(X, X^*))$ has a countable π-base $\Longrightarrow$ A is SCD.

SCD sets: Further examples III (extending the Asplund case)

Theorem

A separable without ℓ_1-sequences $\Longrightarrow (A, \sigma(X, X^*))$ has a countable π-base.

Proof.

- We see $(A, \sigma(X, X^*)) \subset C(T)$ where $T = (B_{X^*}, \sigma(X^*, X))$.
- By Rosenthal ℓ_1 theorem, $(A, \sigma(X, X^*))$ is a relatively compact subset of the space of first Baire class functions on T.
- By a result of Todorčević, $(A, \sigma(X, X^*))$ has a σ-disjoint π-base: $\{V_i : i \in I\}$ is *σ-disjoint* if $I = \bigcup_{n\in\mathbb{N}} I_n$ and each $\{V_i : i \in I_n\}$ is pairwise disjoint.
- A σ-disjoint family of open subsets in a separable space is countable. ✓

Main example

A separable without ℓ_1-sequences $\Longrightarrow A$ is SCD.

SCD spaces: definition and examples

SCD space

X is Slicely Countably Determined (SCD) if so are all of its convex bounded subsets.

Examples of SCD spaces

1. X separable strongly regular. In particular, RNP, CPCP spaces.
2. X separable $X \not\supseteq \ell_1$. In particular, if X^* is separable.

Examples of NOT SCD spaces

1. $C[0,1]$, $L_1[0,1]$, X with Daugavet property.
2. Actually, every X containing (an isomorphic copy of) $C[0,1]$ or $L_1[0,1]$.
3. There is X with Schur property failing to be SCD.

Example (and question), Kadets–M.–Merí–Werner, 2013

- X Banach space with 1-unconditional basis $\implies$ B_X is SCD.
- We do not know whether X is SCD.

SCD spaces: stability properties

Remark

- Every subspace of a SCD space is SCD.
- This is false for quotients.

Theorem

$Z \subset X$. If Z and X/Z are SCD $\Longrightarrow$ X is SCD.

Corollary

X separable NOT SCD $\Longrightarrow$ $X \supset \ell_1$ and

- If $\ell_1 \simeq Y \subset X \Longrightarrow X/Y$ contains a copy of ℓ_1.
- If $\ell_1 \simeq Y_1 \subset X \Longrightarrow$ there is $\ell_1 \simeq Y_2 \subset X$ with $Y_1 \cap Y_2 = 0$.

Corollary

$X_1, \ldots, X_m$ SCD $\Longrightarrow X_1 \oplus \cdots \oplus X_m$ SCD.

SCD spaces: stability properties II

Theorem

$X_1, X_2, \dots$ SCD, E with 1-unconditional basis.

- $E \not\supseteq c_0 \Longrightarrow \left[\bigoplus_{n\in\mathbb{N}} X_n\right]_E$ SCD.
- $E \not\supseteq \ell_1 \Longrightarrow \left[\bigoplus_{n\in\mathbb{N}} X_n\right]_E$ SCD.

More examples

1. $c_0(\ell_1)$ and $\ell_1(c_0)$ are SCD.
2. $c_0\widehat{\otimes}_\varepsilon c_0$, $c_0\widehat{\otimes}_\pi c_0$, $c_0\widehat{\otimes}_\varepsilon \ell_1$, $c_0\widehat{\otimes}_\pi \ell_1$, $\ell_1\widehat{\otimes}_\varepsilon \ell_1$, and $\ell_1\widehat{\otimes}_\pi \ell_1$ are SCD.
3. $K(c_0)$ and $K(c_0, \ell_1)$ are SCD.
4. $\ell_2\widehat{\otimes}_\varepsilon \ell_2 \equiv K(\ell_2)$ and $\ell_2 \oplus_\pi \ell_2 \equiv \mathcal{L}_1(\ell_2)$ are SCD

Applications

Section 3

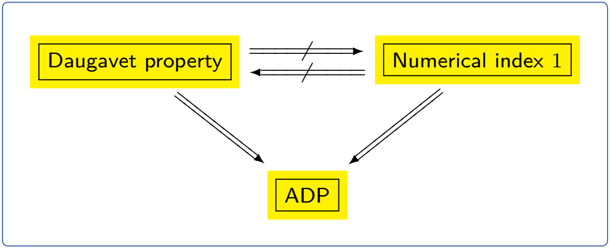
3 Applications
- The DPr, the ADP, numerical index 1, and lushness
- From ADP to lushness
- Daugavet property and projective tensor products

The DPr, the ADP, and numerical index 1

Definition of the properties

1. **Kadets–Shvidkoy–Sirotkin–Werner, 1997:**
   X has the Daugavet property (DPr) if
   $$\|\mathrm{Id} + T\| = 1 + \|T\| \qquad \text{(DE)}$$
   for every rank-one $T \in L(X)$.
   - Then every T not fixing copies of ℓ_1 also satisfies (DE).
2. **Lumer, 1968:** X has numerical index 1 if
   $$\max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta\, T\| = 1 + \|T\| \qquad \text{(aDE)}$$
   for EVERY operator on X.
   - Equivalently,
     $$\|T\| = \sup\{|x^*(Tx)| \colon x \in S_X,\, x^* \in S_{X^*},\, x^*(x) = 1\}$$
     for EVERY $T \in L(X)$.
3. **M.–Oikhberg, 2004:** X has the alternative Daugavet property (ADP) if every rank-one $T \in L(X)$ satisfies (aDE).
   - Then every weakly compact T also satisfies (aDE).

Relations between these properties



Examples

- $C\big([0,1],K(\ell_2)\big)$ has DPr, but has not numerical index 1
- c_0 has numerical index 1, but has not DPr
- $c_0 \oplus_\infty C\big([0,1],K(\ell_2)\big)$ has ADP, neither DPr nor numerical index 1

Remark

For RNP or Asplund spaces, $\boxed{\text{ADP}} \Longrightarrow \boxed{\text{numerical index } 1}$.

An example: the three properties for C^*-algebras and preduals. I

Let V_* be the predual of a von Neumann algebra V.

The Daugavet property of V_* is equivalent to:

- V has no atomic projections, or
- the unit ball of V_* has no extreme points.

V_* has numerical index 1 iff:

- V is commutative, or
- $|v^*(v)| = 1$ for $v \in \operatorname{ext}(B_V)$ and $v^* \in \operatorname{ext}(B_{V^*})$.

The alternative Daugavet property of V_* is equivalent to:

- the atomic projections of V are central, or
- $|v(v_*)| = 1$ for $v \in \operatorname{ext}(B_V)$ and $v_* \in \operatorname{ext}(B_{V_*})$, or
- $V = C \oplus_\infty N$, where C is commutative and N has no atomic projections.

An example: the three properties for C^*-algebras and preduals. II

Let X be a C^*-algebra.

The Daugavet property of X is equivalent to:

- X does not have any atomic projection, or
- the unit ball of X^* does not have any w^*-strongly exposed point.

X has numerical index 1 iff:

- X is commutative, or
- $|x^{**}(x^*)| = 1$ for $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and $x^* \in \operatorname{ext}(B_{X^*})$.

The alternative Daugavet property of X is equivalent to:

- the atomic projections of X are central, or
- $|x^{**}(x^*)| = 1$, for $x^{**} \in \operatorname{ext}(B_{X^{**}})$, and $x^* \in B_{X^*}$ w^*-strongly exposed, or
- $\exists$ a commutative ideal Y such that X/Y has the Daugavet property.

Sufficient conditions for numerical index one

Some sufficient conditions

Let X be a Banach space. Consider:

(a) **Lindenstrauss, 1964:** X has the 3.2.I.P. if the intersection of every family of three mutually intersecting balls is not empty.

(b) **Fullerton, 1961:** X is a CL-space if B_X is the absolutely convex hull of every maximal face of S_X.

(c) **Lima, 1978:** X is an almost-CL-space if B_X is the closed absolutely convex hull of every maximal face of S_X.

$$\text{(a)} \underset{\not\Longleftarrow}{\Longrightarrow} \text{(b)} \underset{\not\Longleftarrow}{\Longrightarrow} \text{(c)} \underset{\not\longleftarrow}{\longrightarrow} \text{numerical index } 1$$

Showing that (c) $\Longrightarrow$ numerical index 1, one realizes that (c) is too much.

Lushness (Boyko–Kadets–M.–Werner, 2007)

X is lush if given $x, y \in S_X$, $\varepsilon > 0$, there is $y^* \in S_{X^*}$ such that

$$x \in S = S(B_X, y^*, \varepsilon) \qquad \operatorname{dist}(y\,, \operatorname{conv}(\mathbb{T}S)) < \varepsilon.$$

A sufficient condition for numerical index 1: lushness

Lushness (Boyko–Kadets–M.–Werner, 2007)

X is lush if given $x, y \in S_X$, $\varepsilon > 0$, there is $y^* \in S_{X^*}$ such that

$$x \in S = S(B_X, y^*, \varepsilon) \qquad \operatorname{dist}(y, \operatorname{conv}(\mathbb{T}S)) < \varepsilon.$$

Theorem (Boyko–Kadets–M.–Werner, 2007)

If X is lush, then X has numerical index 1.

Example (Kadets–M.–Merí–Shepelska, 2009)

There is X with numerical index 1 which is not lush.

Theorem (Kadets–M.–Merí–Payá, 2009)

X lush separable. Then, there is $G \subset \operatorname{ext}(B_{X^*})$ weak-star dense such that

$$B_X = \overline{\operatorname{conv}}\left(\mathbb{T}\left\{x \in B_X : x^*(x) = 1\right\}\right) \qquad (x^* \in G).$$

ADP + SCD $\Longrightarrow$ lushness

Characterization of ADP

X Banach space. TFAE:

- X has ADP (i.e. $\max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta\, T\| = 1 + \|T\|$ for all T rank-one).
- Given $x \in S_X$, a slice S of B_X and $\varepsilon > 0$, there is $y \in S$ with

$$\max_{\theta\in\mathbb{T}} \|x + \theta\, y\| > 2 - \varepsilon.$$

- Given $x \in S_X$, a sequence $\{S_n\}$ of slices of B_X, and $\varepsilon > 0$, there is $y^* \in S_{X^*}$ such that $x \in S(B_X, y^*, \varepsilon)$ and

$$\overline{\mathrm{conv}}\big(\mathbb{T}\, S(B_X, y^*, \varepsilon)\big) \bigcap S_n \neq \emptyset \qquad (n \in \mathbb{N}).$$

Theorem

X ADP + B_X SCD $\Longrightarrow$ given $x \in S_X$ and $\varepsilon > 0$, there is $y^* \in S_{X^*}$ such that

$$x \in S(B_X, y^*, \varepsilon) \quad \text{and} \quad B_X = \overline{\mathrm{conv}}\big(\mathbb{T}\, S(B_X, y^*, \varepsilon)\big).$$

- This clearly implies lushness, and so numerical index 1 (i.e. $\max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta\, T\| = 1 + \|T\|$ for all T).

Some consequences

Corollary

ADP + $\left[\, X \not\supseteq \ell_1\,\right.$ OR strongly regular OR 1-unconditional basis $\left.\right]$
$\implies$ lushness (so numerical index 1).

Corollary

X real + $\dim(X) = \infty$ + ADP $\implies$ $X^* \supseteq \ell_1$.

In particular,

Corollary

X real + $\dim(X) = \infty$ + numerical index 1 $\implies$ $X^* \supseteq \ell_1$.

Open problem (Godefroy, private communication, 1990's)

X with numerical index 1, does X contain c_0 or X contain ℓ_1 **?**

Some consequences II

Proposition (Kadets–M.–Merí–Werner, 2010)

- X with 1-unconditional basis $\Longrightarrow$ B_X is SCD.
- Therefore, X with 1-unconditional basis and ADP $\Longrightarrow$ X is lush.

Theorem (Kadets–M.–Merí–Werner, 2010)

1. The unique Banach spaces with 1-symmetric basis and the ADP are c_0 and ℓ_1.
2. The unique r.i. Banach spaces over $\mathbb{N}$ with the ADP are c_0, ℓ_1 and ℓ_∞.
3. The unique separable r.i. Banach space on $[0,1]$ with the Daugavet property is $L_1[0,1]$.
4. The unique separable r.i. Banach space on $[0,1]$ which is lush is $L_1[0,1]$.

Question

Is it possible to prove the above results (3 and 4) for the ADP **?**

Stability of the Daugavet property by projective tensor product

Open problem (Dirk Werner, 2001)

Suppose that X, Y has the Daugavet property, so does $X\widehat{\otimes}_\pi Y$ **?**

There are some positive answers (mainly by A. Rueda Zoca and collaborators)

- Of course, when $X = L_1(\mu)$ with μ atomless (and in this case Y does not need to have the Daugavet property)
- When X and Y are L_1-predual spaces.
- Some more spaces satisfying a property called WODP...

Another open problem

Suppose $X\widehat{\otimes}_\pi Y$ has the Daugavet property, so does X OR Y **?**

Remark

We cannot expect to get both X AND Y having the Daugavet property:
$L_1[0,1]\widehat{\otimes}_\pi \ell_2 \equiv L_1([0,1],\ell_2)$ has the Daugavet property.

Passing the Daugavet property from a projective tensor product to a factor

Open problem

Suppose $X\widehat{\otimes}_\pi Y$ has the Daugavet property, so does X OR Y **?**

One positive answer (M.–Merí–Quero, 2021)

If $X\widehat{\otimes}_\pi Y$ has the Daugavet property and B_Y is SCD, then X has the Daugavet property.

Another (non-SCD related) answer (M.–Merí–Quero, 2021)

If $X\widehat{\otimes}_\pi Y$ has the Daugavet property and the norm of Y^* is Fréchet smooth at some non-zero point, then X has the Daugavet property.

SCD and HSCD-majorized operators

Section 4

SCD operators

SCD operator

$T \in L(X)$ is an SCD-operator if $T(B_X)$ is an SCD-set.

Examples

T is an SCD-operator when $T(B_X)$ is separable and

1. $T(B_X)$ is RNP (or has CPCP or is strongly regular),
2. $T(B_X)$ has no ℓ_1 sequences,
3. T does not fix copies of ℓ_1

Theorem

- X ADP + T SCD-operator $\implies \max\limits_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta\, T\| = 1 + \|T\|$.
- X DPr + T SCD-operator $\implies \|\mathrm{Id} + T\| = 1 + \|T\|$.

Main corollary

X ADP + T does not fix copies of $\ell_1 \implies \max\limits_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta\, T\| = 1 + \|T\|$.

HSCD-majorized operators (Kadets-Shepelska, 2010)

HSCD and HSDC-majorized operator

- $T \in L(X,Y)$ is an Hereditary-SCD-operator if every convex subset of $T(B_X)$ is an SCD-set.
- $T \in L(X,Y)$ is an HSCD-majorized operator if there is $S \in L(X,Z)$ HSCD-operator such that $\|Tx\| \leqslant \|Sx\|$ for every $x \in X$.

Theorem (Kadets–Shepelska for DPr, Kadets–M.–Merí–Pérez for ADP)

- X DPr + $T \in L(X)$ HSCD-majorized operator $\implies$ $\|\mathrm{Id} + T\| = 1 + \|T\|$.
- X ADP + $T \in L(X)$ HSCD-majorized operator $\implies$ $\max\limits_{\theta \in \mathbb{T}} \|\mathrm{Id} + \theta T\| = 1 + \|T\|$.

Proposition

The class of HSCD-majorized operators is a two-sided operator ideal.

Remark

The class of operators satisfying (DE) or (ADE) is not even a subspace.

Operations with SCD sets

Section 5

Operations with SCD sets

Non-convex sets

- The definition of SCD set does not need convexity, so the same concept can be defined for non-convex sets.
- On the one hand, A is SCD iff $\operatorname{conv}(A)$ is SCD iff $\overline{\operatorname{conv}}(A)$ is SCD.
- On the other hand, some of the results for convex SCD sets ARE FALSE in general:
- (Kadets–Pérez–Werner, 2018) There is A non-convex non-SCD set having a determining sequence of relatively weakly open sets.

Some counterexamples on operations (Kadets–Pérez–Werner, 2018)

- There are A, B closed convex SCD sets such that $A \cap B$ is not SCD.
- There are A, B closed convex SCD sets such that $A + B$ is not SCD.
- There are A, B closed convex SCD sets such that $\overline{\operatorname{conv}}(A \cup B)$ is not SCD.

Open problems

Section 6

Open questions

1. Find more sufficient conditions for a set to be SCD.
2. Is SCD equivalent to the existence of a countable π-base for the weak topology **?**
3. E with (1)-unconditional basis. Is E SCD **?**
4. E with 1-unconditional basis, $\{X_n\}$ a family of SCD spaces.
   Is $[\oplus X_n]_E$ SCD **?**
5. X, Y SCD. Are $X\widehat{\otimes}_\varepsilon Y$ and $X\widehat{\otimes}_\pi Y$ SCD **?**
6. Find a good extension of the SCD property to the nonseparable case.
7. Clarify the relationship between SCD and the Daugavet property:
8. If X fails the SCD, does X contains a subspace isomorphic to a DPr space **?**