On quasi norm attaining operators between Banach spaces

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(joint work with G. Choi, Y. S. Choi, and M. Jung)

Banach Spaces Webinars

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The talk is based on the paper...

G. Choi, Y. S. Choi, M. Jung, and M. Martín
On quasi norm attaining operators between Banach spaces
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Roadmap of the talk

Preliminaries and the new definition

Section 1

Basic notation

X, Y Banach spaces

- $\mathbb{T}$ modulus one scalars
- B_X closed unit ball
- S_X unit sphere
- X^* topological dual
- $\mathcal{L}(X,Y)$ bounded linear operators from X to Y
- $\mathcal{K}(X,Y)$ compact linear operators from X to Y
- $\mathcal{W}(X,Y)$ weakly compact linear operators from X to Y

Norm attaining functionals

Norm attaining functionals

X Banach space, $\mathrm{NA}(X,\mathbb{K}) := \left\{ f \in X^* \colon \|f\| = \max_{x\in S_X} |f(x)| \right\}.$

Some well-known results

- $\mathrm{NA}(X,\mathbb{K})$ weak-star dense (Hanh–Banach theorem),
- X reflexive $\implies$ $\mathrm{NA}(X,\mathbb{K}) = X^*$ (Hanh–Banach theorem),
- X reflexive $\impliedby$ $\mathrm{NA}(X,\mathbb{K}) = X^*$ (James' theorem),

Bishop–Phelps theorem

$\mathrm{NA}(X,\mathbb{K})$ is dense in X^* for every Banach space X.

- The study of norm attaining functionals has many applications,
- and there are many problems being studied nowadays,
- for instance, some results related to lineability properties inside $\mathrm{NA}(X,\mathbb{K})$.

Norm attaining operators

Norm attaining operators

X, Y Banach spaces, $\mathrm{NA}(X,Y) := \left\{ T \in \mathcal{L}(X,Y) \colon \|T\| = \max_{x \in S_X} \|Tx\| \right\}$.

- $T \in \mathrm{NA}(X,Y) \iff T(B_X) \cap \|T\| S_Y \neq \emptyset$.

A few results

- Lindenstrauss, 1963: $\mathrm{NA}(X,Y)$ is not always dense,
- it is dense when X is reflexive or $c_0 \subset Y \subset \ell_\infty$ or Y polyhedral finite dimensional.
- Bourgain, 1977: if X has the RNP $\implies$ $\mathrm{NA}(X,Y)$ dense for every Y,
- Huff, 1981: if X fails the RNP $\implies$ exists X_1, X_2 isomorphic to X such that $\mathrm{NA}(X_1,X_2)$ is not dense.
- Therefore, X has the RNP $\iff$ $\overline{\mathrm{NA}(X_1,Y)} = \mathcal{L}(X_1,Y)$ $\forall X_1 \simeq X$, $\forall Y$.
- If $\overline{\mathrm{NA}(X,Y_1)} = \mathcal{L}(X,Y_1)$ $\forall X$ $\forall Y_1 \simeq Y$ $\implies$ Y RNP, $\Longleftarrow$?
- Gowers, 1990: there is G such that $\mathrm{NA}(G,\ell_p)$ not dense for $1 < p < \infty$.
- Acosta, 1999: ℓ_p above can be substituted by any infinite-dimensional strictly convex space and by ℓ_1 (using different domain spaces).

Norm attaining operators II

Some more results

- Zizler, 1973: $\{T \in \mathcal{L}(X,Y) : T^* \in \mathrm{NA}(Y^*, X^*)\}$ is always dense in $\mathcal{L}(X,Y)$.
- Johnson–Wolfe, 1979: for most "classical" Banach spaces as domain, compact operators can be approximated by norm attaining operators.
  - They use a strong form of the approximation property of X^*.
- Johnson–Wolfe, 1979: for isometric preduals of $L_1(\mu)$ as range or for $L_1(\mu)$ as range (real case), compact operators can be approximated by norm attaining operators.
  - They use Lazar–Lindenstrauss kind of results for Y.
- Diestel, Johnson–Wolfe, 1970's: is $\mathcal{K}(X,Y)$ always contained in $\overline{\mathrm{NA}(X,Y)}$ **?**
- M., 2014: there exist compact operators which cannot be approximated by norm attaining operators.

The main problem which remain open

Are finite rank operators aproximable by norm attaining operators?

Lipschitz maps

Notation

$\mathrm{Lip}_0(X,Y)$ space of Lipschitz maps from X to Y (which are 0 at 0), endowed with the Lipschitz constant as norm.

Strongly norm attaining Lipschitz maps

$F \in \mathrm{Lip}_0(X,Y)$ strongly attains its (Lipschitz) norm if there exist $(x,y) \in X \times X$, $x \neq y$ such that $\dfrac{\|F(x)-F(y)\|}{\|x-y\|} = \|F\|_L$.

Kadets–M.–Soloviova, 2016

Even for $X = Y = \mathbb{R}$, there are Lipschitz functions which cannot be approximated by strongly norm attaining Lipschitz functions.

Two options:

- Study the problem for general metric spaces as X (for other day...).
- Study weaker forms in which a Lipschitz maps may attains its norm.

Godefroy's weaker definition for Lipschitz maps

Norm attaining Lipschitz maps towards vectors (Godefroy, 2015)

$F \in \mathrm{Lip}_0(X,Y)$ attains its norm towards $u \in Y$ if $\|u\| = \|F\|_L$ and there exist $(x_n, y_n) \in X \times X$, $x_n \neq y_n$ such that $\dfrac{F(x_n) - F(y_n)}{\|x_n - y_n\|} \longrightarrow u$.

Example (Godefroy, 2015)

If Y is a renorming of c_0 with the Kadets–Klee property, then no Lipschitz isomorphism from c_0 onto Y can be approximated by Lipschitz maps which attain their norm towards directions of Y.

- This extends a result by Lindenstrauss of 1963 for bounded linear operators,
- the proof is not easy,
- in the words of Gilles, the example shows that even the greater flexibility allowed by non linearity does not always provide norm attaining objects.
- In particular, the set of Lipschitz maps which attain their norm towards directions of Y is not dense in $\mathrm{Lip}_0(c_0, Y)$.
- On the other hand, density holds if $\dim(Y) < \infty$, for instance.

The new concept for bounded linear operators

Quasi norm attaining bounded linear operator

$T \in \mathcal{L}(X,Y)$ quasi attains its norm ($T \in \mathrm{QNA}(X,Y)$) if there exists $(x_n) \subset B_X$ such that $Tx_n \longrightarrow y \in \|T\| S_Y$.

- We say T quasi attains its norm towards y.
- Equivalently, $\overline{T(B_X)} \cap \|T\| S_Y \neq \emptyset$ $\;(T \in \mathrm{NA}(X,Y) \iff T(B_X) \cap \|T\| S_Y \neq \emptyset)$
- Viewing T as Lipschitz map, it is Godefroy's definition.

First remarks

- $\mathrm{NA}(X,Y) \subset \mathrm{QNA}(X,Y)$,
- $\mathcal{K}(X,Y) \subset \mathrm{QNA}(X,Y)$.

Positive examples

The remarks give a lot of positive examples, some of which we will present next.

Negative example (a consequence of Godefroy's example)

Y renorming of c_0 with the Kadets–Klee property $\implies$ $\overline{\mathrm{QNA}(c_0,Y)} \neq \mathcal{L}(c_0,Y)$.

First results on quasi norm attaining operators

Section 2

First positive results I

Remark

- $\mathrm{NA}(X,Y) \subset \mathrm{QNA}(X,Y)$.

Examples

$\mathrm{NA}(X,Y)$ (and so $\mathrm{QNA}(X,Y)$) is dense in $\mathcal{L}(X,Y)$ when...

- X has RNP,
- $c_0 \leqslant Y \leqslant \ell_\infty$,
- $X = L_1(\mu)$ and $Y = L_1(\nu)$,
- $X = L_1(\mu)$ and $Y = L_\infty(\nu)$,
- X Asplund and $Y = C(K)$,
- $X = C(K_1)$ and $Y = C(K_2)$,
- $X = C(K)$ and Y does not contain c_0.

First positive results II

Remark

- $\mathcal{K}(X,Y) \subset \mathrm{QNA}(X,Y)$.

Examples

$\mathcal{K}(X,Y) = \mathcal{L}(X,Y)$ and so $\mathrm{QNA}(X,Y) = \mathcal{L}(X,Y)$ when...

- (Pitt) $X \leqslant \ell_p$ and $Y \leqslant \ell_r$ with $1 \leqslant r < p < \infty$,
- (Rosenthal) $X \leqslant c_0$ and Y does not contain c_0,
- (Rosenthal) $X \leqslant L_p(\mu)$, $Y \leqslant L_r(\nu)$, $1 \leqslant r < p < \infty$ and
- $X = L_1(\mu)$ and $Y = L_\infty(\nu)$,
  - $1 \leqslant r < 2$ and ν is atomic,
  - or $p > 2$ and μ is atomic.

Negative results I

Lemma (R. Payá, in a discussion on this topic)

$T \in \mathrm{QNA}(X,Y)$, T monomorphism $\implies$ $T \in \mathrm{NA}(X,Y)$.

- Let $(x_n) \subset B_X$ and $u \in \|T\| S_Y$ such that $Tx_n \longrightarrow u$.
- Consider $T^{-1}\colon T(X) \longrightarrow X$ bounded linear.
- As $T(X)$ is closed, $u \in T(X)$ and we may consider $x_0 = T^{-1}(u) \in X$.
- Observe that $x_n = T^{-1}(Tx_n) \longrightarrow T^{-1}u = x_0$, hence $\|x_0\| = 1$.
- Besides, $Tx_0 = T(T^{-1}(u)) = u$, so $\|Tx_0\| = \|u\| = \|T\|$ and $T \in \mathrm{NA}(X,Y)$.

Main consequence

If there is $T \in \mathcal{L}(X,Y)$ monomorphism, $T \notin \overline{\mathrm{NA}(X,Y)}$ $\implies$ $\overline{\mathrm{QNA}(X,Y)} \neq \mathcal{L}(X,Y)$.

Negative results II

We use

If there is $T \in \mathcal{L}(X,Y)$ monomorphism, $T \notin \overline{\mathrm{NA}(X,Y)} \implies \overline{\mathrm{QNA}(X,Y)} \neq \mathcal{L}(X,Y)$.

Example 1 (extending Godefroy's first negative result)

X infinite dimensional subspace of c_0, Y strictly convex renorming of c_0
$\implies \quad \mathrm{QNA}(X,Y)$ is not dense in $\mathcal{L}(X,Y)$.

- $T\colon X \longrightarrow Y$ norm attaining at $x_0 \in S_X$,
- As $X \leqslant c_0$, the linear span of $A = \{z \in X\colon \|x_0 \pm z\| \leqslant 1\}$ is of finite-codimension.
- As Y is strictly convex, $T(A) = \{0\}$, so T has finite-rank.
- Now, the formal inclusion of X into Y does not belong to $\overline{\mathrm{NA}(X,Y)}$ and, therefore, it does not belong to $\overline{\mathrm{QNA}(X,Y)}$.

Negative results III

We use

If there is $T \in \mathcal{L}(X,Y)$ monomorphism, $T \notin \overline{\mathrm{NA}(X,Y)} \implies \overline{\mathrm{QNA}(X,Y)} \neq \mathcal{L}(X,Y)$.

Example 2 (improving a result by Johnson–Wolfe)

There exists S such that $\mathrm{QNA}(L_1[0,1], C(S))$ is not dense in $\mathcal{L}(L_1[0,1], C(S))$.

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NORM ATTAINING OPERATORS AND SIMULTANEOUSLY CONTINUOUS RETRACTIONS

JERRY JOHNSON AND JOHN WOLFE[1]

ABSTRACT. A compact metric space S is constructed and it is shown that there is a bounded linear operator $T\colon L^1[0,1] \to C(S)$ which cannot be approximated by a norm attaining operator. Also it is established that there

+ showing that T is a monomorphism!

Negative results. IV

We use

If there is $T \in \mathcal{L}(X,Y)$ monomorphism, $T \notin \overline{\mathrm{NA}(X,Y)} \implies \overline{\mathrm{QNA}(X,Y)} \neq \mathcal{L}(X,Y)$.

Example 3 (improving a result by Bourgain–Huff)

X without the RNP $\implies$ there exist X_1, X_2 isomorphic to X such that $\mathrm{QNA}(X_1, X_2)$ is not dense in $\mathcal{L}(X_1, X_2)$.

MR0577031 (81i:47046) Reviewed

Huff, R.

On nondensity of norm-attaining operators.

Rev. Roumaine Math. Pures Appl. 25 (1980), no. 2, 239–241.

47D15 (46B22)

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From References: 1
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For two Banach spaces X and Y let $L(X,Y)$ denote the space of bounded linear operators from X into Y. If X does not have the Radon-Nikodým property, then there exist renormings X_1 and X_2 of X such that the identity operator on X cannot be approximated by members of $L(X_1, X_2)$ attaining their norms.

Reviewed by J. Bourgain

The Radon–Nikodým property

Section 3

A new positive result

Theorem

$T \in \mathcal{L}(X,Y)$ strong RNP operator, $\varepsilon > 0$. There exists $S \in \mathrm{QNA}(X,Y)$ such that:

- $\|T - S\| < \varepsilon$,
- moreover, there is $z_0 \in \overline{S(B_X)} \cap \|S\| S_Y$ such that whenever $(x_n) \subseteq B_X$ satisfies that $\|Sx_n\| \longrightarrow \|S\|$, we may find a sequence $(\theta_n) \subseteq \mathbb{T}$ such that $S(\theta_n x_n) \longrightarrow z_0$; in particular, there is $\theta_0 \in \mathbb{T}$ and a subsequence $(x_{\sigma(n)})$ of (x_n) such that $Sx_{\sigma(n)} \longrightarrow \theta_0 z_0$.

Tool: Bourgain–Stegall non-linear optimization principle

Suppose D is a bounded RNP set of a Banach space Y and $\phi\colon D \longrightarrow \mathbb{R}$ is upper semicontinuous and bounded above. Then, the set

$$\{y^* \in Y^* : \phi + \operatorname{Re} y^* \text{ strongly exposes } D\}$$

is a dense G_δ subset of Y^*.

A new positive result: Proof

- Apply Bourgain–Stegall to $D := \overline{T(B_X)}$ and $\phi(y) = \|y\|$,
- We get $y_0 \in D$ and $y_0^* \in Y^*$ with $\|y_0^*\| < \varepsilon/\|T\|$ such that $\phi + \operatorname{Re} y_0^*$ strongly exposed D so, by rotating,
  (1) $\|y\| + |y_0^*(y)| \leqslant \|y_0\| + \operatorname{Re} y_0^*(y_0)\ \forall y \in D$;
  (2) if $(x_n) \subset B_X$ satisfies $\|Tx_n\| + |y_0^*(Tx_n)| \longrightarrow \|y_0\| + y_0^*(y_0)$, then exists $(\theta_n) \subset \mathbb{T}$ such that $T(\theta_n x_n) \longrightarrow y_0$.
- $Sx := Tx + y_0^*(Tx)\frac{y_0}{\|y_0\|}$, so $\|S - T\| < \varepsilon$ and $\|Sx\| \leqslant \|y_0\| + y_0^*(y_0)$.
- Write $z_0 = \left(1 + \frac{y_0^*(y_0)}{\|y_0\|}\right) y_0$ and observe that $\|z_0\| = \|y_0\| + y_0^*(y_0)$.
- Moreover, if $(x_n) \subset B_X$ satisfies that $\|Sx_n\| \longrightarrow \|S\|$, we have

$$\|y_0\| + y_0^*(y_0) = \|S\| = \lim_n \left\| Tx_n + y_0^*(Tx_n)\,\frac{y_0}{\|y_0\|} \right\|$$
$$\leqslant \lim_n \big(\|Tx_n\| + |y_0^*(Tx_n)|\big) \leqslant \|y_0\| + y_0^*(y_0).$$

- By (2), we can find $(\theta_n) \subseteq \mathbb{T}$ such that $T(\theta_n x_n) \longrightarrow y_0$, and hence

$$S(\theta_n x_n) = T(\theta_n x_n) + y_0^*\big(T(\theta_n x_n)\big)\,\frac{y_0}{\|y_0\|} \longrightarrow \left(1 + \frac{y_0^*(y_0)}{\|y_0\|}\right) y_0 = z_0.\ \checkmark$$

A new positive result. III

Remark

Let X and Y be Banach spaces and let $T \in \mathcal{L}(X,Y)$ be a strong RNP operator. Consider for $\varepsilon > 0$ the point $y_0 \in \overline{T(B_X)} \subset Y$ and the operator $S \in \mathrm{QNA}(X,Y)$ given in the proof of the Theorem. Then,

(a) $T - S$ is a rank-one operator.

(b) $S(B_X) \subseteq T(B_X) + \{\lambda y_0 \colon \lambda \in \mathbb{K},\ |\lambda| \leqslant \rho\}$ for some $\rho > 0$.

(c) $\overline{S(X)} = \overline{T(X)}$.

(d) S quasi attains its norm toward a point of the form $z_0 = \lambda y_0$ for some $\lambda > 0$.

(e) S^* attains its norm at some $z^* \in S_{Y^*}$ which strongly exposes $\overline{S(B_X)}$ at z_0.

Consequences I

Corollary

If X or Y has the RNP, then $\mathrm{QNA}(X,Y)$ is dense in $\mathcal{L}(X,Y)$.

- The case of X having RNP needs Bourgain's result.
- The case of Y having RNP is the new one and it is false for NA.

Corollary

$\mathrm{QNA}(X,Y) \cap \mathcal{W}(X,Y)$ is always dense in $\mathcal{W}(X,Y)$.

Examples

There are many examples of pair of spaces (X,Y) for which $\mathrm{QNA}(X,Y)$ is dense while $\mathrm{NA}(X,Y)$ not. For instance:

- For $1 < p < \infty$, there exists W such that $\mathrm{NA}(W,\ell_p)$ is not dense, while $\mathrm{QNA}(X,\ell_p)$ is dense for every X.
- In the complex case, $w \in \ell_2 \setminus \ell_1$ decreasing, $\mathrm{NA}(d_*(w,1), d(w,1))$ is not dense in $\mathcal{L}(d_*(w,1), d(w,1))$, while $\mathrm{QNA}(X, d(w,1))$ is dense for every X.

Consequences II. Characterizing the RNP

Corollary

Z Banach space. TFAE:

(a) Z has the RNP.

(b) $\mathrm{QNA}(Z', Y)$ is dense in $\mathcal{L}(Z', Y)$ for every Banach space Y and every equivalent renorming Z' of Z.

(c) $\mathrm{QNA}(X, Z')$ is dense in $\mathcal{L}(X, Z')$ for every Banach space X and every equivalent renorming Z' of Z.

Remarks

- (a) $\iff$ (b) is true for NA,
- (c) $\implies$ (a) is true for NA,
- but (a) $\implies$ (c) is FALSE for NA, as shown by Gowers.

A stronger property and its consequences

UQNA

$T \in \mathcal{L}(X,Y)$ uniquely quasi attains its norm towards $u \in Y$ iff whenever $(x_n) \subset B_X$ satisfies $\|Tx_n\| \longrightarrow \|T\|$ there is $(x_{\sigma(n)})$ and $\theta \in \mathbb{T}$ such that $Tx_{\sigma(n)} \longrightarrow \theta u$.

Corollary

If X or Y has the RNP, then $\mathrm{UQNA}(X,Y)$ is dense in $\mathcal{L}(X,Y)$.

Corollary

- $\mathcal{K}(X,Y) \cap \mathrm{UQNA}(X,Y)$ is dense in $\mathcal{K}(X,Y)$,
- $\mathcal{W}(X,Y) \cap \mathrm{UQNA}(X,Y)$ is dense in $\mathcal{W}(X,Y)$.

Consequence

$\mathrm{UQNA}(X,Y)$ dense in $\mathcal{L}(X,Y)$, $\mathcal{A} \subset B_X$ with $\overline{\mathrm{conv}}(\mathcal{A}) = B_X$

$$\Longrightarrow \ \left\{T \in \mathcal{L}(X,Y)\colon \exists (x_n) \subset \mathcal{A},\ \exists y \in \|T\| S_Y \text{ with } (Tx_n) \longrightarrow y\right\}$$

is dense in $\mathcal{L}(X,Y)$.

An application: a consequence on Lipschitz maps

Norm attaining Lipschitz maps towards vectors (Godefroy, 2015)

$F \in \mathrm{Lip}_0(X, Y)$ attains its norm towards $u \in Y$ if $\|u\| = \|F\|$ and there exist $(x_n, y_n) \in X \times X$, $x_n \neq y_n$ such that $\dfrac{F(x_n) - F(y_n)}{\|x_n - y_n\|} \longrightarrow u$.

Consequence of our results

If Y has RNP, then Lipschitz maps which attains their norms towards vectors are dense in $\mathrm{Lip}_0(X, Y)$ for every X.

- The proof uses Lipschitz-free spaces and linearization of Lipschitz maps,
- and the last result of the previous page.

Further results

Section 4

Characterizing properties using QNA

The idea is to discuss when each of the inclusions in the chain

$$\mathrm{NA}(X,Y) \subseteq \mathrm{QNA}(X,Y) \subseteq \mathcal{L}(X,Y)$$

is an equality.

$\mathrm{NA}(X,Y) = \mathrm{QNA}(X,Y)$

For a Banach space X, TFAE:

- X is reflexive,
- $\mathrm{NA}(X,Y) = \mathrm{QNA}(X,Y)$ for every Y,
- exists $Y \neq \{0\}$ such that $\mathrm{NA}(X,Y) = \mathrm{QNA}(X,Y)$.

$\mathrm{QNA}(X,Y) = \mathcal{L}(X,Y)$

- $\dim(X) = \infty \implies \mathcal{L}(X,c_0) \setminus \mathrm{QNA}(X,c_0) \neq \emptyset$.
- $\dim(Y) = \infty \implies \mathcal{L}(\ell_1,Y) \setminus \mathrm{QNA}(\ell_1,Y) \neq \emptyset$.

Relationship with adjoint operators

Proposition

$T \in \mathrm{QNA}(X, Y) \implies T^* \in \mathrm{NA}(Y^*, X^*)$.

Counterexample

The identity from c_0 to c_0 endowed with Day's norm does not belong to QNA, but its adjoint belongs to NA.

Proposition

$T \in \mathcal{W}(X, Y)$, TFAE:

- $T \in \mathrm{QNA}(X, Y)$,
- $T^* \in \mathrm{NA}(Y^*, X^*)$.

Consequence (new proof using Zizler's result)

$\mathrm{QNA}(X, Y) \cap \mathcal{W}(X, Y)$ is dense in $\mathcal{W}(X, Y)$.

Extending Payá's result and limitations

Lemma (R. Payá)

$T \in \mathrm{QNA}(X,Y)$, T monomorphism $\implies T \in \mathrm{NA}(X,Y)$.

Two extensions

- $T \in \mathrm{QNA}(X,Y)$, $T(X)$ closed, and $\ker T$ proximinal $\implies T \in \mathrm{NA}(X,Y)$.
- $T \in \mathrm{QNA}(X,Y)$, $[\ker T]^\perp \subset \mathrm{NA}(X,\mathbb{K}) \implies T \in \mathrm{NA}(X,Y)$.

Example

The formal identity map T from Gowers' space G to ℓ_2 belongs to QNA, $\ker T = \{0\}$, but $T \notin \overline{\mathrm{NA}(G,\ell_2)}$.

Example

If X is reflexive with $\dim(X)=\infty$, $[\ker T]^\perp \subset \mathrm{NA}(X,\mathbb{K})$ for every $T \in \mathcal{L}(X,Y)$, but $\mathcal{L}(X,c_0) \setminus \mathrm{NA}(X,c_0) \neq \emptyset$.

Remarks and open questions

Section 5

The analogue to Lindenstrauss properties

Properties quasi A and quasi B

- X has property quasi A if $\overline{\mathrm{QNA}(X,Z)} = \mathcal{L}(X,Z)$ for every Z.
- Y has property quasi B if $\overline{\mathrm{QNA}(W,Y)} = \mathcal{L}(W,Y)$ for every W.

- The corresponding properties for NA are called Lindenstrauss properties A and B.

Isomorphic results

- X has property quasi A in every equivalent norm $\iff$ X has RNP,
- Separable spaces (actually WCG) can be renormed to have property (quasi) A.
- Y has property quasi B in every equivalent norm $\iff$ Y has RNP,
- Every Banach space can be renormed to have property (quasi) B.

- We know that property quasi B is different from Lindenstrauss property B, but...

Question

Does property quasi A implies Lindenstrauss property A ?

Uniquely quasi norm attaining operators

UQNA

$T \in \mathcal{L}(X,Y)$ uniquely quasi attains its norm towards $u \in Y$ iff whenever $(x_n) \subset B_X$ satisfies $\|Tx_n\| \longrightarrow \|T\|$ there is $(x_{\sigma(n)})$ and $\theta \in \mathbb{T}$ such that $Tx_{\sigma(n)} \longrightarrow \theta u$.

Example

$\mathrm{Id} \in \mathcal{L}(c_0, c_0)$ does not belong to the closure of $\mathrm{UQNA}(c_0, c_0)$.
Recall that $\mathrm{NA}(c_0, c_0)$ (and so $\mathrm{QNA}(c_0, c_0)$) is dense in $\mathcal{L}(c_0, c_0)$.

Proposition

If Y is LUR, then $\mathrm{QNA}(X,Y) \subseteq \overline{\mathrm{UQNA}(X,Y)}$ for every X.

Question

Find other sufficient conditions on X or Y to get $\mathrm{QNA}(X,Y) \subseteq \overline{\mathrm{UQNA}(X,Y)}$.

On a problem by M. Ostroskii

Open problem (Ostrovskii, 2005)

Does there exist X infinite-dimensional with $\mathrm{NA}(X,X) = \mathcal{L}(X,X)$?

- The only possible candidates for X are separable reflexive spaces without one-complemented subspaces with the AP.

Question

Does there exist X infinite-dimensional with $\mathrm{QNA}(X,X) = \mathcal{L}(X,X)$?

Some remarks

- If X is reflexive, the two problems are the same.
- As $\mathcal{K}(X,X) \subseteq \mathrm{QNA}(X,X)$, one may think in testing spaces with "very few operators" (i.e. $\mathcal{L}(X,X) = \big\{\lambda \mathrm{Id} + S \colon \lambda \in \mathbb{K},\ S \in \mathcal{K}(X,X)\big\}$).
- $T = \lambda \mathrm{Id} + S$ with $\lambda \neq 0$ and $S \in \mathcal{K}(X,X)$, if T belongs to $\mathrm{QNA}(X,X)$, then $T \in \mathrm{NA}(X,X)$.

Applications?

Applications of NA

- There are many applications of the study of $\mathrm{NA}(X, Y)$,
- for instance to differential equations,
- mainly using "variational principles", some of them consequences of the Bourgain–Stegall variational principle.

Main open problem

Find possible applications of QNA, maybe of the theorem on the RNP and UQNA.