Advanced Functional Analysis and its Applications 2020

Numerical index theory

Miguel Martín

UNIVERSIDAD DE GRANADA

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Schedule of the talk

Preliminaries

Section 1

Basic notation I

- $\mathbb{K}$ base field ($\mathbb{R}$ or $\mathbb{C}$):
  - $\mathbb{T}$ modulus-one scalars,
  - $\operatorname{Re} z$ real part of z ($\operatorname{Re} z = z$ if $\mathbb{K} = \mathbb{R}$).

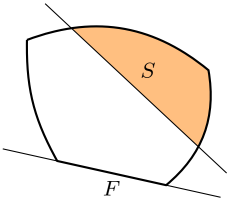
- X, Y Banach spaces:
  - S_X unit sphere, B_X unit ball,
  - X^* dual space,
  - $L(X,Y)$ bounded linear operators,
  - $L(X) := L(X,X)$,
  - $\operatorname{Iso}(X)$ surjective linear isometries.

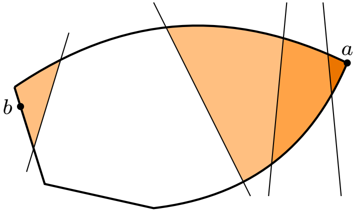
- $T \in L(X,Y)$:
  - $T^* \in L(Y^*,X^*)$ adjoint operator of T.
  - $X = Y$, $\operatorname{Sp}(T)$ spectrum of T.

Basic notation II

X Banach space, $B \subset X$:

- $\|B\| = \sup\{\|b\|\colon b \in B\}$,
- B is rounded if $\mathbb{T}B = B$,
- $\operatorname{conv}(B)$ convex hull of B, $\overline{\operatorname{conv}}(B)$ closed convex hull of B,
- $\operatorname{aconv}(B) = \operatorname{conv}(\mathbb{T}\, B)$ absolutely convex hull of B, $\overline{\operatorname{aconv}}(B) = \overline{\operatorname{conv}}(\mathbb{T}\, B)$,
- $\operatorname{Slice}(B, x^*, \alpha) := \Big\{x \in B\colon\ \operatorname{Re} x^*(x) > \sup \operatorname{Re} x^*(B) - \alpha\Big\}$,
  where $x^* \in X^*$ and $\alpha > 0$,
- $\operatorname{Face}(B, x^*) := \Big\{x \in B\colon\ \operatorname{Re} x^*(x) = \sup \operatorname{Re} x^*(B)\Big\}$,
  where $x^* \in X^*$ attains its supremum on B.
- $\operatorname{ext}(B)$ extreme points of B,
- $\operatorname{dent}(B)$ denting points of B (i.e. those belonging to arbitrarily small slices).





Numerical range: Hilbert spaces

Hilbert space numerical range (Toeplitz, 1918)

- A $n \times n$ real or complex matrix

$$W(A) = \Big\{(Ax \mid x) \colon x \in \mathbb{K}^n,\ (x \mid x) = 1\Big\}.$$

- H real or complex Hilbert space, $T \in L(H)$,

$$W(T) = \Big\{(Tx \mid x) \colon x \in H,\ \|x\| = 1\Big\}.$$

Remark

★ Given $T \in L(H)$ we associate
- a sesquilinear form $\varphi_T(x, y) = (Tx \mid y) \qquad (x, y \in H)$,
- a quadratic form $\widehat{\varphi_T}(x) = \varphi_T(x, x) = (Tx \mid x) \qquad (x \in H)$.

★ Then, $W(T) = \widehat{\varphi_T}(S_H)$.

Numerical range: Hilbert spaces. Properties.

Some properties

H Hilbert space, $T \in L(H)$:

- (Toeplitz-Hausdorff) $W(T)$ is convex.
- $T, S \in L(H)$, $\alpha, \beta \in \mathbb{K}$:
  - $W(\alpha T + \beta S) \subseteq \alpha W(T) + \beta W(S)$;
  - $W(\alpha \mathrm{Id} + S) = \alpha + W(S)$.
- $W(U^*TU) = W(T)$ for every $T \in L(H)$ and every U unitary.
- $\mathrm{Sp}(T) \subseteq \overline{W(T)}$.
- If T is normal, then $\overline{W(T)} = \overline{\mathrm{conv}}\,\mathrm{Sp}(T)$.
- In the real case $(\dim(H) > 1)$, there is $T \in L(H)$, $T \neq 0$ with $W(T) = \{0\}$.
- In the complex case,

  $$\sup\{|(Tx \mid x)| \colon x \in S_H\} \geqslant \frac{1}{2}\,\|T\|.$$

  If T is actually self-adjoint, then

  $$\sup\{|(Tx \mid x)| \colon x \in S_H\} = \|T\|.$$

Numerical range: Hilbert spaces. Motivation.

Some reasons to study numerical ranges

- It gives a "picture" of the matrix/operator which allows to "see" many properties (algebraic or geometrical) of the matrix/operator.
- It is a comfortable way to study the spectrum.
- It is useful to estimate spectral radii of small perturbations of matrices.
- It is useful to work with some concepts like hermitian operator, skew-hermitian operator, dissipative operator. . .

Example

Consider $A = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ \varepsilon & 0 \end{pmatrix}$.

- $\mathrm{Sp}(A) = \{0\}$, $\mathrm{Sp}(B) = \{0\}$.
- $\mathrm{Sp}(A+B) = \{\pm\sqrt{M\varepsilon}\} \subseteq W(A+B) \subseteq W(A) + W(B)$,
- so the spectral radius of $A+B$ is bounded above by $\frac{1}{2}(|M| + |\varepsilon|)$.
- Using norms, we only get $|\mathrm{Sp}(A+B)| \leqslant |M| + |\varepsilon|$

Numerical range: Banach spaces (I)

Banach spaces numerical range (Bauer 1962; Lumer, 1961)

X Banach space, $T \in L(X)$,

$$V(T) = \Big\{x^*(Tx)\colon x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\Big\}$$

Some properties

X Banach space, $T \in L(X)$.

- $V(T)$ is connected but not necessarily convex.
- $T, S \in L(X)$, $\alpha, \beta \in \mathbb{K}$:
  - $V(\alpha T + \beta S) \subseteq \alpha V(T) + \beta V(S)$;
  - $V(\alpha \mathrm{Id} + S) = \alpha + V(S)$.
- $\mathrm{Sp}(T) \subseteq \overline{V(T)}$.
- (Zenger–Crabb) Actually, $\overline{\mathrm{conv}}\Big(\mathrm{Sp}(T)\Big) \subseteq \overline{V(T)}$.
- $\overline{\mathrm{conv}}\,\mathrm{Sp}(T) = \bigcap\{V_p(T)\colon p \text{ equivalent norm}\}$
  where $V_p(T)$ is the numerical range of T in the Banach space (X, p).
- $V(U^{-1}TU) = V(T)$ for every $T \in L(X)$ and every $U \in \mathrm{Iso}(X)$.
- $V(T) \subseteq V(T^*) \subseteq \overline{V(T)}$.

Numerical range: Banach spaces (II)

Observation

The numerical range depends on the base field:

- X complex Banach space $\implies$ $X_{\mathbb{R}}$ real space underlying X.
- $T \in L(X) \implies T_{\mathbb{R}} \in L(X_{\mathbb{R}})$ is T view as a real operator.
- Then $V(T_{\mathbb{R}}) = \operatorname{Re} V(T)$.
- Consequence:
  X complex, then there is $S \in L(X_{\mathbb{R}})$ with $\|S\| = 1$ and $V(S) = \{0\}$.

Some motivation for the numerical range

- It allows to carry to the general case the concepts of hermitian operator, skew-hermitian operator, dissipative operators. . .
- It gives a description of the Lie algebra corresponding to the Lie group of all onto isometries on the space.
- It gives an easy and quantitative proof of the fact that Id is an strongly extreme point of $B_{L(X)}$ (MLUR point).

Numerical radius: definition and properties

Numerical radius

X real or complex Banach space, $T \in L(X)$,

$$\begin{aligned} v(T) &= \sup\left\{|\lambda| \colon \lambda \in V(T)\right\} \\ &= \sup\left\{|x^*(Tx)| \colon x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\right\} \end{aligned}$$

Elementary properties

X Banach space, $T \in L(X)$

- $v(\cdot)$ is a seminorm, i.e.
  - $v(T+S) \leqslant v(T) + v(S)$ for every $T, S \in L(X)$.
  - $v(\lambda T) = |\lambda|\, v(T)$ for every $\lambda \in \mathbb{K}$, $T \in L(X)$.
- $\sup|\operatorname{Sp}(T)| \leqslant v(T)$.
- $v(U^{-1}TU) = v(T)$ for every $U \in \operatorname{Iso}(X)$.

Important property

$$v(T^*) = v(T).$$

Numerical radius: examples

Some examples

1. H real Hilbert space $\dim(H) > 1$
   $\Longrightarrow$ exist $T \in L(X)$ with $v(T) = 0$ and $\|T\| = 1$.
2. H complex Hilbert space $\dim(H) > 1$
   - $v(T) \geqslant \frac{1}{2}\|T\|$,
   - the constant $\dfrac{1}{2}$ is optimal.
3. $X = L_1(\mu) \implies v(T) = \|T\|$ for every $T \in L(X)$.
4. $X^* \equiv L_1(\mu) \implies v(T) = \|T\|$ for every $T \in L(X)$.
5. In particular, this is the case for $X = C(K)$.

Numerical radius: the base field matters

Example

X complex Banach space, define $T \in L(X_{\mathbb{R}})$ by

$$T(x) = i\,x \qquad (x \in X).$$

- $\|T\| = 1$ and $v(T) = 0$ if viewed in $X_{\mathbb{R}}$.
- $\|T\| = 1$ and $V(T) = \{i\}$, so $v(T) = 1$ if viewed in (complex) X.

Theorem (Bohnenblust-Karlin; Glickfeld)

X complex Banach space, $T \in L(X)$:

$$v(T) \geqslant \frac{1}{\mathrm{e}}\|T\|.$$

The constant $\frac{1}{\mathrm{e}}$ is optimal:

$\exists\ X$ two-dimensional complex, $\exists\ T \in L(X)$ with $\|T\| = \mathrm{e}$ and $v(T) = 1$.

Relationship with surjective isometries

The exponential function

X Banach space, $T \in L(X)$:

$$\exp(T) = \sum_{n=0}^{\infty} \frac{1}{n!} T^n$$

where $T^0 = \mathrm{Id}$ and $T^n = T \circ \overset{n)}{\cdots} \circ T$.

First properties

X Banach space, $T, S \in L(X)$.

- $TS = ST \implies \exp(T+S) = \exp(T)\exp(S)$.
- $\exp(T)\exp(-T) = \exp(0) = \mathrm{Id} \implies \exp(T)$ surjective isomorphism.
- $\left\{\exp(\rho T) \colon \rho \in \mathbb{R}_0^+\right\}$ exponential one-parameter semigroup generated by T.

An important property

X Banach space, $T, S \in L(X)$.

- $\|\exp(\lambda T)\| \leqslant \mathrm{e}^{|\lambda|\, v(T)}$ $(\lambda \in \mathbb{K})$.
- $v(T)$ is the best possible constant.

Semigroups of isometries: motivating example

A motivating example

A real or complex $n \times n$ matrix. TFAE:

- A is skew-adjoint (i.e. $A^* = -A$).
- $\operatorname{Re}(Ax \mid x) = 0$ for every $x \in H$.
- $B = \exp(\rho A)$ is unitary for every $\rho \in \mathbb{R}$ (i.e. $B^*B = BB^* = \mathrm{Id}$).

In term of Hilbert spaces

H (n-dimensional) Hilbert space, $T \in L(H)$. TFAE:

- $\operatorname{Re} W(T) = \{0\}$.
- $\exp(\rho T) \in \operatorname{Iso}(H)$ for every $\rho \in \mathbb{R}$.

For general Banach spaces

X Banach space, $T \in L(X)$. TFAE:

- $\operatorname{Re} V(T) = \{0\}$.
- $\exp(\rho T) \in \operatorname{Iso}(X)$ for every $\rho \in \mathbb{R}$.

Semigroups of isometries: characterization

Theorem (Bonsall-Duncan, 1970's; Rosenthal, 1984)

X real or complex Banach space, $T \in L(X)$. TFAE:

- $\operatorname{Re} V(T) = \{0\}$ (T is skew-hermitian, we write $T \in \mathcal{Z}(X)$).
- $\|\exp(\rho T)\| \leqslant 1$ for every $\rho \in \mathbb{R}$.
- $\left\{ \exp(\rho T) \colon \rho \in \mathbb{R}_0^+ \right\} \subset \operatorname{Iso}(X)$.
- T belongs to the tangent space to $\operatorname{Iso}(X)$ at Id.
- $\displaystyle\lim_{\rho \to 0} \frac{\|\operatorname{Id} + \rho T\| - 1}{\rho} = 0$.

Main consequence

If X is a real Banach space such that

$$V(T) = \{0\} \quad \Longrightarrow \quad T = 0,$$

then $\operatorname{Iso}(X)$ is “small”:

- it does not contain any exponential one-parameter semigroup,
- the tangent space of $\operatorname{Iso}(X)$ at Id is zero.

Semigroups of surjective isometries and duality

Remark

X Banach space.

- $T \in \mathrm{Iso}(X) \implies T^* \in \mathrm{Iso}(X^*)$.
- $\mathrm{Iso}(X^*)$ can be bigger than $\mathrm{Iso}(X)$.

A problem

- How much bigger can be $\mathrm{Iso}(X^*)$ than $\mathrm{Iso}(X)$?
- Is it possible that $\mathcal{Z}\big(\mathrm{Iso}(X^*)\big)$ is big while $\mathcal{Z}\big(\mathrm{Iso}(X)\big)$ is trivial?

Example (proved used numerical ranges)

There exists a Banach space $\mathcal{X}$ such that:

- $\mathrm{Iso}\big(\mathcal{X}\big)$ has no exponential one-parameter semigroups.
- $\mathrm{Iso}\big(\mathcal{X}^*\big)$ contains $\mathrm{Iso}(\ell_2)$ (and so it contains infinitely many one-parameter semigroups).

Semigroups of surjective isometries and duality. II

In terms of linear dynamical systems

- In $\mathcal{X}$ there is no $A \in L(\mathcal{X})$ such that the solution to the linear dynamical system

$$x' = A\,x \qquad \left(x : \mathbb{R}_0^+ \longrightarrow \mathcal{X}\right)$$

  (which is $x(t) = \exp(t\,A)(x(0))$) is given by a semigroup of isometries.
- There are infinitely many such A's in $\mathcal{X}^*$, in $\mathcal{X}^{**}\dots$

Further results (Koszmider–M.–Merí., 2009)

- There are unbounded A's on $\mathcal{X}$ such that the solution to the linear dynamical system

$$x'(t) = A\,x(t)$$

  is a one-parameter C_0 semigroup of isometries.
- However, there is $\mathcal{Y}$ such that $\operatorname{Iso}(\mathcal{Y}) = \{-\mathrm{Id}, \mathrm{Id}\}$ and $\operatorname{Iso}(\mathcal{Y}^*)$ contains $\operatorname{Iso}(\ell_2)$.
- Therefore, there is no semigroups in $\operatorname{Iso}(X)$, but there are infinitely many exponential one-parameter semigroups in $\operatorname{Iso}(X^*)$.

Numerical index of Banach spaces

Section 2

Numerical index of Banach spaces: definitions

Numerical radius

X Banach space, $T \in L(X)$. The numerical radius of T is

$$v(T) = \sup\left\{|x^*(Tx)|\colon x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\right\}$$

Remark

The numerical radius is a continuous seminorm in $L(X)$. Actually, $v(\cdot) \leqslant \|\cdot\|$

Numerical index (Lumer, 1968)

X Banach space, the numerical index of X is

$$\begin{aligned} n(X) &= \inf\left\{v(T)\colon T \in L(X),\ \|T\| = 1\right\} \\ &= \max\left\{k \geqslant 0\colon k\,\|T\| \leqslant v(T)\ \ \forall\, T \in L(X)\right\} \\ &= \inf\left\{M \geqslant 0\colon \|\exp(\rho T)\| \leqslant \mathrm{e}^{\rho M}\ \forall \rho \in \mathbb{R}, \forall T \in L(X)\, \|T\| = 1\right\} \end{aligned}$$

Numerical index of Banach spaces: basic properties

Recalling some basic properties

- $n(X) = 1$ iff v and $\|\cdot\|$ coincide.
- $n(X) = 0$ iff v is not an equivalent norm in $L(X)$

- X complex $\implies n(X) \geqslant 1/\mathrm{e}$.

  (Bohnenblust–Karlin, 1955; Glickfeld, 1970)

- Actually,

$$\{n(X)\colon X \text{ complex},\ \dim(X) = 2\} = [\mathrm{e}^{-1}, 1]$$
$$\{n(X)\colon X \text{ real},\ \dim(X) = 2\} = [0, 1]$$

(Duncan–McGregor–Pryce–White, 1970)

Numerical index of Banach spaces: examples (I)

Some examples

1. H Hilbert space, $\dim(H) > 1$,

$$\begin{array}{ll} n(H) = 0 & \text{if } H \text{ is real} \\ n(H) = 1/2 & \text{if } H \text{ is complex} \end{array}$$

2. $n\big(L_1(\mu)\big) = 1 \qquad \mu$ positive measure

   $n\big(C(K)\big) = 1 \qquad K$ compact Hausdorff space

   (Duncan et al., 1970)

3. If A is a C^*-algebra $\Longrightarrow \begin{cases} n(A) = 1 & A \text{ commutative} \\ n(A) = 1/2 & A \text{ not commutative} \end{cases}$

   (Huruya, 1977; Kaidi–Morales–Rodríguez, 2000)

4. If A is a function algebra $\Longrightarrow n(A) = 1$

   (Werner, 1997)

Numerical index of Banach spaces: some examples (II)

More examples

5. For $n \geqslant 2$, the unit ball of X_n is a $2n$ regular polygon:

$$n(X_n) = \begin{cases} \tan\left(\frac{\pi}{2n}\right) & \text{if } n \text{ is even,} \\ \\ \sin\left(\frac{\pi}{2n}\right) & \text{if } n \text{ is odd.} \end{cases}$$

(M.–Merí, 2007)

6. Every finite-codimensional subspace of $C[0,1]$ has numerical index 1

(Boyko–Kadets–M.–Werner, 2007)

Numerical index of Banach spaces: some examples (III)

Even more examples

7. Numerical index of L_p-spaces, $1 < p < \infty$:
   - $n\Big(L_p[0,1]\Big) = n(\ell_p) = \lim\limits_{m\to\infty} n\Big(\ell_p^{(m)}\Big)$.

     (Ed-Dari, 2005 & Ed-Dari-Khamsi, 2006)
   - $n\Big(\ell_p^{(2)}\Big)$ **?**
   - Very recent:

     $$n\Big(\ell_p^{(2)}\Big) = M_p \qquad (3/2 \leqslant p \leqslant 3)$$

     and $M_p = v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \max\limits_{t\in[0,1]} \dfrac{|t^{p-1}-t|}{1+t^p}$

     (Merí–Quero, 2020)
   - In the real case, $n\Big(L_p(\mu)\Big) \geqslant \dfrac{M_p}{8\mathrm{e}}$.
   - In particular, $n\Big(L_p(\mu)\Big) > 0$ for $p \neq 2$.

     (M.–Merí–Popov, 2009)

Numerical index: open problems on computing

Some open problems

1. Compute $n\Big(L_p[0,1]\Big)$ for $1 < p < \infty$, $p \neq 2$.
2. Is $n\Big(\ell_p^{(2)}\Big) = M_p$ (real case) for all p's **?**
3. Is $n\Big(\ell_p^{(2)}\Big) = \left(p^{\frac{1}{p}}\, q^{\frac{1}{q}}\right)^{-1}$ (complex case) **?**
4. Compute the numerical index of real C^*-algebras.
5. Compute the numerical index of more classical Banach spaces: $C^m[0,1]$, $\mathrm{Lip}_0(K)$, Lorentz spaces, Orlicz spaces...

Stability properties

Direct sums of Banach spaces (M.–Payá, 2000)

$$n\Big([\oplus_{\lambda\in\Lambda}X_\lambda]_{c_0}\Big) = n\Big([\oplus_{\lambda\in\Lambda}X_\lambda]_{\ell_1}\Big) = n\Big([\oplus_{\lambda\in\Lambda}X_\lambda]_{\ell_\infty}\Big) = \inf_\lambda n(X_\lambda)$$

Consequences

- There is a real Banach space X such that
$$v(T) > 0 \quad \text{when } T \neq 0,$$
but $n(X) = 0$
(i.e. $v(\cdot)$ is a norm on $L(X)$ which is not equivalent to the operator norm).
- For every $t \in [0,1]$, there exist a real X_t isomorphic to c_0 (or ℓ_1 or ℓ_∞) with $n(X_t) = t$.
- For every $t \in [\mathrm{e}^{-1},1]$, there exist a complex Y_t isomorphic to c_0 (or ℓ_1 or ℓ_∞) with $n(Y_t) = t$.

Stability properties (II)

Vector-valued function spaces (López-M.-Merí-Payá-Villena, 2000's)

E Banach space, μ positive σ-finite measure, K compact space. Then

$$n\Big(C(K,E)\Big) = n\Big(C_w(K,E)\Big) = n\Big(L_1(\mu,E)\Big) = n\Big(L_\infty(\mu,E)\Big) = n(E),$$

and $n\Big(C_{w^*}(K,E^*)\Big) \leqslant n(E)$ (this inequality may be strict).

L_p-spaces (Askoy–Ed-Dari–Khamsi, 2007)

$$n\Big(L_p([0,1],E)\Big) = n\Big(\ell_p(E)\Big) = \lim_{m\to\infty} n\Big(E \oplus_p \overset{m}{\cdots} \oplus_p E\Big).$$

Stability properties (III)

Tensor products (Lima, 1980)

There is no general formula for $n\big(X\widetilde{\otimes}_\varepsilon Y\big)$ nor for $n\big(X\widetilde{\otimes}_\pi Y\big)$:

- $n\big(\ell_1^{(4)}\widetilde{\otimes}_\pi\,\ell_1^{(4)}\big) = n\big(\ell_\infty^{(4)}\widetilde{\otimes}_\varepsilon\,\ell_\infty^{(4)}\big) = 1.$
- $n\big(\ell_1^{(4)}\widetilde{\otimes}_\varepsilon\,\ell_1^{(4)}\big) = n\big(\ell_\infty^{(4)}\widetilde{\otimes}_\pi\,\ell_\infty^{(4)}\big) < 1.$

Inequalities for tensor products and ideals of operators (M.-Merí-Quero, 2020)

X, Y Banach spaces:

- $n\big(X\widetilde{\otimes}_\pi Y\big) \leqslant \min\{n(X), n(Y)\}$,
- $n\big(X\widetilde{\otimes}_\varepsilon Y\big) \leqslant \min\{n(X), n(Y)\}$.
- $\mathcal{Z}$ ideal of $L(X,Y)$ $\implies$ $n(\mathcal{Z}) \leqslant \min\{n(X), n(Y)\}$,
- in particular, $n\big(L(X,Y)\big) \leqslant \min\{n(X), n(Y)\}$.
- $n\big(K(X,Y)\big) \leqslant \min\{n(X^*), n(Y)\}$,
- $n\big(W(X,Y)\big) \leqslant \min\{n(X^*), n(Y)\}$.

Numerical index and duality

Proposition

X Banach space, $T \in L(X)$. Then

- $\sup \operatorname{Re} V(T) = \lim\limits_{\alpha\to 0^+} \dfrac{\|\mathrm{Id} + \alpha\, T\| - 1}{\alpha}$.
- Then, $v(T^*) = v(T)$ for every $T \in L(X)$.
- Therefore, $n(X^*) \leqslant n(X)$.

(Duncan–McGregor–Pryce–White, 1970)

Question (From the 1970's)

Is $n(X) = n(X^*)$ **?**

Negative answer (Boyko–Kadets–M.–Werner, 2007)

Consider the space

$$X = \Big\{(x, y, z) \in c \oplus_\infty c \oplus_\infty c\colon\ \lim x + \lim y + \lim z = 0\Big\}.$$

Then, $n(X) = 1$ but $n(X^*) < 1$.

The isomorphic point of view

Renorming and numerical index (Finet–M.–Payá, 2003)

$(X, \|\cdot\|)$ (separable or reflexive) Banach space, $\dim(X) > 1$. Then

- Real case:
$$[0, 1[\subseteq \{n(X, |\cdot|) \colon |\cdot| \simeq \|\cdot\|\}$$
- Complex case:
$$[\mathrm{e}^{-1}, 1[\subseteq \{n(X, |\cdot|) \colon |\cdot| \simeq \|\cdot\|\}$$

Open question

The result is known to be true when X has a long biorthogonal system.
Is it true in general **?**

Remark

In some sense, any other value of $n(X)$ but **1** is isomorphically trivial.
★ What about the value **1** **?**

Banach spaces with numerical index one

Section 3

Banach spaces with numerical index one

Numerical index one

Recall that X has numerical index one ($n(X) = 1$) iff

$$\|T\| = \sup\Big\{|x^*(Tx)|\colon x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1\Big\}$$

(i.e. $v(T) = \|T\|$) for every $T \in L(X)$.
Equivalently, Id is a "spear operator" (we will see this concept later on).

Examples

$C(K)$, $L_1(\mu)$, $A(\mathbb{D})$, H^∞, finite-codimensional subspaces of $C[0,1]\ldots$

This is a property of X which is very complicated to work with as one has to deal with **all** the operators on the space.

Leading open questions

X Banach space with numerical index one $\implies$ $X \supset c_0$ or $X \supset \ell_1$ **?** $X^* \supset \ell_1$ **?**

How to deal with numerical index one property?

One the one hand: weaker properties

- In a general Banach space, we only can construct compact (aproximable) operators.
- Actually, we only may easily calculate the norm of rank-one operators.
- Most of the results we know for Banach spaces with numerical index one are actually true for Banach spaces with the alternative Daugavet property (ADP), that is, those Banach spaces satisfying:
  - $v(T) = \|T\|$ for every rank-one T,
  - equivalently, $\max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta T\| = 1 + \|T\|$ for every T rank-one.

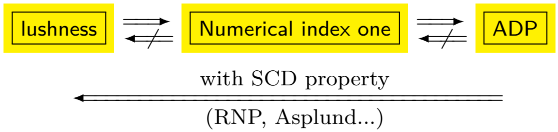
One the other hand: stronger properties

- We do not know any operator-free characterization of Banach spaces with numerical index one.
- When we know that a Banach space has numerical index one (or that it can be renormed with numerical index one), we actually prove more.
- There are some sufficient geometrical conditions.
- The weakest of such properties is called lushness.

How to deal with numerical index one property?

Relationship between the properties

- One of the key ideas to get interesting results for Banach spaces with numerical index one is to study when the three properties below are equivalent.
- A very interesting property appears: the slicely countably determination,
- it will be studied in the next chapter.



The numerical index one has isomorphic consequences

Question

Does every Banach space admit an equivalent norm with numerical index one **?**

Negative answer (López–M.–Payá, 1999)

Not every Banach space can be renormed to have numerical index one.
Concretely:

- If X is real, RNP, $\dim(X) = \infty$, and $n(X) = 1$, then $X \supset \ell_1$.

On the proof of the 1999 results

Lemma

X Banach space, $n(X) = 1$
$\Longrightarrow$ $|x_0^*(x_0)| = 1$ for every $x_0^* \in \operatorname{ext}(B_{X^*})$ and every $x_0 \in \operatorname{dent}(B_X)$.

Proposition

X real, $A \subset S_X$ infinite with $|x^*(a)| = 1$ $\forall x^* \in \operatorname{ext}(B_{X^*})$, $\forall a \in A$.
$\Longrightarrow$ $X \supseteq c_0$ or $X \supseteq \ell_1$.

Main consequence

X real, RNP, $\dim(X) = \infty$, and $n(X) = 1 \implies X \supseteq \ell_1$.

Sufficient conditions for numerical index one

Some sufficient conditions

Let X be a Banach space. Consider:

(a) **Lindenstrauss, 1964:** X has the 3.2.I.P. if the intersection of every family of three mutually intersecting balls is not empty.

(b) **Fullerton, 1961:** X is a CL-space if B_X is the absolutely convex hull of every maximal face of S_X.

(c) **Lima, 1978:** X is an almost-CL-space if B_X is the closed absolutely convex hull of every maximal face of S_X.

$$\boxed{\text{(a)}} \;\underset{\not\Longleftarrow}{\Longrightarrow}\; \boxed{\text{(b)}} \;\underset{\not\Longleftarrow}{\Longrightarrow}\; \boxed{\text{(c)}} \;\underset{\not\Longleftarrow}{\Longrightarrow}\; \boxed{n(X) = 1}$$

Showing that (c) $\Longrightarrow$ $n(X) = 1$, one realizes that (c) is too much.

Lushness (Boyko–Kadets–M.–Werner, 2007)

X is lush if given $x, y \in S_X$, $\varepsilon > 0$, there is $x^* \in S_{X^*}$ such that

$$x \in \operatorname{Slice}(B_X, x^*, \varepsilon) \quad \text{and} \quad \operatorname{dist}\Big(y, \operatorname{aconv}\big(\operatorname{Slice}(B_X, x^*, \varepsilon)\big)\Big) < \varepsilon.$$

Definition and first property

Lushness (Boyko–Kadets–M.–Werner, 2007)

X is lush if given $x, y \in S_X$, $\varepsilon > 0$, there is $x^* \in S_{X^*}$ such that

$$x \in \operatorname{Slice}(B_X, x^*, \varepsilon) \quad \text{and} \quad \operatorname{dist}\Big(y, \operatorname{aconv}\Big(\operatorname{Slice}(B_X, x^*, \varepsilon)\Big)\Big) < \varepsilon.$$

Theorem (Boyko–Kadets–M.–Werner, 2007)

X lush $\implies n(X) = 1$.

Reformulations of lushness and applications

Proposition (Boyko–Kadets–M.–Merí, 2009)

X Banach space. TFAE:

- X is lush,
- Every separable $E \subset X$ is contained in a separable lush Y with $E \subset Y \subset X$.

Separable lush spaces (Kadets–M.–Meri–Payá, 2009; Lee–M., 2012)

X **separable**. TFAE:

- X is lush.
- There is $G \subseteq S_{X^*}$ norming for X such that

$$B_X = \overline{\text{aconv}}\,(\text{Face}(B_X, x^*)) \qquad (x^* \in G).$$

Therefore, $|x^{**}(x^*)| = 1\ \forall x^{**} \in \text{ext}\,(B_{X^{**}})\ \forall x^* \in G$.

An important consequence

Showed in the previous slide...

X lush **separable**, $\dim(X) = \infty \implies$ there is $G \in S_{X^*}$ infinite such that

$$|x^{**}(x^*)| = 1 \qquad \big(x^{**} \in \operatorname{ext}(B_{X^{**}}),\ x^* \in G\big).$$

Proposition (López–M.–Payá, 1999)

X real, $A \subset S_X$ infinite with $|x^*(a)| = 1\ \forall x^* \in \operatorname{ext}(B_{X^*})$, $\forall a \in A$.
$\implies X \supseteq c_0$ or $X \supseteq \ell_1$.

Main consequence

X real lush, $\dim(X) = \infty \implies X^* \supseteq \ell_1$.

Proof.

- There is $E \subseteq X$ infinite-dimensional, separable, and lush.
- Then $E^* \supseteq c_0$ or $E^* \supseteq \ell_1 \implies E^* \supseteq \ell_1$.
- By the “lifting” property of $\ell_1 \implies X^* \supseteq \ell_1$. ✓

Lushness is not equivalent to numerical index one

Example (Kadets–M.–Merí–Shepelska, 2009)

There is a separable Banach space $\mathcal{X}$ such that

- $\mathcal{X}^*$ is lush but $\mathcal{X}$ is not lush.
- Since $n(\mathcal{X}^*) = 1$, also $n(\mathcal{X}) = 1$.
- But the set
$$\left\{x^* \in S_{\mathcal{X}^*} : |x^{**}(x^*)| = 1 \text{ for every } x^{**} \in \operatorname{ext}\left(B_{\mathcal{X}^{**}}\right)\right\}$$
is empty.

Remark

We cannot expect to show that $X^* \supseteq \ell_1$ when $n(X) = 1$ using only the ideas developed for lush spaces, something more is needed.

Slicely countably determined Banach spaces

Section 4

Two classical concepts: Radon-Nikodým property and Asplund spaces

The Radon-Nikodým property or RNP (1930's)

- X has the RNP iff the Radon-Nikodým theorem is valid for X-valued meassures;
- Equivalently [1960's], every bcc subset contains a denting point.

X Asplund $\iff$ X^* RNP

Reflexive (say) $\Longrightarrow$ (RNP and Asplund)

(RNP or Asplund) $\Longrightarrow$ **???**

Asplund spaces (1960's)

- X is an Asplund space if every continuous convex real-valued function defined on an open subset of X is Frechet-differentiable on a dense subset;
- Equivalently [1970's], every separable subspace has separable dual.

SCD sets and spaces: Definitions and examples

SCD sets

$A \subset X$ bounded convex is slicely countably determined (SCD) if there is a sequence $\{S_n : n \in \mathbb{N}\}$ of slices of A satisfying one of the following equivalent conditions:

- every slice of A contains one of the S_n's,
- $A \subseteq \overline{\operatorname{conv}}(B)$ if $B \subseteq A$ satisfies $B \cap S_n \neq \emptyset \ \forall n$,
- given $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in S_n \ \forall n \in \mathbb{N}$, $A \subseteq \overline{\operatorname{conv}}\big(\{x_n : n \in \mathbb{N}\}\big)$.

SCD spaces

X is Slicely Countably Determined (SCD) if so are all its bounded convex subsets.

Avilés–Kadets–M.–Merí–Shepelska, 2010

Remarks

- A is SCD iff $\overline{A}$ is SCD.
- If A is SCD, then it is separable.

Examples of SCD sets and spaces

Examples of sets

$A \subset X$ **separable** bounded and convex.

1. (Easy): A RNP $\implies$ A is SCD,
2. (Easy): A Asplund $\implies$ A is SCD,
3. (Main): $A \not\supset \ell_1$ $\implies$ A is SCD,
4. $B_{C[0,1]}$ and $B_{L_1[0,1]}$ are not SCD.

Examples of spaces

X **separable** Banach space.

1. X RNP $\implies$ X is SCD,
2. X Asplund $\implies$ X is SCD,
3. $X \not\supset \ell_1$ $\implies$ X is SCD,
4. $C[0,1]$ and $L_1[0,1]$ are not SCD.

- The proofs of the easy ones are straightforward. . .
- The proof of the main one relies on a deep result of S. Todorčević which needs "forcing".

SCD spaces: definition and examples

SCD space

X is Slicely Countably Determined (SCD) if so are all of its convex bounded subsets.

Examples of SCD spaces

1. X separable strongly regular. In particular, RNP, CPCP spaces.
2. X separable $X \not\supseteq \ell_1$. In particular, if X^* is separable.

Examples of NOT SCD spaces

1. $C[0,1]$, $L_1[0,1]$
2. There is X with the Schur property which is not SCD.

Remark

- Every subspace of a SCD space is SCD.
- This is false for quotients.

ADP + SCD $\Longrightarrow$ numerical index 1

Characterizations of the ADP

X Banach space. TFAE:

- X has ADP (i.e. $\max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta T\| = 1 + \|T\|$ for all T rank-one).
- Given $x \in S_X$, a slice S of B_X and $\varepsilon > 0$, there is $y \in S$ with
$$\max_{\theta\in\mathbb{T}} \|x + \theta\, y\| > 2 - \varepsilon.$$
- Given $x \in S_X$, a sequence $\{S_n\}$ of slices of B_X, and $\varepsilon > 0$, there is $y^* \in S_{X^*}$ such that $x \in \mathrm{Slice}(B_X, y^*, \varepsilon)$ and
$$\overline{\mathrm{conv}}\Big(\mathbb{T}\ \mathrm{Slice}(B_X, y^*, \varepsilon)\Big) \cap S_n \neq \emptyset \qquad (n \in \mathbb{N}).$$

Theorem

X ADP + B_X SCD $\Longrightarrow$ given $x \in S_X$ and $\varepsilon > 0$, there is $y^* \in S_{X^*}$ such that
$$x \in \mathrm{Slice}(B_X, y^*, \varepsilon) \quad \text{and} \quad B_X = \overline{\mathrm{conv}}\Big(\mathbb{T}\ \mathrm{Slice}(B_X, y^*, \varepsilon)\Big).$$

★ This implies lushness and so, numerical index 1.

Some consequences

Corollary

- ADP + strongly regular $\Longrightarrow$ numerical index 1 (actually, lushness).
- ADP + $X \not\supseteq \ell_1$ $\Longrightarrow$ numerical index 1 (actually, lushness).

Main consequence

X real + $\dim(X) = \infty$ + ADP $\Longrightarrow$ $X^* \supseteq \ell_1$.

In particular,

Corollary

X real + $\dim(X) = \infty$ + numerical index 1 $\Longrightarrow$ $X^* \supseteq \ell_1$.

Open question

X real, $\dim(X) = \infty$, $n(X) = 1$ $\Longrightarrow$ $X \supset c_0$ or $X \supset \ell_1$ **?**

The numerical index with respect to an operator

Section 5

Motivation

Geometry of the space of operators

X Banach space

- The numerical range of $T \in L(X)$ represent the geometry of the unit ball of $L(X)$ at Id in the direction of T: $\sup \operatorname{Re} V(T) = \lim\limits_{\alpha \to 0^+} \dfrac{\|\mathrm{Id} + \alpha\, T\| - 1}{\alpha}$.
- Actually, $n(X) > 0 \iff \mathrm{Id}$ is a geometrically unitary element of $B_{L(X)}$...
- A point $u \in S_Z$ is unitary if the linear span of the set $\{z^* \in S_{Z^*} : z^*(u) = 1\}$ coincides with the whole of Z^*.
- Equivalently, exists $k > 0$ such that $\max_{\theta \in \mathbb{T}} \|u + \theta z\| \geqslant 1 + k\|z\| \ \forall z \in Z$.
- The study of unitary elements has been very important in many results of functional analysis as, for instance, in Vidav's characterization of C^*-algebras.

Question

Can we do the same for an arbitrary norm one operator between Banach spaces **?**

That is, is there a notion of numerical range, numerical radius, numerical index...
for an arbitrary operator which helps to study when the operator is a unitary **?**

Spatial numerical range

Bauer–Lumer (spatial) Numerical range

X Banach space, $T \in L(X)$,

$$V(T) = \{x^*(Tx) \colon x \in S_X,\, x^* \in S_{X^*},\, x^*(\mathrm{Id}\, x) = 1\}$$

★ $G \in L(X,Y)$ with $\|G\| = 1$, $T \in L(X,Y)$, how to define $V_G(T)$?
The first idea (not working):

$$V_G(T) = \{y^*(Tx) \colon x \in S_X,\, y^* \in S_{Y^*},\, y^*(Gx) = 1\}$$

(Approximate spatial) Numerical range with respect to G (Ardalani, 2014)

X, Y Banach spaces, $G \in L(X,Y)$ with $\|G\| = 1$, $T \in L(X,Y)$

$$V_G(T) = \bigcap_{\delta>0} \overline{\{y^*(Tx) \colon x \in S_X,\, y^* \in S_{Y^*},\, \operatorname{Re} y^*(Gx) > 1-\delta\}}$$

For $G = \mathrm{Id}$, by the Bishop–Phelps–Bollobás theorem (Ardalani, 2014)

$$V_{\mathrm{Id}}(T) = \overline{V(T)} \qquad \text{for every } T \in L(X)$$

Intrinsic Numerical range

(Bonsall-Duncan, 1971)

Let X be a Banach space. Then for every $T \in L(X)$

$$\overline{\operatorname{conv}}\, V(T) = \{\Phi(T) \colon \Phi \in L(X)^*,\ \|\Phi\| = \Phi(\mathrm{Id}) = 1\}.$$

Consequently, $v(T) = \max\{|\Phi(T)| \colon \Phi \in L(X)^*,\ \|\Phi\| = \Phi(\mathrm{Id}) = 1\}$.

Intrinsic (or algebraic) numerical range

X Banach space, $T \in L(X)$,

$$\widetilde{V}(T) = \{\Phi(T) \colon \Phi \in L(X)^*,\ \|\Phi\| = \Phi(\mathrm{Id}) = 1\}$$

Intrinsic numerical range with respect to G

X, Y Banach spaces, $G \in L(X,Y)$ with $\|G\| = 1$, $T \in L(X,Y)$

$$\widetilde{V}_G(T) = \{\Phi(T) \colon \Phi \in L(X,Y)^*,\ \|\Phi\| = \Phi(G) = 1\}$$

The relationship

Two possible numerical ranges

X, Y Banach spaces, $G \in L(X,Y)$ with $\|G\| = 1$, $T \in L(X,Y)$

$$V_G(T) = \bigcap_{\delta>0} \overline{\{y^*(Tx)\colon x \in S_X,\, y^* \in S_{Y^*},\, \operatorname{Re} y^*(Gx) > 1-\delta\}}$$

$$\widetilde{V}_G(T) = \{\Phi(T)\colon \Phi \in L(X,Y)^*,\ \|\Phi\| = \Phi(G) = 1\}$$

Relationship (M., 2016)

X, Y be Banach spaces, $G \in L(X,Y)$ with $\|G\| = 1$, then

$$\widetilde{V}_G(T) = \operatorname{conv} V_G(T) \qquad \text{for every } T \in L(X,Y)$$

Both concepts produce the same numerical radius:

Numerical radius with respect to G

X, Y Banach spaces, $G \in L(X,Y)$ with $\|G\| = 1$, $T \in L(X,Y)$

$$v_G(T) = \sup\{|\lambda|\colon \lambda \in V_G(T)\} = \sup\{|\lambda|\colon \lambda \in \widetilde{V}_G(T)\}$$

Numerical index with respect to an operator

Numerical index with respect to G

X, Y Banach spaces, $G \in L(X,Y)$ with $\|G\| = 1$,

$$n_G(X,Y) = \inf\{v_G(T)\colon T \in S_{L(X,Y)}\} = \max\{k \geqslant 0\colon k\|T\| \leqslant v_G(T)\}$$

We recuperate the classical numerical index

$$n_{\mathrm{Id}}(X,X) = n(X)$$

Characterization

For $k \in [0,1]$, TFAE:

- $n_G(X,Y) \geqslant k$,
- $\inf\limits_{\delta>0} \sup\{|y^*(Tx)|\colon x \in S_X,\ y^* \in S_{Y^*},\ \operatorname{Re} y^*(Gx) > 1-\delta\} \geqslant k\|T\| \quad \forall\, T \in L(X,Y)$,
- $\max\limits_{|\theta|=1} \|G + \theta\, T\| \geqslant 1 + k\|T\| \quad \forall\, T \in L(X,Y)$.

Consequence

$$n_G(X,Y) > 0 \iff G \text{ is a (geometrically) unitary element of } L(X,Y)$$

Some interesting examples I

Set of values

There exists X (real and complex versions) such that

$$\Big\{n_G(X,X)\colon G\in L(X,X),\ \|G\|=1\Big\}=[0,1].$$

Hilbert spaces

H_1, H_2 Hilbert spaces of dimension at least two,

- Real case: $n_G(H_1,H_2)=0$ for all $G\in L(H_1,H_2)$ with $\|G\|=1$,
- Complex case: $n_G(H_1,H_2)\in\{0,1/2\}$ for all $G\in L(H_1,H_2)$ with $\|G\|=1$.

Actually...

$G\in L(X,Y)$ with $\|G\|=1$, if X or Y is a real Hilbert space

$$\Longrightarrow \quad n_G(X,Y)=0.$$

★ There are more spaces with this property...

Some interesting examples II

L_p-spaces

$G \in L(X,Y)$ with $\|G\| = 1$, if X or Y is a $L_p(\mu)$-space $(1 < p < \infty)$,

$$\Longrightarrow \quad n_G(X,Y) \leqslant \begin{cases} \sup_{t\in[0,1]} \frac{|t^{p-1}-t|}{1+t^p} & \text{real case} \\ p^{-1/p} q^{-1/q} & \text{complex case} \end{cases}$$

Spaces of integrable functions

μ_1, μ_2 σ-finite measures,

$$n_G\big(L_1(\mu_1), L_1(\mu_2)\big) \in \{0,1\} \text{ for all } G \in L\big(L_1(\mu_1), L_1(\mu_2)\big) \text{ with } \|G\| = 1.$$

Spaces of essentially bounded functions

μ_1, μ_2 σ-finite measures,

$$n_G\big(L_\infty(\mu_1), L_\infty(\mu_2)\big) \in \{0,1\} \text{ for all } G \in L\big(L_\infty(\mu_1), L_\infty(\mu_2)\big) \text{ with } \|G\| = 1.$$

Spaces of continuous functions

For SOME pairs of compact Hausdorff topological spaces K_1 and K_2:

$$n_G\big(C(K_1), C(K_2)\big) \in \{0,1\} \text{ for all } G \in L\big(C(K_1), C(K_2)\big) \text{ with } \|G\| = 1.$$

Sums of Banach spaces

Proposition

Let $\{X_\lambda : \lambda \in \Lambda\}$, $\{Y_\lambda : \lambda \in \Lambda\}$ be two families of Banach spaces and let $G_\lambda \in L(X_\lambda, Y_\lambda)$ with $\|G_\lambda\| = 1$ for every $\lambda \in \Lambda$. Let E be one of the Banach spaces c_0, ℓ_∞ or ℓ_1, let $X = \left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_E$ and $Y = \left[\bigoplus_{\lambda\in\Lambda} Y_\lambda\right]_E$ and define the operator $G\colon X \longrightarrow Y$ by

$$G\left[(x_\lambda)_{\lambda\in\Lambda}\right] = (G_\lambda x_\lambda)_{\lambda\in\Lambda}$$

for every $(x_\lambda)_{\lambda\in\Lambda} \in \left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_E$. Then

$$n_G(X,Y) = \inf_\lambda n_{G_\lambda}(X_\lambda, Y_\lambda).$$

Moreover, for $1 < p < \infty$

$$n_G\left([\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_p}, [\oplus_{\lambda\in\Lambda} Y_\lambda]_{\ell_p}\right) \leqslant \inf_\lambda n_{G_\lambda}(X_\lambda, Y_\lambda).$$

Composition operators

Theorem

Let X, Y be Banach spaces, and $G \in L(X,Y)$ with $\|G\| = 1$.

- K compact, consider $\widetilde{G}\colon C(K,X) \longrightarrow C(K,Y)$ given by $\widetilde{G}(f) = G \circ f$; then

$$n_{\widetilde{G}}\big(C(K,X), C(K,Y)\big) = n_G(X,Y).$$

- μ measure, consider $\widetilde{G}\colon L_1(\mu,X) \longrightarrow L_1(\mu,Y)$ given by $\widetilde{G}(f) = G \circ f$; then

$$n_{\widetilde{G}}\big(L_1(\mu,X), L_1(\mu,Y)\big) = n_G(X,Y).$$

- μ σ-finite, consider $\widetilde{G}\colon L_\infty(\mu,X) \longrightarrow L_\infty(\mu,Y)$ given by $\widetilde{G}(f) = G \circ f$; then

$$n_{\widetilde{G}}\big(L_\infty(\mu,X), L_\infty(\mu,Y)\big) = n_G(X,Y).$$

Besides, for vector-valued L_p-spaces one inequality holds:

$$n_{\widetilde{G}}\big(L_p(\mu,X), L_p(\mu,Y)\big) \leqslant n_G(X,Y)$$

for $1 < p < \infty$, $\widetilde{G}$ defined analogously.

Examples of spear operators

Spear operator (Ardalani, 2014; Kadets, Martín, Merí, Pérez, 2018)

$$G \text{ spear operator} \iff n_G(X,Y) = 1 \iff \max_{|\theta|=1} \|G + \theta\, T\| = 1 + \|T\| \ \forall\, T \in L(X,Y).$$

Some interesting examples of spear operators

- Fourier transform (for example, $\mathcal{F} : L_1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$);
- The inclusion $A(\mathbb{D}) \longrightarrow C(\mathbb{T})$;
- The identity operator on $C(K)$, $L_1(\mu)\ldots$
- $G : X \longrightarrow c_0$ spear iff $\left|x^{**}\Big(G^*(e_n)\Big)\right| = 1$ for $n \in \mathbb{N}$ and $x^{**} \in \operatorname{ext}(B_{X^{**}})$;
- $G : \ell_1 \longrightarrow Y$ spear iff $\left|y^*\Big(G(e_n)\Big)\right| = 1$ for $n \in \mathbb{N}$ and $y^* \in \operatorname{ext}(B_{Y^*})$;
- If $\dim(X) < \infty$, G spear iff $\left|y^*\Big(Gx\Big)\right| = 1$ for $y^* \in \operatorname{ext}(B_{Y^*})$ and $x \in \operatorname{ext}(B_X)$;
- If $\dim(Y) < \infty$, G spear iff $\left|x^{**}\Big(G^*(y^*)\Big)\right| = 1$ for $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and $y^* \in \operatorname{ext}(B_{X^*})$;

Studying spear operators

Spear operator (Ardalani, 2014; Kadets, Martín, Merí, Pérez, 2018)

G spear operator $\iff n_G(X,Y)=1 \iff \max\limits_{|\theta|=1} \|G+\theta\, T\| = 1+\|T\| \;\forall\, T\in L(X,Y).$

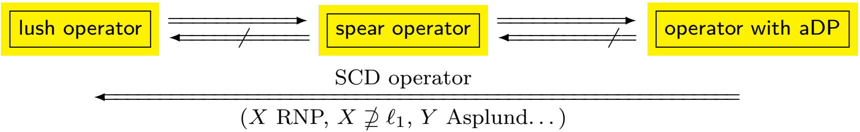
Remark

To work with spear operators, two other concepts are introduced:

- lush operator,
- the alternative Daugavet property (aDP),

★ Both are geometric properties (related to G)

★ They are related as follows:



Spear operators: consequences

Some isomorphic and isometric consequences

X, Y Banach spaces, $G \in L(X,Y)$ spear operator,

- if $\dim(G(X)) = \infty$ and X is real, then $X^* \supset \ell_1$,
- if X^* is strictly convex, then $X = \mathbb{K}$,
- if X^* is smooth, then $X = \mathbb{K}$,
- if B_X contains a WLUR point, then $X = \mathbb{K}$,
- if Y^* is strictly convex, then $Y = \mathbb{K}$,
- if B_Y contains a WLUR point, then $Y = \mathbb{K}$.

Norm attainment

- If G is lush, G attains its norm; actually:

$$B_X = \overline{\operatorname{conv}}\Big\{x \in S_X : \|Gx\| = 1\Big\},$$

- There are examples of aDP operators which do not attain the norm,
- What about spear operators **?**

The second numerical index

Section 6

The base field does matter for the numerical index

(Bohnenblust-Karlin, Glickfeld-1970)

$n(X) \geqslant 1/\mathrm{e}$ for every complex Banach space X

Examples in the real case

- $n(H) = 0$ for H real Hilbert space with $\dim(H) \geqslant 2$
- $n(X_{\mathbb{R}}) = 0$ for X complex Banach space
- But there is X such that $n(X) = 0$ and v is a norm (Martín–Payá, 2000)

In the first two cases there is $T \in L(X) \setminus \{0\}$ with $v(T) = 0$:

- $(x_1, x_2, x_3, \ldots) \longmapsto (-x_2, x_1, 0, \ldots)$,
- $x \longmapsto ix$

Observation

$$v(T) = 0 \iff \exp(\rho T) \text{ is an onto isometry for every } \rho \in \mathbb{R}$$

The second numerical index

Lie Algebra

X real Banach space

$\mathcal{Z}(X) := \{S \in L(X) \colon v(S) = 0\}$ (it is a closed subspace of $L(X)$)

Then, for all $T + \mathcal{Z}(X) \in L(X)/\mathcal{Z}(X)$ we may consider two norms:

$$\|T + \mathcal{Z}(X)\| := \inf\Big\{\|T - S\| \colon S \in \mathcal{Z}(X)\Big\}$$
$$v(T + \mathcal{Z}(X)) := \inf\Big\{v(T - S) \colon S \in \mathcal{Z}(X)\Big\} = v(T)$$

It is immediate that $v(T) \leqslant \|T + \mathcal{Z}(X)\|$ for every $T \in L(X)$

Second numerical index

$$\begin{aligned} n'(X) &:= \inf\Big\{v(T) \colon T \in L(X)\,, \|T + \mathcal{Z}(X)\| = 1\Big\} \\ &= \max\Big\{k \geqslant 0 \colon k\|T + \mathcal{Z}(X)\| \leqslant v(T)\ \forall T \in L(X)\Big\} \end{aligned}$$

Obviously $0 \leqslant n'(X) \leqslant 1$

The second numerical index

Observations

- If $\mathcal{Z}(X) = \{0\}$ (in particular if $n(X) > 0$), then $n'(X) = n(X)$
- $n(X) \leqslant n'(X)$ (observe that $v(T) \leqslant \|T + \mathcal{Z}(X)\| \leqslant \|T\|$)
- On $L(X)/\mathcal{Z}(X)$, both $\| \cdot + \mathcal{Z}(X)\|$ and $v(\cdot)$ are norms

Further observation

There is no third numerical index

Some examples

- $n'(X) > 0$ when X is finite-dimensional
- But there is a Banach space X with $n(X) = 0$ and $n'(X) = 0$

Main example

Theorem

Let H be a Hilbert space. Then, $n'(H) = 1$.

Proof

Fixed $T \in L(H)$ we have to show that

$$v(T) = \|T + \mathcal{Z}(H)\| \quad \left(= \left\| \frac{T + T^*}{2} \right\|\right)$$

Facts

- $S \in \mathcal{Z}(H) \iff S = -S^*$
- $T = T^* \implies v(T) = \|T\|$

Absolute norm on $\mathbb{R}^m$ and absolute sum of Banach spaces

Absolute norm

A norm $\|\cdot\|$ on $\mathbb{R}^m$ is absolute if

- $\|(a_1,\dots,a_m)\| = \Big\|(|a_1|,\dots,|a_m|)\Big\|$ for every $(a_1,\dots,a_m)\in\mathbb{R}^m$.
- $\|e_k\| = 1$ for every $k=1,\dots,m$ where $e_k = (0,\dots,0,\underbrace{1}_{k},0,\dots,0)$.

Absolute sum

Let E be $\mathbb{R}^m$ endowed with an absolute norm. We write $\big[X_1\oplus\cdots\oplus X_m\big]_E$ for the E-sum of the Banach spaces $X_1,\dots,X_m$. That is, the space $X_1\times\cdots\times X_m$ endowed with the complete norm $\|(x_1,\dots,x_n)\| = \Big\|(\|x_1\|,\dots,\|x_m\|)\Big\|_E$.

When E is $\mathbb{R}^2$ endowed with an absolute norm $\|\cdot\|_a$ we just write $X_1\oplus_a X_2 = [X_1\oplus X_2]_E$.

Relationship of n' with absolute sums

Proposition

Let $X = X_1 \oplus_a X_2$, where $\oplus_a \neq \oplus_2$ is an absolute sum. Then,

$$n'(X) \leqslant \min\{n'(X_1), n'(X_2)\}.$$

Corollary

Let $\{X_\lambda \colon \lambda \in \Lambda\}$ be a family of Banach spaces, $1 \leqslant p \leqslant \infty$ with $p \neq 2$. Then

$$n'\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{\ell_p}\Big) \leqslant \inf\{n'(X_\lambda) \colon \lambda \in \Lambda\}.$$

Examples (equality does not hold)

$$n'(\ell_2^2 \oplus_\infty \mathbb{R}) \leqslant \frac{\sqrt{3}}{2} < 1 \qquad \text{and} \qquad n'(\ell_2^2 \oplus_1 \mathbb{R}) \leqslant \frac{\sqrt{3}}{2} < 1$$

Relationship of n' with absolute sums

Proposition

Let X_1, X_2 be Banach spaces and write $X = X_1 \oplus_\infty X_2$ or $X = X_1 \oplus_1 X_2$.

- If $n(X_1) > 0$ and $n(X_2) > 0$, then $n'(X) = n(X) = \min\{n(X_1), n(X_2)\}$.
- If $n(X_1) > 0$ and $n(X_2) = 0$, then $n'(X) \geqslant \min\left\{n(X_1), \frac{n'(X_2)}{n'(X_2)+1}\right\}$.
- If $n(X_1) = 0$ and $n(X_2) = 0$, then

$$n'(X) \geqslant \min\left\{\frac{n'(X_1)}{n'(X_1)+1}, \frac{n'(X_2)}{n'(X_2)+1}\right\}.$$

Example

$$\tfrac{1}{2} \leqslant n'(\ell_2^2 \oplus_\infty \mathbb{R}) \leqslant \tfrac{\sqrt{3}}{2} \qquad \text{and} \qquad \tfrac{1}{2} \leqslant n'(\ell_2^2 \oplus_1 \mathbb{R}) \leqslant \tfrac{\sqrt{3}}{2}$$

A family of examples

Example

For every $\theta \in (0, 1/2]$, there is a four-dimensional Banach space X_θ such that $n(X_\theta) = 0$ and $n'(X_\theta) = \theta$.

Let Y_θ be a two-dimensional space with $n(Y_\theta) = \theta$ and take $X_\theta = Y_\theta \oplus_\infty \ell_2^2$. Then:

- $n(X_\theta) \leqslant n(\ell_2^2) = 0$
- $n'(X_\theta) \leqslant n'(Y_\theta) = n(Y_\theta) = \theta$
- $n'(X_\theta) \geqslant \min\left\{ n(Y_\theta), \dfrac{n'(\ell_2^2)}{n'(\ell_2^2) + 1} \right\} = \min\left\{\theta, \frac{1}{2}\right\} = \theta$

More examples (low dimensions)

- $\dim(X) = 2,\ n(X) = 0 \implies n'(X) = 1,$
- $\{n'(X) \colon n(X) = 0,\ \dim(X) = 3\} \supset [1/\mathrm{e}, 1/2]$ and it is NOT an interval,
- $\{n'(X) \colon n(X) = 0,\ \dim(X) = 4\} \supset (0, 1/2]$.

n' is not continuous with respect Banach-Mazur distance

Example

For $1 < p < \infty$, let $X_p = \ell_p^2 \oplus_p \ell_2^2$ (observe that $n(X_p) = 0$ for every p).

- Then $n'(X_p) \leqslant n'(\ell_p^2) = n(\ell_p^2) \quad$ for $p \neq 2$
- Therefore $\lim\limits_{p\to 2} n'(X_p) \leqslant \lim\limits_{p\to 2} n(\ell_p) = 0$
- On the other hand $n'(X_2) = n'(\ell_2^4) = 1$

Another example

For $1 < p < \infty$, let $X_p = \ell_p^2 \oplus_1 \ell_2^2$ (observe that $n(X_p) = 0$ for every p).

- Then $n'(X_p) \leqslant n'(\ell_p^2) = n(\ell_p^2) \quad$ for $p \neq 2$
- Therefore $\lim\limits_{p\to 2} n'(X_p) \leqslant \lim\limits_{p\to 2} n(\ell_p) = 0$
- On the other hand $\frac{1}{2} \leqslant n'(X_2) < 1$

Observation

Continuity of $n'(\cdot)$ holds if $\mathcal{Z}(X)$ does not change

Spaces with absolute norm and $n'(X) = 1$

Theorem

Let X be $\mathbb{R}^m$ endowed with an absolute norm. Suppose that $n(X) = 0$ and $n'(X) = 1$. Then, X is a Hilbert space.

Observation

The result is more general and it can be extended to Banach spaces with (long) one-unconditional basis.

Vector valued spaces

Proposition

Let X be a Banach space, L locally compact Hausdorff, K compact Hausdorff, Ω completely regular Hausdorff, and μ positive measure. Then

- $n'\big(C_0(L,X)\big) \leqslant n'(X)$
- $n'\big(C_w(K,X)\big) \leqslant n'(X)$
- $n'\big(C_b(\Omega,X)\big) \leqslant n'(X)$
- $n'\big(L_1(\mu,X)\big) \leqslant n'(X)$
- $n'\big(L_\infty(\mu,X)\big) \leqslant n'(X)$

Example

Let K be a compact Hausdorff topological space with at least two points. Then

$$n'(C(K,\ell_2^2)) \leqslant \frac{\sqrt{3}}{2} < 1.$$

Duality

Observation

Let X be a Banach space. If every element in $\mathcal{Z}(X^*)$ is the transpose of an element in $\mathcal{Z}(X)$ then $n'(X^*) \leqslant n'(X)$

Proposition

Suppose that one of the following holds

- The norm of X^* is Fréchet-smooth on a dense set (e.g. $X = \ell_\infty$);
- B_X is the closed convex hull of the $w - \|\cdot\|$ continuity points of Id (in particular, X RNP, X CPCP, X LUR, X has a Kadec norm, $X = X_1 \widetilde{\otimes}_\pi X_2$ where X_1, X_2 RNP, or $X = L(R)$ where R is reflexive);
- $X^* \not\supseteq \ell_1$;
- X is isomorphic to a subspace of a separable L-embedded space;
- X is the (unique) predual of a von Neumann algebra.

Then $n'(X^*) \leqslant n'(X)$

Duality II

On the other hand,

Example

Given $0 \leqslant \alpha \leqslant \beta \leqslant 1/2$, there is a Banach space $X_{\alpha,\beta}$ with $n(X_{\alpha,\beta}) = 0$ such that

$$n'(X_{\alpha,\beta}) = \beta \quad \text{and} \quad n'(X_{\alpha,\beta}^*) = \alpha.$$

An application

Definition (Guirao-Kozhushkina, 2013; Kim-Lee-Martín,2014)

X Banach space.

- X has the Bishop-Phelps-Bollobás property for numerical radius if for every $0 < \varepsilon < 1$, there is $\eta(\varepsilon) > 0$ such that whenever $T \in L(X)$ and $(x, x^*) \in \Pi(X)$ satisfy $v(T) = 1$ and $|x^*Tx| > 1 - \eta(\varepsilon)$, there exist $S \in L(X)$ and $(y, y^*) \in \Pi(X)$ such that

$$v(S) = |y^*Sy| = 1, \quad \|T - S\| < \varepsilon, \quad \|x - y\| < \varepsilon, \quad \text{and} \quad \|x^* - y^*\| < \varepsilon.$$

- X has the weak-Bishop-Phelps-Bollobás property for numerical radius if for every $0 < \varepsilon < 1$, there is $\eta(\varepsilon) > 0$ such that whenever $T \in L(X)$ and $(x, x^*) \in \Pi(X)$ satisfy $v(T) = 1$ and $|x^*Tx| > 1 - \eta(\varepsilon)$, there exist $S \in L(X)$ and $(y, y^*) \in \Pi(X)$ such that

$$v(S) = |y^*Sy|, \quad \|T - S\| < \varepsilon, \quad \|x - y\| < \varepsilon, \quad \text{and} \quad \|x^* - y^*\| < \varepsilon.$$

An application

Proposition (Kim–Lee–Martín, 2014)

X Banach space with $n(X) > 0$. Then, the weak-Bishop-Phelps-Bollobás property for numerical radius implies the Bishop-Phelps-Bollobás property for numerical radius

Actually...

X Banach space with $n'(X) > 0$. Then, the weak-Bishop-Phelps-Bollobás property for numerical radius implies the Bishop-Phelps-Bollobás property for numerical radius

Proposition (Kim–Lee–Martín, 2014)

X uniformly convex and uniformly smooth $\Longrightarrow$ X has the weak-Bishop-Phelps-Bollobás property for numerical radius

Corollary

(Real) Hilbert spaces have the Bishop-Phelps-Bollobás property for numerical radius

Some open problems on the second numerical index

- Which is the set of values of $n'(X)$ for Banach spaces X with $n(X)=0$? Does it cover the interval $[0,1]$?
  - We know that it covers the interval $[0,1/2]$ and contains 1. It can be done (except for the value cero) with four-dimensional spaces.
  - If $\dim(X)=2$ and $n(X)=0$ then $X=\ell_2^2$.
  - If $\dim(X)=3$ and $n(X)=0$ then $X=\ell_2^2\oplus_a\mathbb{R}$. In this case we know that it is NOT an interval.
- Is $n'(X\oplus_2 Y)\leqslant\min\{n'(Y),n'(W)\}$?
- Let μ be a positive measure, X a Banach space and $1<p<\infty$. Is it true that $n'(L_p(\mu,X))\leqslant n'(X)$?
- Is $n'(X^*)\leqslant n'(X)$ for every Banach space X?
- Are Hilbert spaces the unique Banach spaces X with $n(X)=0$ and $n'(X)=1$?
- X complex, what is the meaning of $n'(X_{\mathbb{R}})$?
- $X=\mathbb{C}\oplus_a\mathbb{C}$. What is the value of $n'(X_{\mathbb{R}})$?
  - $\oplus_a=\oplus_2 \implies n'(X_{\mathbb{R}})=1$,
  - $\oplus_a=\oplus_1 \implies \frac{1}{2}\leqslant n'(X_{\mathbb{R}})\leqslant\frac{\sqrt{3}}{2}$.