Strongly norm attaining Lipschitz maps

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Contents

Preliminaries

Some notation

X, Y real Banach spaces

B_X closed unit ball

S_X unit sphere

X^* topological dual

$\mathcal{L}(X, Y)$ Banach space of all bounded linear operators from X to Y

$\mathcal{L}(X)$ Banach algebra of all bounded linear operators from X to X

Main definition and leading problem

Lipschitz function

M, N (complete) metric spaces. A map $F\colon M \longrightarrow N$ is Lipschitz if there exists a constant $k > 0$ such that

$$d(F(p), F(q)) \leqslant k\, d(p, q) \quad \forall\, p, q \in M$$

The least constant so that the above inequality works is called the Lipschitz constant of F, denoted by $L(F)$:

$$L(F) = \sup \left\{ \frac{d(F(p), F(q))}{d(p, q)} : p \neq q \in M \right\}$$

- If $N = Y$ is a normed space, then $L(\cdot)$ is a seminorm in the vector space of all Lipschitz maps from M into Y.
- F attain its Lipschitz number if the supremum defining it is actually a maximum.

Leading problem

Let M be a metric space, let Y be a Banach and let $F\colon M \longrightarrow Y$ be a Lipschitz map. Can F be approximated by Lipschitz functions from M to Y which attain their Lipschitz number?

First examples

Finite sets

If M is finite, obviously every Lipschitz map attains its Lipschitz number.

★ This characterizes finiteness of M.

Example (Kadets–Martín–Soloviova, 2016)

$M = [0,1]$, $A \subseteq [0,1]$ closed with empty interior and positive Lebesgue measure. Then, the Lipschitz function $f\colon [0,1] \longrightarrow \mathbb{R}$ given by

$$f(t) = \int_0^t \chi_A(s)\,ds,$$

cannot be approximated by Lipschitz functions which attain their Lipschitz number.

Objective

To extend those results (to more interesting ones).

More definitions

Pointed metric space

M is *pointed* if it carries a distinguished element called base point.

Space of Lipschitz maps

M pointed metric space, Y Banach space.
$\mathrm{Lip}_0(M,Y)$ is the Banach space of all Lipschitz maps from M to Y which are zero at the base point, endowed with the Lipschitz number as norm.

Strongly norm attaining Lipschitz map

M pointed metric space. $F \in \mathrm{Lip}_0(M,Y)$ strongly attains its norm, writing $F \in \mathrm{SNA}(M,Y)$, if there exist $p \neq q \in M$ such that

$$L(F) = \|F\| = \frac{\|F(p) - F(q)\|}{d(p,q)}.$$

Our objective is then

to study when $\mathrm{SNA}(M,Y)$ is norm dense in the Banach space $\mathrm{Lip}_0(M,Y)$

Some more definitions

Evaluation functional, Lipschitz-free space, molecule

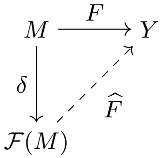
M pointed metric space.

- $p \in M$, $\delta_p \in \mathrm{Lip}_0(M,\mathbb{R})^*$ given by $\delta_p(f) = f(p)$ is the evaluation functional at p;
- $\mathcal{F}(M) := \overline{\mathrm{span}}\{\delta_p : p \in M\} \subseteq \mathrm{Lip}_0(M,\mathbb{R})^*$ is the Lipschitz-free space of M;
- For $p \neq q \in M$, $m_{p,q} := \frac{\delta_p - \delta_q}{d(p,q)} \in \mathcal{F}(M)$ is a molecule;
- $\mathrm{Mol}\,(M) := \{m_{p,q} : p, q \in M,\ p \neq q\}$.
- $B_{\mathcal{F}(M)} = \overline{\mathrm{conv}}(\mathrm{Mol}\,(M))$.

Very important property (Arens-Eells, Kadets, Godefroy-Kalton, Weaver...)

M pointed metric space.

- $\delta : M \rightsquigarrow \mathcal{F}(M)$, $p \longmapsto \delta_p$, is an isometric embedding;
- $\mathcal{F}(M)^* \cong \mathrm{Lip}_0(M,\mathbb{R})$;
- Actually, Y Banach space, $F \in \mathrm{Lip}_0(M,Y)$, $\exists$ (a unique) $\widehat{F} \in \mathcal{L}(\mathcal{F}(M),Y)$ such that $F = \widehat{F} \circ \delta$, and so $\|\widehat{F}\| = \|F\|$.



★ In particular, $\mathrm{Lip}_0(M,Y) \cong \mathcal{L}(\mathcal{F}(M),Y)$.

Two ways of attaining the norm

We have two ways of attaining the norm

M pointed metric space, Y Banach space, $F \in \mathrm{Lip}_0(M,Y) \cong \mathcal{L}(\mathcal{F}(M),Y)$.

- $\widehat{F} \in \mathrm{NA}(\mathcal{F}(M),Y)$ if exists $\xi \in B_{\mathcal{F}(M)}$ such that $\|F\| = \|\widehat{F}\| = \|\widehat{F}(\xi)\|$;
- $F \in \mathrm{SNA}(M,Y)$ if exists $m_{p,q} \in \mathrm{Mol}\,(M)$ such that

$$\|F\| = \|\widehat{F}\| = \|\widehat{F}(m_{p,q})\| = \frac{\|F(p)-F(q)\|}{d(p,q)}.$$

Clearly, $\mathrm{SNA}(M,Y) \subseteq \mathrm{NA}(\mathcal{F}(M),Y)$.

- Therefore, if $\mathrm{SNA}(M,Y)$ is dense in $\mathrm{Lip}_0(M,Y)$, then $\mathrm{NA}(\mathcal{F}(M),Y)$ is dense in $\mathcal{L}(\mathcal{F}(M),Y)$;
- But the opposite direction is NOT true:

Example

- $\overline{\mathrm{NA}(\mathcal{F}(M),\mathbb{R})} = \mathcal{L}(\mathcal{F}(M),\mathbb{R})$ for every M by the Bishop–Phelps theorem,
- But $\overline{\mathrm{SNA}([0,1],\mathbb{R})} \neq \mathrm{Lip}_0([0,1],\mathbb{R})$.

A little of geometry of the unit ball of $\mathcal{F}(M)$ (A–G–GL–P–P–R–W)

Preserved extreme point

$\xi \in B_{\mathcal{F}(M)}$, TFAE:

- ξ is extreme in $B_{\mathcal{F}(M)^{**}}$,
- ξ is a denting point,
- $\xi = m_{p,q}$ and for every $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t. $d(p,t) + d(t,q) - d(p,q) > \delta$ when $d(p,t), d(t,q) \geqslant \varepsilon$.

★ M boundedly compact, it is equivalent to:

- $d(p,q) < d(p,t) + d(t,q)$ $\forall t \notin \{p,q\}$.

Strongly exposed point

$\xi \in B_{\mathcal{F}(M)}$, TFAE:

- ξ strongly exposed point,
- $\xi = m_{p,q}$ and $\exists$ $\rho = \rho(p,q) > 0$ such that
$$\frac{d(p,t) + d(t,q) - d(p,q)}{\min\{d(p,t), d(t,q)\}} \geqslant \rho$$
when $t \notin \{p,q\}$.

Concave metric space

M is concave if $m_{p,q}$ is a preserved extreme point for all $p \neq q$.

★ Examples: $y = x^3$, S_X if X unif. convex…

Uniform Gromov rotundity

$\mathcal{M} \subset \mathrm{Mol}\,(M)$ is uniformly Gromov rotund if $\exists \rho_0 > 0$ such that
$$\frac{d(p,t) + d(t,q) - d(p,q)}{\min\{d(p,t), d(t,q)\}} \geqslant \rho_0$$
when $m_{p,q} \in \mathcal{M}$, $t \notin \{p,q\}$.

$\Longleftrightarrow$ M is a set of uniformly strongly exposed points (same relation $\varepsilon - \delta$)

★ $\mathrm{Mol}\,(M)$ uniformly Gromov rotund when:

- $M = ([0,1], |\cdot|^{\theta})$,
- M finite and concave,
- $1 \leqslant d(p,q) \leqslant D < 2$ $\forall p,q \in M$, $p \neq q$.

★ $\mathbb{T}$ is concave but $\mathrm{Mol}\,(\mathbb{T})$ not u. Gromov r.

Negative results

Negative results

Previous result (Kadets–Martín–Soloviova, 2016)

If M is metrically convex (or "geodesic"), then $\mathrm{SNA}(M,\mathbb{R})$ is not dense in $\mathrm{Lip}_0(M,\mathbb{R})$.

Definition (length space)

Let M be a metric space. We say that M is length if $d(p,q)$ is equal to the infimum of the length of the rectifiable curves joining p and q for every pair of points $p,q \in M$.

★ Equivalently (Avilés, García, Ivankhno, Kadets, Martínez, Prochazka, Rueda, Werner)

- M is local (i.e. the Lipschitz constant of every function can be approximated in pairs of arbitrarily closed points);
- The unit ball of $\mathcal{F}(M)$ has no strongly exposed points;
- $\mathrm{Lip}_0(M,\mathbb{R})$ (and so $\mathcal{F}(M)$) has the Daugavet property.

Theorem

Let M be a length pointed metric space. Then,

$$\overline{\mathrm{SNA}(M,\mathbb{R})} \neq \mathrm{Lip}_0(M,\mathbb{R})$$

Other type of negative results

Observation

All the previous examples of M's such that $\mathrm{SNA}(M, \mathbb{R})$ is not dense in $\mathrm{Lip}_0(M, \mathbb{R})$ are arc-connected metric spaces and "almost convex".

Let's present two different kind of examples:

Example

M "fat" Cantor set, then $\overline{\mathrm{SNA}(M, \mathbb{R})} \neq \mathrm{Lip}_0(M, \mathbb{R})$ and M is totally disconnected.

Example

$M = \mathbb{T}$, then $\overline{\mathrm{SNA}(M, \mathbb{R})} \neq \mathrm{Lip}_0(M, \mathbb{R})$.

Positive results

Possible sufficient conditions

Observation (previously commented)

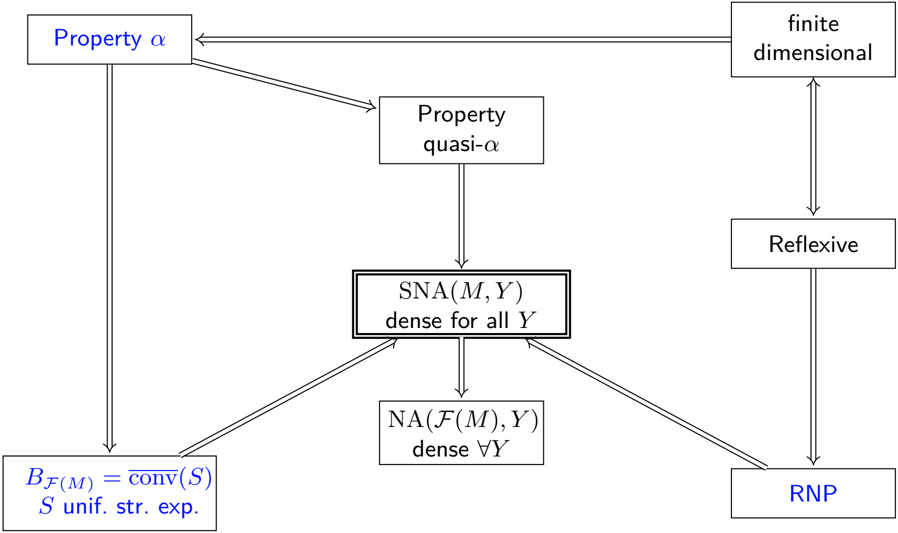
$\mathrm{SNA}(M,Y)$ dense in $\mathrm{Lip}_0(M,Y) \implies \mathrm{NA}(\mathcal{F}(M),Y)$ dense in $\mathcal{L}(\mathcal{F}(M),Y)$.

Therefore, it is reasonable to discuss the known sufficient conditions for a Banach space X to have $\overline{\mathrm{NA}(X,Y)} = \mathcal{L}(X,Y)$ for every Y:

- RNP,
- Property α,
- Property quasi-α,
- the existence of a norming set of uniformly strongly exposed points.

In the next slice we will relate all these properties for Lipschitz-free spaces:

Sufficient conditions for the density of $\mathrm{SNA}(M, Y)$ for every Y: relations



The RNP

Theorem (García-Lirola–Petitjean–Procházka–Rueda-Zoca, 2018)

Let M be a pointed metric space. Assume that $\mathcal{F}(M)$ has the RNP. Then,

$$\overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y) \quad \text{for every Banach space } Y$$

Proof

- Bourgain, 1977:
  X RNP $\implies \{T \in \mathcal{L}(X,Y) \colon T$ strongly exposes $B_X\}$ is dense in $\mathcal{L}(X,Y)$;
- T strongly exposing operator, then T attains its norm at a strongly exposed point;
- Weaver, 1999: strongly exposed points of $B_{\mathcal{F}(M)}$ are molecules.

$\mathcal{F}(M)$ has the RNP when...

- $M = (N, d^\theta)$ for (N,d) boundedly compact and $0 < \theta < 1$ (Weaver, 1999 - 2018);
- M is uniformly discrete (Kalton, 2004);
- M is countable and compact (Dalet, 2015);
- $M \subset \mathbb{R}$ with Lebesgue measure 0 (Godard, 2010).

Property alpha

Property alpha

X Banach space. X has property α if there exist a balanced subset $\{x_\lambda\}_{\lambda\in\Lambda} \subseteq X$ and a subset $\{x_\lambda^*\}_{\lambda\in\Lambda} \subseteq X^*$ such that

- $\|x_\lambda\| = \|x_\lambda^*\| = |x_\lambda^*(x_\lambda)| = 1 \quad \forall\, \lambda \in \Lambda$;
- There exists $0 \leqslant \rho < 1$ such that $|x_\lambda^*(x_\mu)| \leqslant \rho \quad \forall\, x_\mu \neq \pm x_\lambda$;
- $\overline{\mathrm{co}}(\{x_\lambda\}_{\lambda\in\Lambda}) = B_X$.

- Introduced by Schachermayer in 1983 as a sufficient condition for X to get $\overline{\mathrm{NA}(X,Y)} = \mathcal{L}(X,Y)$ for every Y;
- Every separable Banach space X can be renormed with property α;
- (Godun–Troyanski, 1993): this result extends to Banach spaces with long biorthogonal systems.
- (Schachermayer, 1983): If X has property α, then

$$\{T \in \mathcal{L}(X,Y) \colon T \text{ attains its norm at one } x_k \}$$

is dense in $\mathcal{L}(X,Y)$ for every Y.

Property alpha and density of $\mathrm{SNA}(M,Y)$

Theorem

M metric space such that $\mathcal{F}(M)$ has property α. Then,

$$\overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y) \quad \text{for every Banach space } Y.$$

Examples of M's such that $\mathcal{F}(M)$ has property alpha

- M finite,
- $M \subset \mathbb{R}$ with Lebesgue measure 0,
- $1 \leqslant d(p,q) \leqslant D < 2$ for all $p,q \in M$, $p \neq q$.

Characterization in the case of concave metric spaces

M concave metric space. TFAE:

- $\mathcal{F}(M)$ has property α.
- M is uniformly discrete, bounded, and $\mathrm{Mol}\,(M)$ is uniformly Gromov rotund.

A norming uniformly Gromov rotund set of molecules

Theorem

M pointed metric space, $A \subset \mathrm{Mol}\,(M)$ uniformly Gromov rotund (i.e. A is a set of uniformly strongly exposed points) such that $\overline{\mathrm{co}}(A) = B_{\mathcal{F}(M)}$.

$$\Longrightarrow \quad \overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y) \quad \text{for every Banach space } Y.$$

Examples

- $\mathcal{F}(M)$ with property α (with $A = \{\pm x_\lambda \colon \lambda \in \Lambda\}$);
- $M = ([0,1], |\cdot|^\theta)$ (with $A = \mathrm{Mol}\,(M)$). This one does not have property α.

Particular case (uniformly Gromov concave metric spaces)

M pointed metric space. Suppose that

$$\frac{d(p,t)+d(t,q)-d(p,q)}{\min\{d(p,t),d(t,q)\}} \geqslant \rho_0 > 0 \qquad \forall p \neq q \neq t.$$

Then, $\overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y)$ for every Banach space Y.

★ We will see that something stronger happens.

A further example

An important example

There exists M such that $\mathrm{SNA}(M, Y)$ is dense in $\mathrm{Lip}_0(M, Y)$ for every Y, but $\mathcal{F}(M)$ does not have the RNP, does not contain a norming uniformly Gromov rotund set of molecules, and does not has property α.

Indeed, write $A_0 = [0,1] \times \{0\} \subset \mathbb{R}^2$ and for each $n \in \mathbb{N}$, write

$$A_n = \left\{ \left(\frac{k}{2^n}, \frac{1}{2^n} \right) : k \in \{0, \ldots, 2^n\} \right\} \subseteq \mathbb{R}^2.$$

Then, the desired metric space is $M = \bigcup_{n \in \mathbb{N} \cup \{0\}} A_n$, where the metric of M comes from the Euclidean metric.

A modification

Considering in M above the metric coming from the ℓ_1 metric of $\mathbb{R}^2$, then $\mathcal{F}(M)$ has property α.

★ Compare with the case of M being concave.

Further results

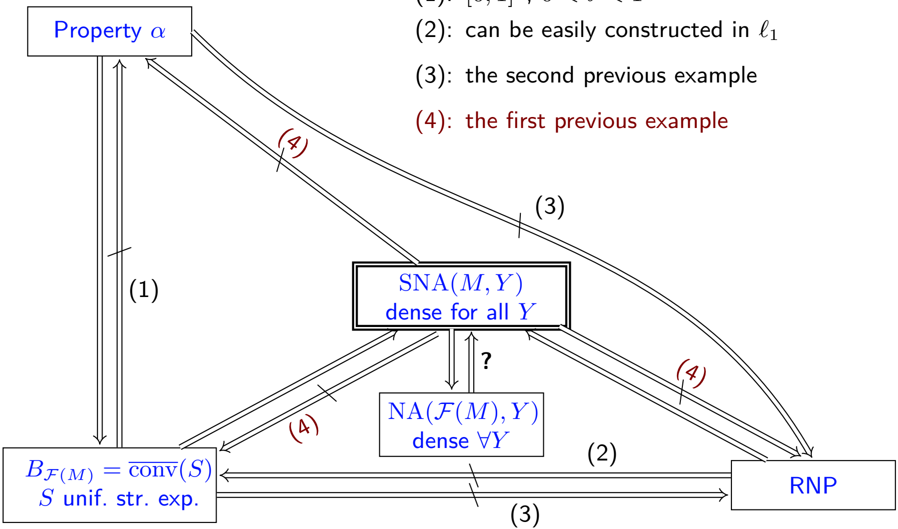
Summarizing the relations

(1): $[0,1]^\theta$, $0 < \theta < 1$

(2): can be easily constructed in ℓ_1

(3): the second previous example

(4): the first previous example



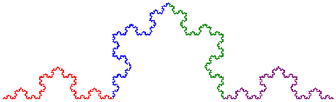
Two paradigmatic examples

Koch curve

Let $M_1 = ([0,1], |\cdot|^\theta)$, $0 < \theta < 1$.

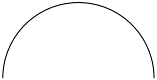
- $\mathcal{F}(M_1)$ has RNP, so $\overline{\mathrm{SNA}(M_1, Y)} = \mathrm{Lip}_0(M_1, Y)\ \forall Y$.
- Every molecule is strongly exposed,
- even more, $\mathrm{Mol}\,(M_1)$ is uniformly Gromov rotund.

★ For $\theta = \log(3)/\log(4)$, M_1 is bi-Lipschitz equivalent to the Koch curve:



The unit circle

Let M_2 be the upper half of the unit circle:



- We know that $\mathrm{SNA}(M_2, \mathbb{R})$ is not dense in $\mathrm{Lip}_0(M_2, \mathbb{R})$.
- So, $\mathcal{F}(M_2)$ has NOT the RNP.
- However, every molecule is strongly exposed...
- but NO subset $A \subset \mathrm{Mol}\,(M_2)$ which is uniformly Gromov rotund can be norming for $\mathrm{Lip}_0(M_2, \mathbb{R})$.

Microscopically, an small piece of M_1 is equivalent to M_1 itself.

Microscopically, an small piece of M_2 is very closed to be an interval.

From scalar-valued to vector-valued and viceversa

From vector-valued to scalar-valued

M metric space, $\mathrm{SNA}(M,Y)$ dense in $\mathrm{Lip}_0(M,Y)$ for some Y
$\implies$ $\mathrm{SNA}(M,\mathbb{R})$ dense in $\mathrm{Lip}_0(M,\mathbb{R})$

★ We do not know if the density for scalar functions implies the density for all vector-valued maps, but there are some cases in which this happens:

From scalar-valued to vector-valued

M metric space such that $\overline{\mathrm{SNA}(M,\mathbb{R})} = \mathrm{Lip}_0(M,\mathbb{R})$, Y Banach space.

- If Y has property β (e.g. $c_0 \leqslant Y \leqslant \ell_\infty$), then $\overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y)$.
- For compact Lipschitz maps, the same is true for $Y = C(K)$.

★ These results are proved using the concepts of ACK_ρ-spaces and Γ-flat operators from Cascales–Guirao–Kadets–Soloviova, 2018.

The strongly Lipschitz BPB property

The strongly Lipschitz BPB property: definition

The strongly Lipchitz Bishop-Phelps-Bollobás

M metric space, Y Banach space. (M,Y) has the Lip-BPBp if for every $\varepsilon>0$ there is $\eta>0$ such that for $F_0\in \mathrm{Lip}_0(M,Y)$ with $\|F_0\|=1$, $p\neq q\in M$ s.t.

$$\frac{\|F_0(p)-F_0(q)\|}{d(p,q)}>1-\eta,$$

there exist $F\in \mathrm{Lip}_0(M,Y)$ and $x\neq y\in M$ such that $1=\|F\|=\frac{\|F(x)-F(y)\|}{d(x,y)}$ and

$$\|F_0-F\|<\varepsilon \quad \text{and} \quad \frac{d(p,x)+d(q,y)}{d(p,q)}<\varepsilon \;\; (\text{or } \|m_{p,q}-m_{x,y}\|<\varepsilon).$$

★ It is the Lipschitz version of the so-called BPBp for linear operators:

The BPB property for linear operators (Acosta–Aron–García–Maestre, 2008)

X, Y Banach spaces. (X,Y) has the BPBp if for every $\varepsilon>0$ there is $\eta>0$ such that whenever $T\in\mathcal{L}(X,Y)$, $\|T\|=1$, $x\in S_X$ satisfy $\|Tx\|>1-\eta$, there exist $S\in\mathcal{L}(X,Y)$, $y\in S_X$ verifying that

$$1=\|S\|=\|Sy\| \qquad \text{and} \qquad \|x-y\|,\ \|T-S\|<\varepsilon.$$

Preliminary examples: finite metric spaces

Finite metric spaces

M finite, if $(\mathcal{F}(M), Y)$ has the (linear) BPBp
$\implies$ (M, Y) has the Lip-BPBp.

Positive example

M finite, $\dim(Y) < \infty \implies (M, Y)$ has the Lip-BPBp.

Negative examples

- Exists Y (infinite-dimensional) such that $(\{0, 1, 2\}, Y)$ fails Lip-BPBp.
- $M = \mathbb{N}$ with the distance inherited from $\mathbb{R}$, then $(M, \mathbb{R})$ fails Lip-BPBp.

The main result

Positive result (uniformly Gromov concave metric spaces)

$\operatorname{Mol}(M)$ uniformly Gromov rotund, Y arbitrary $\implies$ (M, Y) has Lip-BPBp.

Some example of uniformly Gromov concave metric spaces

- $M = [0,1]^\theta$ for $0 < \theta < 1$,
- M finite and concave,
- $1 \leqslant d(p,q) \leqslant D < 2$ for every $p, q \in M$, $p \neq q$,
- M concave such that $\mathcal{F}(M)$ has property α.

A family of new examples

(M, d) arbitrary metric space, $0 < \theta < 1$ $\implies$ (M, d^θ) is u. Gromov concave.

Remark

- In particular, $\operatorname{SNA}\big((M, d^\theta), Y\big)$ is always dense,
- It is not known whether $\mathcal{F}(M, d^\theta)$ always has the RNP.

Weak density

Weak density

Theorem

M metric space $\implies \mathrm{SNA}(M,\mathbb{R})$ is weakly sequentially dense in $\mathrm{Lip}_0(M,\mathbb{R})$.

Previously known

- $\mathcal{F}(M)$ RNP;
- Kadets–Martín–Soloviova, 2016: when M is length.

The tool

$\{f_n\} \subset \mathrm{Lip}_0(M,\mathbb{R})$ bounded with pairwise disjoint supports $\implies \{f_n\}$ weakly null.

Observations

- The linear span of $\mathrm{SNA}(M,\mathbb{R})$ is always norm-dense in $\mathrm{Lip}_0(M,\mathbb{R})$;
- $\mathcal{F}(M)$ RNP $\implies \mathrm{Lip}_0(M,\mathbb{R}) = \mathrm{SNA}(M,\mathbb{R}) - \mathrm{SNA}(M,\mathbb{R})$.

A by-product of our construction

Theorem

If M' is infinite or M is discrete but no uniformly discrete or M is compact (infinite) $\Longrightarrow$ then the norm of $\mathcal{F}(M)^{**}$ is octahedral.

Octahedral norm

The norm of X is octahedral iff $\forall\, Y \leqslant X$ finite-dimensional, $\forall \varepsilon > 0$, $\exists x \in S_X$ s.t.

$$\|y + \lambda x\| \geqslant (1-\varepsilon)\big(\|y\| + |\lambda|\big) \qquad \big(y \in Y,\ \lambda \in \mathbb{R}\big).$$

Equivalently

If M' is infinite or M is discrete but no uniformly discrete or M is compact (infinite) $\Longrightarrow$ every convex combination of slices of $B_{\mathrm{Lip}_0(M,\mathbb{R})}$ has diameter two.

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