Strongly norm attaining Lipschitz maps

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Contents

1 Preliminaries
2 A compilation of negative and positive results
3 Two new examples
4 References

B. Cascales, R. Chiclana, L. García-Lirola, M. Martín, A. Rueda
On strongly norm attaining Lipschitz maps
J. Funct. Anal. (2019)

R. Chiclana, L. García-Lirola, M. Martín, A. Rueda
Examples and applications of the density of strongly norm attaining Lipschitz maps
Preprint (2019)

R. Chiclana, M. Martín
The Bishop-Phelps-Bollobás property for Lipschitz maps
Nonlinear Analysis (2019)

Preliminaries

Some notation

X, Y real Banach spaces

B_X closed unit ball

S_X unit sphere

X^* topological dual

$\mathcal{L}(X, Y)$ Banach space of all bounded linear operators from X to Y

Main definition and leading problem

Lipschitz function

M, N (complete) metric spaces. A map $F\colon M \longrightarrow N$ is Lipschitz if there exists a constant $k > 0$ such that

$$d(F(p), F(q)) \leqslant k\, d(p, q) \quad \forall\, p, q \in M$$

The least constant so that the above inequality works is called the Lipschitz constant of F, denoted by $L(F)$:

$$L(F) = \sup \left\{ \frac{d(F(p), F(q))}{d(p, q)} : p \neq q \in M \right\}$$

- If $N = Y$ is a normed space, then $L(\cdot)$ is a seminorm in the vector space of all Lipschitz maps from M into Y.
- F attain its Lipschitz number if the supremum defining it is actually a maximum.

Leading problem (Godefroy, 2015)

Study the metric spaces M and the Banach spaces Y such that the set of Lipschitz maps which attain their Lipschitz number is dense in the set of all Lipschitz maps.

More definitions

Pointed metric space

M is *pointed* if it carries a distinguished element called base point.

Space of Lipschitz maps

M pointed metric space, Y Banach space.
$\mathrm{Lip}_0(M,Y)$ is the Banach space of all Lipschitz maps from M to Y which are zero at the base point, endowed with the Lipschitz number as norm.

Strongly norm attaining Lipschitz map

M pointed metric space. $F \in \mathrm{Lip}_0(M,Y)$ strongly attains its norm, writing $F \in \mathrm{SNA}(M,Y)$, if there exist $p \neq q \in M$ such that

$$L(F) = \|F\| = \frac{\|F(p) - F(q)\|}{d(p,q)}.$$

Our objective is then

to study when $\mathrm{SNA}(M,Y)$ is norm dense in the Banach space $\mathrm{Lip}_0(M,Y)$

First examples

Finite sets

If M is finite, obviously every Lipschitz map attains its Lipschitz number.
★ This characterizes finiteness of M.

Example (Kadets–Martín–Soloviova, 2016)

$M = [0,1]$, $A \subseteq [0,1]$ closed with empty interior and positive Lebesgue measure. Then, the Lipschitz function $f\colon [0,1] \longrightarrow \mathbb{R}$ given by

$$f(t) = \int_0^t \chi_A(s)\, ds,$$

cannot be approximated by Lipschitz functions which attain their Lipschitz number.

Example (Godefroy, 2015)

M compact, $\mathrm{lip}_0(M)$ strongly separates M (e.g. M usual Cantor set, $M = [0,1]^\theta$)
$\Longrightarrow$ $\mathrm{SNA}(M,Y)$ dense in $\mathrm{Lip}_0(M,Y)$ for every finite-dimensional Y.

Some more definitions

Evaluation functional, Lipschitz-free space, molecule

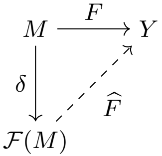
M pointed metric space.

- $p \in M$, $\delta_p \in \mathrm{Lip}_0(M,\mathbb{R})^*$ given by $\delta_p(f) = f(p)$ is the evaluation functional at p;
- $\mathcal{F}(M) := \overline{\mathrm{span}}\{\delta_p : p \in M\} \subseteq \mathrm{Lip}_0(M,\mathbb{R})^*$ is the Lipschitz-free space of M;
- $\delta : M \rightsquigarrow \mathcal{F}(M)$, $p \longmapsto \delta_p$, is an isometric embedding;
- For $p \neq q \in M$, $m_{p,q} := \frac{\delta_p - \delta_q}{d(p,q)} \in \mathcal{F}(M)$ is a molecule, $\|m_{p,q}\| = 1$;
- $\mathrm{Mol}\,(M) := \{m_{p,q} : p,q \in M,\ p \neq q\}$.
- $B_{\mathcal{F}(M)} = \overline{\mathrm{conv}}(\mathrm{Mol}\,(M))$.

Very important property (Arens-Eells, Kadets, Godefroy-Kalton, Weaver…)

M pointed metric space.

- $\mathcal{F}(M)^* \cong \mathrm{Lip}_0(M,\mathbb{R})$;
- Actually, Y Banach space, $F \in \mathrm{Lip}_0(M,Y)$,
  $\exists$ (a unique) $\widehat{F} \in \mathcal{L}(\mathcal{F}(M),Y)$ such that $F = \widehat{F} \circ \delta$,
  and so $\|\widehat{F}\| = \|F\|$.



★ In particular, $\mathrm{Lip}_0(M,Y) \cong \mathcal{L}(\mathcal{F}(M),Y)$.

Two ways of attaining the norm

We have two ways of attaining the norm

M pointed metric space, Y Banach space, $F \in \mathrm{Lip}_0(M, Y) \cong \mathcal{L}(\mathcal{F}(M), Y)$.

- $\widehat{F} \in \mathrm{NA}(\mathcal{F}(M), Y)$ if exists $\xi \in B_{\mathcal{F}(M)}$ such that $\|F\| = \|\widehat{F}\| = \|\widehat{F}(\xi)\|$;
- $F \in \mathrm{SNA}(M, Y)$ if exists $m_{p,q} \in \mathrm{Mol}\,(M)$ such that

$$\|F\| = \|\widehat{F}\| = \|\widehat{F}(m_{p,q})\| = \frac{\|F(p) - F(q)\|}{d(p,q)}.$$

Clearly, $\mathrm{SNA}(M, Y) \subseteq \mathrm{NA}(\mathcal{F}(M), Y)$.

- Therefore, if $\mathrm{SNA}(M, Y)$ is dense in $\mathrm{Lip}_0(M, Y)$, then $\mathrm{NA}(\mathcal{F}(M), Y)$ is dense in $\mathcal{L}(\mathcal{F}(M), Y)$;
- But the opposite direction is NOT true:

Example

- $\overline{\mathrm{NA}(\mathcal{F}(M), \mathbb{R})} = \mathcal{L}(\mathcal{F}(M), \mathbb{R})$ for every M by the Bishop–Phelps theorem,
- But $\overline{\mathrm{SNA}([0,1], \mathbb{R})} \neq \mathrm{Lip}_0([0,1], \mathbb{R})$.

A little of geometry of the unit ball of $\mathcal{F}(M)$ (A–C–C–G–GL–M–P–P–R–W)

Preserved extreme point

$\xi \in B_{\mathcal{F}(M)}$, TFAE:

- ξ is extreme in $B_{\mathcal{F}(M)^{**}}$,
- ξ is a denting point,
- $\xi = m_{p,q}$ and for every $\varepsilon > 0\ \exists\ \delta > 0$ s.t. $d(p,t) + d(t,q) - d(p,q) > \delta$ when $d(p,t), d(t,q) \geqslant \varepsilon$.

★ M boundedly compact, it is equivalent to:

- $d(p,q) < d(p,t) + d(t,q)\ \forall t \notin \{p,q\}$.

Strongly exposed point

$\xi \in B_{\mathcal{F}(M)}$, TFAE:

- ξ strongly exposed point,
- $\xi = m_{p,q}$ and $\exists\ \rho = \rho(p,q) > 0$ such that

$$\frac{d(p,t) + d(t,q) - d(p,q)}{\min\{d(p,t), d(t,q)\}} \geqslant \rho$$

  when $t \notin \{p,q\}$.

Concave metric space

M is concave if $m_{p,q}$ is a preserved extreme point for all $p \neq q$.

★ Examples: $y = x^3$, S_X if X unif. convex. . .

Uniform Gromov rotundity

$\mathcal{M} \subset \mathrm{Mol}\,(M)$ is uniformly Gromov rotund if $\exists \rho_0 > 0$ such that

$$\frac{d(p,t) + d(t,q) - d(p,q)}{\min\{d(p,t), d(t,q)\}} \geqslant \rho_0$$

when $m_{p,q} \in \mathcal{M}$, $t \notin \{p,q\}$.

$\Longleftrightarrow$ $\mathcal{M}$ is a set of uniformly strongly exposed points (same relation $\varepsilon - \delta$)

★ $\mathrm{Mol}\,(M)$ uniformly Gromov rotund (aka M is uniformly Gromov concave) when:

- $M = ([0,1], |\cdot|^{\theta})$,
- M finite and concave,
- $1 \leqslant d(p,q) \leqslant D < 2\ \forall p,q \in M,\ p \neq q$.

A compilation of negative and positive results

Negative results I

First extension of the case of $[0,1]$ (Kadets–Martín–Soloviova, 2016)

If M is metrically convex (or “geodesic”), then $\mathrm{SNA}(M,\mathbb{R})$ is not dense in $\mathrm{Lip}_0(M,\mathbb{R})$.

Definition (length space)

Let M be a metric space. M is length if $d(p,q)$ is equal to the infimum of the length of the rectifiable curves joining p and q for every pair of points $p,q \in M$.

★ Equivalently (Avilés, García, Ivankhno, Kadets, Martínez, Prochazka, Rueda, Werner)

- M is local (i.e. the Lipschitz constant of every function can be approximated in pairs of arbitrarily closed points);
- The unit ball of $\mathcal{F}(M)$ has no strongly exposed points;
- $\mathrm{Lip}_0(M,\mathbb{R})$ (and so $\mathcal{F}(M)$) has the Daugavet property.

Theorem

M length metric space $\implies \overline{\mathrm{SNA}(M,\mathbb{R})} \neq \mathrm{Lip}_0(M,\mathbb{R})$

Negative results II

A different kind of example

M "fat" Cantor set in $[0,1]$, then $\overline{\mathrm{SNA}(M,\mathbb{R})} \neq \mathrm{Lip}_0(M,\mathbb{R})$ and M is totally disconnected.

Theorem

Actually, if $M \subset \mathbb{R}$ is compact and has positive measure, then $\mathrm{SNA}(M,\mathbb{R})$ is NOT dense in $\mathrm{Lip}_0(M,\mathbb{R})$.

Possible sufficient conditions

Observation (previously commented)

$\mathrm{SNA}(M,Y)$ dense in $\mathrm{Lip}_0(M,Y) \implies \mathrm{NA}(\mathcal{F}(M),Y)$ dense in $\mathcal{L}(\mathcal{F}(M),Y)$.

Therefore, it is reasonable to discuss the known sufficient conditions for a Banach space X to have $\overline{\mathrm{NA}(X,Y)} = \mathcal{L}(X,Y)$ for every Y:

- containing a norming and uniformly strongly exposed set (Lindenstrauss, 1963),
- RNP (Bourgain, 1977),
- Property α (Schachermayer, 1983) ,
- Property quasi-α (Choi–Song, 2008).

Main result

EACH of these properties on $\mathcal{F}(M)$ implies $\overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y)$ for all Y.

How to prove the positive results?

Common scheme of all the proofs

- Consider the original proof of
  "(P) on $\mathcal{F}(M)$ implies that $\overline{\mathrm{NA}(\mathcal{F}(M),Y)} = \mathcal{L}(\mathcal{F}(M),Y)$ for all Y's".
- Check, in each case, what is the set of points where the constructed set of norm attaining operators actually attain their norm.
- It happens that, in all the cases, this set is contained in the set of strongly exposed points or, maybe, in its closure.
- Strongly exposed points are molecules (Weaver, 1999),
  and the set of molecules is norm closed (GL-P-P-RZ, 2018).

Examples of positive results

$\mathcal{F}(M)$ has the RNP when...

- $M = (N, d^\theta)$ for (N, d) boundedly compact and $0 < \theta < 1$ (Weaver, 1999 - 2018);
- M is uniformly discrete (Kalton, 2004);
- M is countable and boundedly compact (Dalet, 2015);
- $M \subset \mathbb{R}$ with Lebesgue measure 0 (Godard, 2010).

$\mathcal{F}(M)$ has property α when...

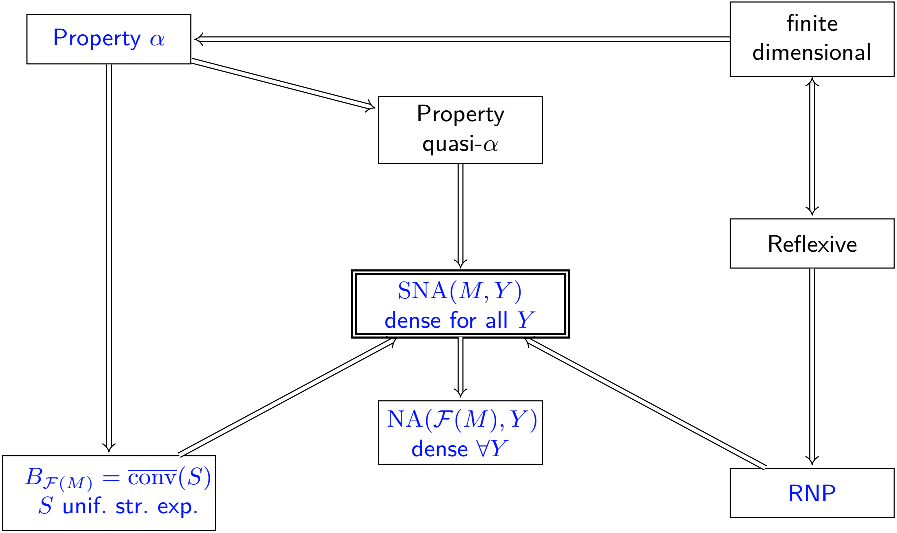
- M finite,
- $M \subset \mathbb{R}$ with Lebesgue measure 0,
- $1 \leqslant d(p, q) \leqslant D < 2$ for all $p, q \in M$, $p \neq q$.

Other example for which $\mathrm{SNA}(M, Y)$ is dense...

- $M = (N, d^\theta)$ for (N, d) arbitrary and $0 < \theta < 1$.

★ In this case, we actually have a “Bishop–Phelps–Bollobás” type result.

Sufficient conditions for the density of $\mathrm{SNA}(M,Y)$ for every Y: relations



Weak density

Theorem

M metric space $\implies$ $\mathrm{SNA}(M,\mathbb{R})$ is weakly sequentially dense in $\mathrm{Lip}_0(M,\mathbb{R})$.

Previously known

- When it is norm dense... (e.g. $\mathcal{F}(M)$ RNP);
- Kadets–Martín–Soloviova, 2016: when M is length.

The tool

$\{f_n\} \subset \mathrm{Lip}_0(M,\mathbb{R})$ bounded with pairwise disjoint supports $\implies$ $\{f_n\}$ weakly null.

Observations

- The linear span of $\mathrm{SNA}(M,\mathbb{R})$ is always norm-dense in $\mathrm{Lip}_0(M,\mathbb{R})$;
- $\mathcal{F}(M)$ RNP $\implies$ $\mathrm{Lip}_0(M,\mathbb{R}) = \mathrm{SNA}(M,\mathbb{R}) - \mathrm{SNA}(M,\mathbb{R})$.

Two new examples

Some questions and an open problem

Some questions

(Q1) Does $\mathcal{F}(M)$ have the RNP if $\overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y)\ \forall Y$?

(Q2) Is $\mathrm{SNA}(M,\mathbb{R})$ dense in $\mathrm{Lip}_0(M,\mathbb{R})$ if $B_{\mathcal{F}(M)} = \overline{\mathrm{co}}\big(\text{str-exp}\left(B_{\mathcal{F}(M)}\right)\big)$?

(Q3) Does $\mathrm{SNA}(M,\mathbb{R})$ fail to be dense in $\mathrm{Lip}_0(M,\mathbb{R})$ if $\mathcal{F}(M)$ contains an isomorphic (or even isometric) copy of $L_1[0,1]$?

Open problem (Godefroy, 2015)

Let M be compact and suppose that $\overline{\mathrm{SNA}(M,\mathbb{R})} = \mathrm{Lip}_0(M,\mathbb{R})$.
Does $\mathrm{lip}_0(M)$ strongly separates M (and, in particular, $\mathcal{F}(M)$ has the RNP)?

In dimension one, everything works...

$M \subset \mathbb{R}$ compact, then the questions and the open problem have positive answer.

but not always

We will see that the questions and the open problem have negative answer in general by using compact subsets of $\mathbb{R}^2$.

The first example: the torus

Theorem

Let $M = \mathbb{T}$ endowed with the distance inherited from the Euclidean plane.

- $\mathrm{SNA}(M, \mathbb{R})$ is not dense in $\mathrm{Lip}_0(M, \mathbb{R})$;
- M is Gromov concave (i.e. every molecule is strongly exposed).

Remarks

- It is a (negative) solution to (Q2);
- $\mathcal{F}(\mathbb{T})$ satisfies that the set of strongly exposed points is norming, but strongly exposing functionals are not dense in the dual.

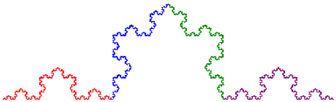
Two different curves

Koch curve

Let $M_1 = ([0,1], |\cdot|^\theta)$, $0 < \theta < 1$.

- $\mathcal{F}(M_1)$ has RNP, so $\overline{\mathrm{SNA}(M_1, Y)} = \mathrm{Lip}_0(M_1, Y)$ $\forall Y$.
- Every molecule is strongly exposed,
- even more, M_1 is uniformly Gromov concave.

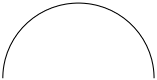
★ For $\theta = \log(3)/\log(4)$, M_1 is bi-Lipschitz equivalent to the Koch curve:



Microscopically, an small piece of M_1 is equivalent to M_1 itself.

The unit circle

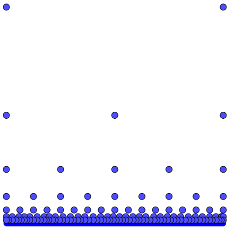
Let M_2 be the upper half of the unit circle:



- We know that $\mathrm{SNA}(M_2, \mathbb{R})$ is not dense in $\mathrm{Lip}_0(M_2, \mathbb{R})$.
- So, $\mathcal{F}(M_2)$ has NOT the RNP.
- However, every molecule is strongly exposed...
- but NO subset $A \subset \mathrm{Mol}\,(M_2)$ which is uniformly Gromov rotund can be norming for $\mathrm{Lip}_0(M_2, \mathbb{R})$.

Microscopically, an small piece of M_2 is very closed to be an interval.

The second example: the "rain"



Consider the compact subset of $\mathbb{R}^2$ given by

$$M := ([0,1] \times \{0\}) \cup \bigcup_{n=0}^{\infty} \left\{ \left(\frac{k}{2^n}, \frac{1}{2^n} \right) : k \in \{0, \dots, 2^n\} \right\}$$

Let $\mathfrak{M}_p$ be the set M endowed with the distance inherited from $(\mathbb{R}^2, \|\cdot\|_p)$ for $p = 1, 2$.

Theorem

$\mathrm{SNA}(\mathfrak{M}_p, Y)$ is dense in $\mathrm{Lip}_0(\mathfrak{M}_p, Y)$ for all Y and $p = 1, 2$. Moreover:

1. $\mathcal{F}(\mathfrak{M}_p)$ fails the RNP (it contains a complemented copy of $L_1[0,1]$!);
2. $\mathcal{F}(\mathfrak{M}_1)$ has property α;
3. $\mathcal{F}(\mathfrak{M}_2)$ does not contain any norming and uniformly strongly exposed set.

Remark

This gives negative answer to both (Q1) and (Q3), and to Godefroy's problem.

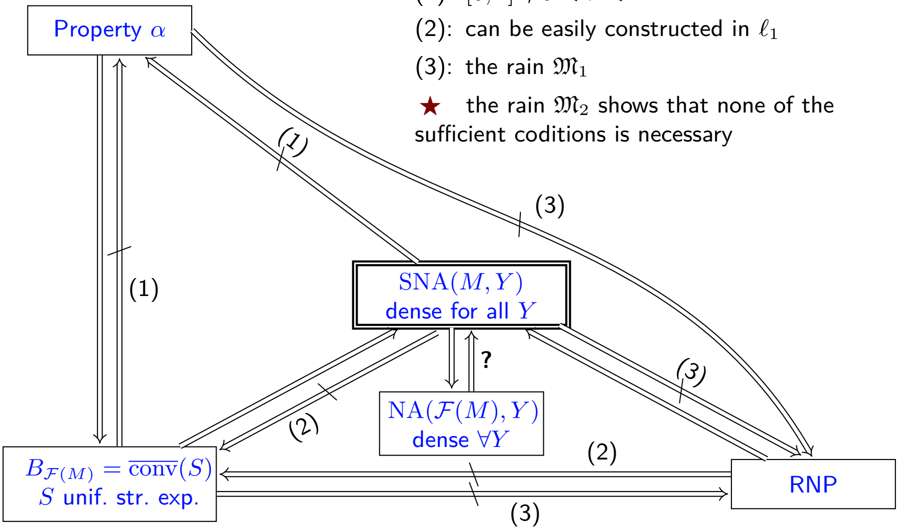
Summarizing the relations

(1): $[0, 1]^\theta$, $0 < \theta < 1$

(2): can be easily constructed in ℓ_1

(3): the rain $\mathfrak{M}_1$

★ the rain $\mathfrak{M}_2$ shows that none of the sufficient coditions is necessary



Some open problems

Two old results

- (Bourgain, 1977; Huff, 1980): X has the RNP iff $\overline{\mathrm{NA}(X',Y)} = \mathcal{L}(X',Y)$ for every Y and every renorming X' of X.
- (Schachermayer, 1983): X separable (WCG) $\implies$ exists X' renorming of X such that $\overline{\mathrm{NA}(X',Y)} = \mathcal{L}(X',Y)$ for every Y.

Open problem 1

M (compact) metric space such that $\overline{\mathrm{SNA}(M',Y)} = \mathrm{Lip}_0(M',Y)$ for every Y and every M' bi-Lipschitz equivalent to M.

★ Does $\mathcal{F}(M)$ have the RNP?

Open problem 2

M (compact) metric space.

★ Does there exists M' bi-Lipschitz equivalent to M such that $\overline{\mathrm{SNA}(M',Y)} = \mathrm{Lip}_0(M',Y)$ for every Y?

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