Vector space structure in the set of norm attaining functionals and proximinality of subspaces of finite-codimension

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Roadmap of the talk

Preliminaries

Section 1

Basic notation

X, Y **real** Banach spaces

B_X closed unit ball

S_X unit sphere

X^* topological dual

$\mathcal{L}(X, Y)$ bounded linear operators from X to Y

$\mathcal{L}(X)$ bounded linear operators from X to X

$\mathcal{K}(X)$ compact linear operators from X to X

Norm attaining functionals

Norm attaining functionals

$x^* \in X^*$ attains its norm when

$$\exists\, x \in X,\ \|x\| = 1 \ :\ x^*(x) = \|x^*\|$$

★ $\mathrm{NA}(X) = \{x^* \in X^* \colon x^*$ attains its norm$\}$

First results

- $\dim(X) < \infty \implies \mathrm{NA}(X) = X^*$ (Heine-Borel),
- X reflexive $\implies \mathrm{NA}(X) = X^*$ (Hahn-Banach),
- X non-reflexive $\implies \mathrm{NA}(X) \neq X^*$ (James),
- $\mathrm{NA}(X)$ is always norm dense in X^* (Bishop-Phelps).

First examples

- $\mathrm{NA}(c_0) = c_{00} \leqslant \ell_1$,
- $\mathrm{NA}(\ell_1) = \Big\{x \in \ell_\infty \colon \|x\|_\infty = \max_n\{|x(n)|\}\Big\}$.

Lineability of $\mathrm{NA}(X)$

Examples

- $\mathrm{NA}(c_0) = c_{00} \leqslant \ell_1$,
- $\mathrm{NA}(\ell_1) = \big\{ x \in \ell_\infty \colon \|x\|_\infty = \max_n \{|x(n)|\} \big\}$.

- Note that $\mathrm{NA}(c_0)$ is a linear space, but $\mathrm{NA}(\ell_1)$ is not.
- However, $\mathrm{NA}(\ell_1)$ contains the infinite-dimensional space c_0.

Lineability

Recall that a subset S of a vector space V is called lineable if $S \cup \{0\}$ contains an infinite-dimensional subspace.

More examples

- $\mathrm{NA}(L_1(\mu))$ is a (dense) linear subspace,
- $\mathrm{NA}(C(K))$ is lineable.

Lineability of $\mathrm{NA}(X)$ II

Main question

Lineability of $\mathrm{NA}(X)$**?**

More concretely,

Problems (G. Godefroy, 2001)

(G_∞) Does $\mathrm{NA}(X)$ always contain an infinite-dimensional linear subspace?

(G) Does $\mathrm{NA}(X)$ always contain a linear subspace of dimension 2?

The case of dimension 1 is taken care of by the Hahn-Banach theorem!

Note that (G_∞) holds in all classical spaces.

Why lineability of $\mathrm{NA}(X)$ is interesting?

Theorem (Petunin–Plichko, 1974)

X separable Banach space, $W \leqslant X^*$ (norm) closed, separating, $W \subset \mathrm{NA}(X)$
$\Longrightarrow$ W is an isometric predual of X.

Two remarks

- "Norming for free": X separable, $W \leqslant X^*$ closed, $W \subset \mathrm{NA}(X)$, separating $\Longrightarrow$ W is 1-norming.
- Separability is needed: for $X = \ell_1([0,1])$, there is a closed, separating subspace of X^* contained in $\mathrm{NA}(X)$ which is not even norming.

Examples of application

- M compact metric space, $\mathrm{lip}_0(M)$ is an isometric predual of $\mathcal{F}(M)$ as soon as it separates it.
- X separable reflexive space with the (compact) approximation property. Then, $\mathcal{K}(X)^{**} = \mathcal{L}(X)$ and X has the metric (compact) approximation property.

Proximinality

Proximinal subspace

$Y \leqslant X$ is proximinal (in X) iff

$$\forall x \in X \ \exists y_0 \in Y \ : \quad \|x - y_0\| = \inf\{\|x - y\| : y \in Y\} = \mathrm{dist}(x, Y)$$

- It is a notion related to best approximation
  - by least square approximation (Gauss, Legendre, 1800's),
  - of functions by polynomials of a fix degree (Chebyshev, 1853)...
- Y proximinal iff $q(B_X) = B_{X/Y}$ $\qquad (q : X \longrightarrow X/Y$ quotient map)
- If Y is reflexive, then it is proximinal in any superspace.
- $x^* \in \mathrm{NA}(X) \iff \ker x^*$ proximinal.

Problem (I. Singer, 1974)

(S) Is there always a proximinal subspace of codimension 2?

Proximinality and norm attaining functionals

The two main problems

(S) Does there always exist a proximinal subspace of codimension 2?

(G) Does $\mathrm{NA}(X)$ always contain a linear subspace of dimension 2?

Important observation (Garkavi, 1967)

$$Y \leqslant X \text{ proximinal, } X/Y \text{ reflexive } \implies Y^{\perp} \subset \mathrm{NA}(X).$$

★ $Y \leqslant X$ such that X/Y reflexive is called factor reflexive.

Therefore. . .

If (S) has positive answer, then so does (G).

The converse to Garkavi's result is not true (Phelps, 1963)

There exist $X = C(K)$ and finite codimensional Y such that $Y^{\perp} \subset \mathrm{NA}(X)$ but Y is not proximinal.

Read's and Rmoutil's results

Section 2

Read's and Rmoutil's theorems

Theorem (Read, 2013)

There is a counterexample X_R to (S).

As (S) $\Rightarrow$ (G), X_R is a natural candidate for a counterexample to (G).

Actually,

Theorem (Rmoutil, 2015)

- $\dim X_R/Y < \infty, \quad Y^\perp \subset \mathrm{NA}(X_R) \implies X_R/Y$ strictly convex.
- (Known result) X/Y strictly convex and $Y^\perp \subset \mathrm{NA}(X) \implies Y$ proximinal.
- Consequently, X_R is also a counterexample to (G).

A simplification of Rmoutil's proof by Kadets/López/M.:

Proposition

X_R^{**} is strictly convex; hence *all* quotients of X_R are strictly convex.

Read's construction

X_R is a renorming of c_0:

Let $\Omega = \{(s_n)\colon (s_n) \text{ has finite support, all } s_n \in \mathbb{Q}\} \subset \ell_1$.
Enumerate $\Omega = \{u_1, u_2, \dots\}$ so that every element is repeated infinitely often.
Take a sequence of integers (a_n) such that

$$a_k > \max \operatorname{supp} u_k, \quad a_k \geqslant \|u_k\|_{\ell_1}.$$

Renorm c_0 by

$$p(x) = \|x\|_\infty + \sum_k 2^{-a_k^2} |\langle u_k - e_{a_k}, x\rangle|.$$

Then Read shows that (c_0, p) fails (S), and Rmoutil shows, relying on Read's work, that (c_0, p) fails (G).

The proof of Read's theorem is not trivial at all!!!!!

Our construction

Section 3

A new, direct approach to (G)

We four are more used to norm attainment than to proximinality, so we changed the point of view:

We want to show directly that certain Banach spaces have a renorming failing (G) and *hence* have a renorming failing (S).

Let $R\colon X \longrightarrow \ell_1$ be continuous; we renorm X by

$$p(x) = \|x\| + \|Rx\|_{\ell_1}.$$

More precisely, let $[Rx](n) = 2^{-n} v_n^*(x)$, with $(v_n^*) \subset B_{X^*}$, so

$$p(x) = \|x\| + \sum_{n=1}^{\infty} \frac{v_n^*(x)}{2^n}.$$

(Note that Read's renorming is of this type.)

Aim

Under suitable assumptions, the v_n^* can be chosen so that (X, p) fails (G) (and hence fails (S)).

A tentative calculation

$p(x) = \|x\| + \sum 2^{-n}|v_n^*(x)|$. Then $B_{(X^*,p^*)} = B_{X^*} + \sum 2^{-n}[-v_n^*, v_n^*]$ (Minkowski sum)

Let $x^* \in \mathrm{NA}_1(X,p)$ be norm attaining at x; then

$$x^* = x_0^* + \sum 2^{-n} t_n v_n^*$$

for some $x_0^* \in \mathrm{NA}_1(X)$ and $t_n = \operatorname{sign} v_n^*(x)$ whenever $v_n^*(x)$ is nonzero.
Write the same decomposition for $y^* \in \mathrm{NA}_1(X,p)$, norm attaining at y:

$$y^* = y_0^* + \sum 2^{-n} t_n' v_n^*.$$

Let's try to prove that $x^* + y^* \notin \mathrm{NA}(X,p)$: Otherwise we would have a similar decomposition for $z^* = (x^* + y^*)/\|x^* + y^*\|$:

$$z^* = z_0^* + \sum 2^{-n} s_n v_n^*.$$

Sort the items, setting $\lambda = \|x^* + y^*\|$:

$$0 = x^* + y^* - \lambda z^* = [x_0^* + y_0^* - \lambda z_0^*] + \left[\sum (t_n + t_n' - \lambda s_n) v_n^*\right]$$

Wish list

$$0 = [x_0^* + y_0^* - \lambda z_0^*] + \left[\sum (t_n + t_n' - \lambda s_n) v_n^*\right]$$

We now **wish** to select the v_n^* to be sort of "orthogonal" to $\mathrm{span}(\mathrm{NA}(X))$ (which contains the first bracket) so that both brackets vanish.

In addition we **wish** the v_n^* to have some Schauder basis character so that we can deduce from $\sum (t_n + t_n' - \lambda s_n) v_n^* = 0$ that all $t_n + t_n' - \lambda s_n = 0$.

Finally we **wish** the support points x and y to be distinct, and we **wish** the span of the v_n^* to be dense enough to separate x and y for many n, i.e., $v_n^*(x) < 0 < v_n^*(y)$ and thus $t_n + t_n' = 0$ fairly often, while at the same time $s_n \neq 0$ for at least one of those n.

This contradiction would show that $x^* + y^* \notin \mathrm{NA}(X, p)$.

Modest subspaces

Definition: operator range, (weak*) modest subspace

- V, W infinite-dimensional Banach spaces, $T\colon V \longrightarrow W$ bounded injective. Then $T(V)$ is called an operator range.
- $Z \leqslant W$ is modest if there is a separable dense operator range Y with $Y \cap Z = \{0\}$.
- If W is a dual space, then $Z \leqslant W$ is weak* modest if there is a separable weak* dense operator range Y with $Y \cap Z = \{0\}$.

Note that the choice of V in the definition of a modest subspace is at our discretion since

$$E, F \text{ separable} \implies \exists \text{ continuous injection } S\colon E \longrightarrow F \text{ with dense range.}$$

Example

$$c_{00} := \{(s_n)\colon (s_n) \text{ has finite support}\} \text{ is modest in } \ell_1.$$

Indeed, let $A_r(\mathbb{D})$ the real Banach space of those function of the disk algebra which takes real valued on the real axis;
define $T\colon A_r(\mathbb{D}) \longrightarrow \ell_1$ by $[Tf](n) = 2^{-n} f(2^{-n})$; then T has dense range and every non-null sequence in $T(A_r(\mathbb{D}))$ can only take the value 0 finitely many times.

Main Theorem

Theorem

If $\mathrm{span}(\mathrm{NA}(X))$ is weak* modest, then X has a renorming that fails (G) and, consequently, fails (S). (We call such an equivalent norm a Read norm.)

Recall ansatz: $p(x) = \|x\| + \sum 2^{-n}|v_n^*(x)|$; how to choose the v_n^*?

Lemma (where the technicalities appears)

Let $Y \leqslant X^*$ be a separable operator range. Then there is an injective operator $S\colon \ell_1 \longrightarrow X^*$ such that, for $v_n^* = S(e_n)$, the set $\{v_n^*/\|v_n^*\|\}$ is dense in S_Y.

With this choice of v_n^* it is possible to fulfill our wishes: the v_n^* are "orthogonal" to $\mathrm{NA}(X)$ (wish #1), they are an injective image of a Schauder basis (wish #2) and sufficiently dense (wish #4). As for wish #3, if $x = y$, then $x \neq -y$ and one should look at $x^* - y^*$!

Thus we can show that for linearly independent $x^*, y^* \in \mathrm{NA}(X, p)$ of norm 1, at most one of $x^* \pm y^*$ can be in $\mathrm{NA}(X, p)$.

First consequence

Example (we recuperate Read's and Rmoutil's results)

c_0 admits an equivalent Read norm, that is, a norm failing (G) and hence failing (S).

Indeed, $\mathrm{NA}(c_0) = c_{00}$ is modest in ℓ_1.

Note

The original construction by Read is NOT a particular case of ours:

Indeed, both norms are of the form $p(x) = \|x\| + \sum 2^{-n}|v_n^*(x)|$, but

- in the original Read's construction, the v_n^*'s belong to $\mathrm{NA}(c_0)$,
- in our construction, the v_n^*'s are "orthogonal" to $\mathrm{NA}(c_0)$.

More consequences I

Proposition

A separable Banach space containing a copy of c_0 admits a Read norm.

Indeed, renorm X so that $X = c_0 \oplus_\infty E$; then $X^* = \ell_1 \oplus_1 E^*$ and $\mathrm{NA}(X) \subset \mathrm{NA}(c_0) \times E^*$. The latter can be shown to be a weak* modest subspace.

Example

$C[0,1]$ admits an equivalent Read norm.

Norms with additional properties

X separable containing c_0. Then for each $0 < \varepsilon < 2$ there is a Read norm p_ε on X with the following properties:

- p_ε is strictly convex and smooth,
- p_ε^* is strictly convex,
- p_ε^* is $(2-\varepsilon)$-rough; i.e., every slice of $B_{(X,p_\varepsilon)}$ has diameter $\geqslant 2-\varepsilon$,
- If X^* is separable, then p_ε can be found to get p_ε^{**} strictly convex.

More consequences II

Theorem

A Banach space which isomorphically embeds into ℓ_∞ and contains a copy of c_0 admits an equivalent Read norm.

Example

ℓ_∞ admits an equivalent Read norm.

Norms with additional properties

X containing a copy of c_0 and isomorphic to a subspace of ℓ_∞. Then, for each $0 < \varepsilon < 2$ there is a Read norm p_ε on X so that

- p_ε is strictly convex,
- p_ε^* is $(2-\varepsilon)$-rough; i.e., every slice of $B_{(X,p_\varepsilon)}$ has diameter $\geqslant 2-\varepsilon$,
- actually, every convex combination of slices has diameter $\geqslant 2-\varepsilon$.

Limitation of the construction

$\ell_\infty(\Gamma)$ with Γ uncountable does not admit a Read norm (by a result of Partington)

An n-by-n version of the results

Section 4

An n-by-n version of the results

Problem (I. Singer, 1974), exact formulation

Let $1 < n < \infty$. Does every infinite-dimensional normed linear space (or, in particular, Banach space) contain a proximal subspace of codimension n?

Observations

- If X contains n-codimensional proximinal subspaces, then it contains k-codimensional proximinal subspaces for all $k \leqslant n$.
- But, is it possible to construct X_n having proximinal subspaces of codimension n but not higher?

Example

Let $1 \leqslant n < \infty$. There exists X_n satisfying:

- X_n contains n-codimensional proximinal subspaces,
- but no $(n+1)$-codimensional proximinal subspace;
- $\mathrm{NA}(X_n)$ contains subspaces of dimension n,
- but no subspace of dimension $n+1$.

One possibility:

$$X_{n+1} = X_1 \oplus_2 X_n$$

An infinite version of the results

$$X_\infty = \left[\bigoplus_{n\in\mathbb{N}} X_n\right]_{c_0}$$

X_n **non-separable** with Read norm

Example

Exists X_∞ non-separable satisfying:

- $\mathrm{NA}(X_\infty)$ contains infinite-dimensional separable subspaces,
- but $\mathrm{NA}(X_\infty)$ contains no non-separable subspaces,
- and every closed subspace contained in $\mathrm{NA}(X_\infty)$ is finite-dimensional;
- X_∞ contains n-codimensional proximinal subspaces for every $n \in \mathbb{N}$,
- but every proximinal factor reflexive subspace is of finite-codimension.

Open problem

Is there exist a Banach space X such that $\mathrm{NA}(X)$ contains linear subspaces of every finite dimensions but $\mathrm{NA}(X)$ contains no infinite-dimensional linear subspaces?

A brushstroke on norm attaining operators of finite rank

Section 5

Norm attaining operators

Norm attaining operators

X, Y Banach spaces, $\mathrm{NA}(X,Y) := \{T \in \mathcal{L}(X,Y) \colon \|T\| = \max_{x \in S_X} \|Tx\|\}$.

A few results

- Lindenstrauss, 1963: $\mathrm{NA}(X,Y)$ is not always dense,
- it is dense when X is reflexive or $c_0 \subset Y \subset \ell_\infty$ or Y polyhedral finite dimensional.
- Bourgain, 1977: $\mathrm{NA}(X,Y)$ dense if X RNP (and certain reciprocate).
- Gowers, 1990: for $1 < p < \infty$, exists X_p such that $\mathrm{NA}(X_p, \ell_p)$ is not dense.
- Acosta, 1999: ℓ_p above can be substituted by any infinite-dimensional strictly convex space and by ℓ_1.
- Johnsoh–Wolfe, 1979: for most "classical" Banach spaces as domain, compact operators can be approximated by norm attaining operators.
- M., 2014: there exist compact operators which cannot be approximated by norm attaining operators.

Open problem

Is every finite rank operator aproximable by norm attaining operators?

Norm attaining operators of finite rank: existence I

Open problem

Is every finite rank operator aproximable by norm attaining operators?

But, actually, we do not know whether...

...given X and Y of dimension greater than 2, there always exists $T \in \mathrm{NA}(X, Y)$ of rank-two.

Observation

If there is $T \in \mathrm{NA}(X, \ell_2)$ which is not rank-one, then for every Y with $\dim(Y) \geqslant 2$, there is $S \in \mathrm{NA}(X, Y)$ of rank-two.

A characterization

There are surjective operators in $\mathrm{NA}(X, \ell_2^{(2)})$ iff there exist $f \in \mathrm{NA}(X)$ with $\|f\| = 1$ and $h \in B_{X^*} \setminus \{0\}$ such that

$$\limsup_{t \to 0} \frac{\|f + th\| - 1}{t^2} < \infty.$$

Norm attaining operators of finite rank: existence II

"Easy" sufficient conditions for existence

X Banach space, $\mathrm{NA}(X,Y)$ contains a rank-two operator for all Y's with $\dim(Y) \geqslant 2$ provided one of the following conditions holds:

- there is a norm-one projection $P \in \mathcal{L}(X)$ with two-dimensional range,
- X contains a two-codimensional proximinal subspace,
- $\mathrm{NA}(X)$ contains two-dimensional subspaces,
- X is not smooth.

What happens with smooth spaces with Read norms?

Theorem

If $\mathrm{NA}(X)$ contains non-trivial cones, then $\mathrm{NA}(X,Y)$ contains rank-two operators for all Y's with $\dim(Y) \geqslant 2$.

Example

This applies to all known (smooth) spaces with Read norms.

Norm attaining operators of finite rank: density

Open problem

Is every finite rank operator aproximable by norm attaining operators?

A sufficient condition

If $\mathrm{NA}(X)$ contains a dense linear subspace, then finite rank operators with domain X can be approximated by finite rank norm attaining operators.

- If, besides, X^* has the AP, then compact operators with domain X can be approximated by finite rank norm attaining operators.

This applies to...

- $X = L_1(\mu)$ (density known from Diestel–Uhl 1976),
- stronger version applies to $X = C_0(L)$ (density known from Johnson-Wolfe 1979),
- if $X^* \equiv \ell_1$ (density known from M. 2016),
- subspaces of c_0 with monotone Schauder basis (density known from M. 2016),
- finite-codimensional proximinal subspaces of c_0 or $\mathcal{K}(\ell_2)$,
- c_0-sums of reflexive spaces.

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Section 6

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