Vector space structure in the set of norm attaining functionals and proximinality of subspaces of finite-codimension

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Supported by MTM2015-65020-P (MINECO/FEDER, UE), PGC2018-093794-B-I00 (MCIU/AEI/FEDER, UE),

FQM-185 (Junta de Andalucía/FEDER, UE), Plan propio de investigación de la Universidad de Granada

Banach Spaces and their Applications

International conference dedicated to the 70th anniversary of Professor A. M. Plichko

Roadmap of the talk

Preliminaries

Section 1

Basic notation

X, Y **real** Banach spaces

B_X closed unit ball

S_X unit sphere

X^* topological dual

$\mathcal{L}(X, Y)$ bounded linear operators from X to Y

$\mathcal{L}(X)$ bounded linear operators from X to X

$\mathcal{K}(X)$ compact linear operators from X to X

Norm attaining functionals

Norm attaining functionals

$x^* \in X^*$ attains its norm when

$$\exists\, x \in X,\ \|x\| = 1 \ :\ x^*(x) = \|x^*\|$$

★ $\mathrm{NA}(X) = \{x^* \in X^* \colon x^*$ attains its norm$\}$

First results

- $\dim(X) < \infty \implies \mathrm{NA}(X) = X^*$ (Heine-Borel),
- X reflexive $\implies \mathrm{NA}(X) = X^*$ (Hahn-Banach),
- X non-reflexive $\implies \mathrm{NA}(X) \neq X^*$ (James),
- $\mathrm{NA}(X)$ is always norm dense in X^* (Bishop-Phelps).

First examples

- $\mathrm{NA}(c_0) = c_{00} \leqslant \ell_1$,
- $\mathrm{NA}(\ell_1) = \left\{x \in \ell_\infty \colon \|x\|_\infty = \max_n \{|x(n)|\}\right\}$.

Lineability of $\mathrm{NA}(X)$

Examples

- $\mathrm{NA}(c_0) = c_{00} \leqslant \ell_1$,
- $\mathrm{NA}(\ell_1) = \left\{ x \in \ell_\infty \colon \|x\|_\infty = \max_n \{|x(n)|\} \right\}$.

- Note that $\mathrm{NA}(c_0)$ is a linear space, but $\mathrm{NA}(\ell_1)$ is not.
- However, $\mathrm{NA}(\ell_1)$ contains the infinite-dimensional space c_0.

Lineability

Recall that a subset S of a vector space V is called lineable if $S \cup \{0\}$ contains an infinite-dimensional subspace.

More examples

- $\mathrm{NA}(L_1(\mu))$ is a (dense) linear subspace,
- $\mathrm{NA}(C(K))$ is lineable.

Lineability of $\mathrm{NA}(X)$ II

Main question

Lineability of $\mathrm{NA}(X)$**?**

More concretely,

Problems (G. Godefroy, 2001)

(G_∞) Does $\mathrm{NA}(X)$ always contain an infinite-dimensional linear subspace?

(G) Does $\mathrm{NA}(X)$ always contain a linear subspace of dimension 2?

The case of dimension 1 is taken care of by the Hahn-Banach theorem!

Note that (G_∞) holds in all classical spaces.

Why lineability of $\mathrm{NA}(X)$ is interesting?

Theorem (Petunin–Plichko, 1974)

X separable Banach space, $W \leqslant X^*$ (norm) closed, separating, $W \subset \mathrm{NA}(X)$
$\Longrightarrow$ W is an isometric predual of X.

Two remarks

- "Norming for free": X separable, $W \leqslant X^*$ closed, $W \subset \mathrm{NA}(X)$, separating $\Longrightarrow$ W is 1-norming.
- Separability is needed: for $X = \ell_1([0,1])$, there is a closed, separating subspace of X^* contained in $\mathrm{NA}(X)$ which is not even norming.

Examples of application

- M compact metric space, $\mathrm{lip}_0(M)$ is an isometric predual of $\mathcal{F}(M)$ as soon as it separates it.
- X separable reflexive space with the (compact) approximation property.
  Then, $\mathcal{K}(X)^{**} = \mathcal{L}(X)$ and X has the metric (compact) approximation property.

Proximinality

Proximinal subspace

$Y \leqslant X$ is proximinal iff

$$\forall x \in X \; \exists y_0 \in Y \quad : \quad \|x - y_0\| = \inf\{\|x - y\| : y \in Y\} = \operatorname{dist}(x, Y)$$

- Y proximinal iff $q(B_X) = B_{X/Y}$ ($q : X \longrightarrow X/Y$ quotient map)
- $x^* \in \mathrm{NA}(X) \iff \ker x^*$ proximinal.

Problem (I. Singer, 1974)

(S) Is there always a proximinal subspace of codimension 2?

Proximinality and norm attaining functionals

The two main problems

(S) Does there always exist a proximinal subspace of codimension 2?

(G) Does $\mathrm{NA}(X)$ always contain a linear subspace of dimension 2?

Important observation (Garkavi, 1967)

$$Y \leqslant X \text{ proximinal, } X/Y \text{ reflexive } \implies Y^{\perp} \subset \mathrm{NA}(X).$$

★ $Y \leqslant X$ such that X/Y reflexive is called factor reflexive.

Therefore...

If (S) has positive answer, then so does (G).

The converse result is not true

There exist X and finite codimensional Y such that $Y^{\perp} \subset \mathrm{NA}(X)$ but Y is not proximinal (Phelps, 1963)

Read's and Rmoutil's results

Section 2

Read's and Rmoutil's theorems

Theorem (Read, 2013)

There is a counterexample X_R to (S).

As (S) $\Rightarrow$ (G), X_R is a natural candidate for a counterexample to (G).

Actually,

Theorem (Rmoutil, 2015)

- $\dim X_R/Y < \infty, \quad Y^\perp \subset \mathrm{NA}(X_R) \implies X_R/Y$ strictly convex.
- (Known result) X/Y strictly convex and $Y^\perp \subset \mathrm{NA}(X) \implies Y$ proximinal.
- Consequently, X_R is also a counterexample to (G).

A simplification of Rmoutil's proof by Kadets/López/M.:

Proposition

X_R^{**} is strictly convex; hence *all* quotients of X_R are strictly convex.

Read's construction

X_R is a renorming of c_0:

Let $\Omega = \{(s_n) \colon (s_n) \text{ has finite support, all } s_n \in \mathbb{Q}\} \subset \ell_1$.
Enumerate $\Omega = \{u_1, u_2, \ldots\}$ so that every element is repeated infinitely often.
Take a sequence of integers (a_n) such that

$$a_k > \max \operatorname{supp} u_k, \quad a_k \geqslant \|u_k\|_{\ell_1}.$$

Renorm c_0 by

$$p(x) = \|x\|_\infty + \sum_k 2^{-a_k^2} |\langle u_k - e_{a_k}, x\rangle|.$$

Then Read shows that (c_0, p) fails (S), and Rmoutil shows, relying on Read's work, that (c_0, p) fails (G).

The proof of Read's theorem is not trivial at all!!!!!

Our construction

Section 3

A new, direct approach to (G)

We four are more used to norm attainment than to proximinality, so we changed the point of view:

We want to show directly that certain Banach spaces have a renorming failing (G) and *hence* have a renorming failing (S).

Let $R\colon X \longrightarrow \ell_1$ be continuous; we renorm X by

$$p(x) = \|x\| + \|Rx\|_{\ell_1}.$$

More precisely, let $[Rx](n) = 2^{-n} v_n^*(x)$, with $(v_n^*) \subset B_{X^*}$, so

$$p(x) = \|x\| + \sum_{n=1}^{\infty} \frac{v_n^*(x)}{2^n}.$$

(Note that Read's renorming is of this type.)

Aim

Under suitable assumptions, the v_n^* can be chosen so that (X,p) fails (G) (and hence fails (S)).

A tentative calculation

$p(x) = \|x\| + \sum 2^{-n}|v_n^*(x)|$. Then $B_{(X^*,p^*)} = B_{X^*} + \sum 2^{-n}[-v_n^*, v_n^*]$ (Minkowski sum)

Let $x^* \in \mathrm{NA}_1(X,p)$ be norm attaining at x; then

$$x^* = x_0^* + \sum 2^{-n} t_n v_n^*$$

for some $x_0^* \in \mathrm{NA}_1(X)$ and $t_n = \operatorname{sign} v_n^*(x)$ whenever $v_n^*(x)$ is nonzero.
Write the same decomposition for $y^* \in \mathrm{NA}_1(X,p)$, norm attaining at y:

$$y^* = y_0^* + \sum 2^{-n} t_n' v_n^*.$$

Let's try to prove that $x^* + y^* \notin \mathrm{NA}(X,p)$: Otherwise we would have a similar decomposition for $z^* = (x^* + y^*)/\|x^* + y^*\|$:

$$z^* = z_0^* + \sum 2^{-n} s_n v_n^*.$$

Sort the items, setting $\lambda = \|x^* + y^*\|$:

$$0 = x^* + y^* - \lambda z^* = [x_0^* + y_0^* - \lambda z_0^*] + \left[\sum (t_n + t_n' - \lambda s_n) v_n^*\right]$$

Wish list

$$0 = [x_0^* + y_0^* - \lambda z_0^*] + \left[\sum (t_n + t_n' - \lambda s_n) v_n^*\right]$$

We now **wish** to select the v_n^* to be sort of "orthogonal" to $\mathrm{span}(\mathrm{NA}(X))$ (which contains the first bracket) so that both brackets vanish.

In addition we **wish** the v_n^* to have some Schauder basis character so that we can deduce from $\sum (t_n + t_n' - \lambda s_n) v_n^* = 0$ that all $t_n + t_n' - \lambda s_n = 0$.

Finally we **wish** the support points x and y to be distinct, and we **wish** the span of the v_n^* to be dense enough to separate x and y for many n, i.e., $v_n^*(x) < 0 < v_n^*(y)$ and thus $t_n + t_n' = 0$ fairly often, while at the same time $s_n \neq 0$ for at least one of those n.

This contradiction would show that $x^* + y^* \notin \mathrm{NA}(X, p)$.

Modest subspaces

Definition: operator range, (weak*) modest subspace

- V, W infinite-dimensional Banach spaces, $T\colon V \longrightarrow W$ bounded injective. Then $T(V)$ is called an operator range.
- $Z \leqslant W$ is modest if there is a separable dense operator range Y with $Y \cap Z = \{0\}$.
- If W is a dual space, then $Z \leqslant W$ is weak* modest if there is a separable weak* dense operator range Y with $Y \cap Z = \{0\}$.

Note that the choice of V in the definition of a modest subspace is at our discretion since

$$E, F \text{ separable} \quad \Longrightarrow \quad \exists \text{ continuous injection } S\colon E \longrightarrow F \text{ with dense range.}$$

Example

$$c_{00} := \{(s_n)\colon (s_n) \text{ has finite support}\} \text{ is modest in } \ell_1.$$

Indeed, let $A_r(\mathbb{D})$ the real Banach space of those function of the disk algebra which takes real valued on the real axis;
define $T\colon A_r(\mathbb{D}) \longrightarrow \ell_1$ by $[Tf](n) = 2^{-n} f(2^{-n})$; then T has dense range and every non-null sequence in $T(A_r(\mathbb{D}))$ can only take the value 0 finitely many times.

Main Theorem

Theorem

If $\mathrm{span}(\mathrm{NA}(X))$ is weak* modest, then X has a renorming that fails (G) and, consequently, fails (S). (We call such an equivalent norm a Read norm.)

Recall ansatz: $p(x) = \|x\| + \sum 2^{-n}|v_n^*(x)|$; how to choose the v_n^*?

Lemma

Let $Y \leqslant X^*$ be a separable operator range. Then there is an injective operator $S\colon \ell_1 \longrightarrow X^*$ such that, for $v_n^* = S(e_n)$, the set $\{v_n^*/\|v_n^*\|\}$ is dense in S_Y.

With this choice of v_n^* it is possible to fulfill our wishes: the v_n^* are "orthogonal" to $\mathrm{NA}(X)$ (wish #1), they are an injective image of a Schauder basis (wish #2) and sufficiently dense (wish #4). As for wish #3, if $x = y$, then $x \neq -y$ and one should look at $x^* - y^*$!

Thus we can show that for linearly independent $x^*, y^* \in \mathrm{NA}(X, p)$ of norm 1, at most one of $x^* \pm y^*$ can be in $\mathrm{NA}(X, p)$.

First consequence

Example (we recuperate Read's and Rmoutil's results)

c_0 admits an equivalent Read norm, that is, a norm failing (G) and hence failing (S).

Indeed, $\mathrm{NA}(c_0) = c_{00}$ is modest in ℓ_1.

Note

The original construction by Read is NOT a particular case of ours:

Indeed, both norms are of the form $p(x) = \|x\| + \sum 2^{-n} |v_n^*(x)|$, but

- in the original Read's construction, the v_n^*'s belong to $\mathrm{NA}(c_0)$,
- in our construction, the v_n^*'s are "orthogonal" to $\mathrm{NA}(c_0)$.

More consequences I

Proposition

A separable Banach space containing a copy of c_0 admits a Read norm.

Indeed, renorm X so that $X = c_0 \oplus_\infty E$; then $X^* = \ell_1 \oplus_1 E^*$ and $\mathrm{NA}(X) \subset \mathrm{NA}(c_0) \times E^*$. The latter can be shown to be a weak* modest subspace.

Example

$C[0,1]$ admits an equivalent Read norm.

Norms with additional properties

X separable containing c_0. Then for each $0 < \varepsilon < 2$ there is a Read norm p_ε on X with the following properties:

- p_ε is strictly convex and smooth,
- p_ε^* is strictly convex,
- p_ε^* is $(2-\varepsilon)$-rough; i.e., every slice of $B_{(X,p_\varepsilon)}$ has diameter $\geqslant 2-\varepsilon$,
- If moreover X^* is separable, then p_ε^{**} is strictly convex.

More consequences II

Theorem

A Banach space which isomorphically embeds into ℓ_∞ and contains a copy of c_0 admits an equivalent Read norm.

Example

ℓ_∞ admits an equivalent Read norm.

Norms with additional properties

X containing a copy of c_0 and isomorphic to a subspace of ℓ_∞. Then, for each $0 < \varepsilon < 2$ there is a Read norm p_ε on X so that

- p_ε is strictly convex,
- p_ε^* is $(2-\varepsilon)$-rough; i.e., every slice of $B_{(X,p_\varepsilon)}$ has diameter $\geqslant 2-\varepsilon$,
- actually, every convex combination of slices has diameter $\geqslant 2-\varepsilon$.

Remark

$\ell_\infty(\Gamma)$ with Γ uncountable does not admit a Read norm (by a result of Partington)

An n-by-n version of the results

Section 4

An n-by-n version of the results

Problem (I. Singer, 1974), exact formulation

Let $1 < n < \infty$. Does every infinite-dimensional normed linear space (or, in particular, Banach space) contain a proximal subspace of codimension n?

Observations

- If X contains n-codimensional proximinal subspaces, then it contains k-codimensional proximinal subspaces for all $k \leqslant n$.
- But, is it possible to construct X_n having proximinal subspaces of codimension n but not higher?

Example

Let $1 \leqslant n < \infty$. There exists X_n satisfying:

- X_n contains n-codimensional proximinal subspaces,
- but no $(n+1)$-codimensional proximinal subspace;
- $\mathrm{NA}(X_n)$ contains subspaces of dimension n,
- but no subspace of dimension $n+1$.

One possibility:

$$X_{n+1} = X_1 \oplus_2 X_n$$

An infinite version of the results

Example 1

$$X_\infty = \left[\bigoplus_{n\in\mathbb{N}} X_n\right]_{c_0}$$

X_n non-separable with Read norm

Exists X_∞ non-separable satisfying:

- $\mathrm{NA}(X_\infty)$ contains infinite-dimensional separable subspaces,
- but $\mathrm{NA}(X_\infty)$ contains no non-separable subspaces,
- and every closed subspace contained in $\mathrm{NA}(X_\infty)$ is finite-dimensional;
- X_∞ contains n-codimensional proximinal subspaces for every $n \in \mathbb{N}$,
- but every proximinal factor reflexive subspace is of finite-codimension.

Example 2

$$\widetilde{X}_\infty = \left[\bigoplus_{n\in\mathbb{N}} X_n\right]_{\ell_2}$$

X_n non-separable with Read norm

Exists $\widetilde{X}_\infty$ non-separable satisfying:

- $\mathrm{NA}(\widetilde{X}_\infty)$ contains infinite-dimensional closed separable subspaces,
- but $\mathrm{NA}(\widetilde{X}_\infty)$ contains no non-separable subspaces;
- $\widetilde{X}_\infty$ contains a factor reflexive proximinal subspace Y such that $\widetilde{X}_\infty/Y$ is infinite-dimensional and separable,
- and if Y is a factor reflexive proximinal subspace of $\widetilde{X}_\infty$, then $\widetilde{X}_\infty/Y$ is separable.

A brushstroke on norm attaining operators of finite rank

Section 5

Norm attaining operators

Norm attaining operators

X, Y Banach spaces, $\mathrm{NA}(X,Y) := \{T \in \mathcal{L}(X,Y) \colon \|T\| = \max_{x\in S_X} \|Tx\|\}$.

A few results

- Lindenstrauss, 1963: $\mathrm{NA}(X,Y)$ is not always dense,
- it is dense when X is reflexive or $c_0 \subset Y \subset \ell_\infty$ or Y polyhedral finite dimensional.
- Bourgain, 1977: $\mathrm{NA}(X,Y)$ dense if X RNP (and certain reciprocate).
- Gowers, 1990: for $1 < p < \infty$, exists X_p such that $\mathrm{NA}(X_p, \ell_p)$ is not dense.
- Acosta, 1999: ℓ_p above can be substituted by any infinite-dimensional strictly convex space and by ℓ_1.
- Johnsoh–Wolfe, 1979: for most "classical" Banach spaces as domain, compact operators can be approximated by norm attaining operators.
- M., 2014: there exist compact operators which cannot be approximated by norm attaining operators.

Open problem

Is every finite rank operator aproximable by norm attaining operators?

Norm attaining operators of finite rank: existence I

Open problem

Is every finite rank operator aproximable by norm attaining operators?

But, actually, we do not know whether...

...given X and Y of dimension greater than 2, there always exists $T \in \mathrm{NA}(X,Y)$ of rank-two.

Observation

If there is $T \in \mathrm{NA}(X,\ell_2)$ which is not rank-one, then for every Y with $\dim(Y) \geqslant 2$, there is $S \in \mathrm{NA}(X,Y)$ of rank-two.

A characterization

There are surjective operators in $\mathrm{NA}(X,\ell_2^{(2)})$ iff there exist $f \in \mathrm{NA}(X)$ with $\|f\| = 1$ and $h \in B_{X^*} \setminus \{0\}$ such that

$$\limsup_{t\to 0} \frac{\|f+th\|-1}{t^2} < \infty.$$

Norm attaining operators of finite rank: existence II

“Easy” sufficient conditions for existence

X Banach space, $\mathrm{NA}(X,Y)$ contains a rank-two operator for all Y's with $\dim(Y) \geqslant 2$ provided one of the following conditions holds:

- there is a norm-one projection $P \in \mathcal{L}(X)$ with two-dimensional range,
- X contains a two-codimensional proximinal subspace,
- $\mathrm{NA}(X)$ contains two-dimensional subspaces,
- X is not smooth.

What happens with smooth spaces with Read norms?

Theorem

If $\mathrm{NA}(X)$ contains non-trivial cones, then $\mathrm{NA}(X,Y)$ contains rank-two operators for all Y's with $\dim(Y) \geqslant 2$.

Example

This applies to all known (smooth) spaces with Read norms.

Norm attaining operators of finite rank: density

Open problem

Is every finite rank operator aproximable by norm attaining operators?

A sufficient condition

If $\mathrm{NA}(X)$ contains a dense linear subspace, then finite rank operators with domain X can be approximated by finite rank norm attaining operators.

- If, besides, X^* has the AP, then compact operators with domain X can be approximated by finite rank norm attaining operators.

This applies to...

- $X = L_1(\mu)$ (density known from Diestel–Uhl 1976),
- stronger version applies to $X = C_0(L)$ (density known from Johnson-Wolfe 1979),
- if $X^* \equiv \ell_1$ (density known from M. 2016),
- subspaces of c_0 with monotone Schauder basis (density known from M. 2016),
- finite-codimensional proximinal subspaces of c_0 or $\mathcal{K}(\ell_2)$,
- c_0-sums of reflexive spaces.

Bibliography

Section 6

Bibliography

V. Kadets, G. López, and M. Martín
Some geometric properties of Read's space
J. Funct. Anal. (2018)

V. Kadets, G. López, M. Martín, and D. Werner
Equivalent norms with an extremely nonlineable set of norm attaining functionals
J. Inst. Math. Jussieu (to appear). Doi: 10.1017/S1474748018000087

V. Kadets, G. López, M. Martín, and D. Werner
Norm attaining operators of finite rank
Preprint (2019), Arxiv: 1905.08272

M. Martín
Vector space structure in the set of norm attaining functionals and proximinality of subspaces of finite-codimension
Preprint (2019), Arxiv: 1805.05979

C. Read
Banach spaces with no proximinal subspaces of codimension 2
Israel J. Math. (2018)

M. Rmoutil
Norm-attaining functionals need not contain 2-dimensional subspaces
J. Funct. Anal. (2017)