Spear operators and the numerical index with respect to an operator

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Contents

Some notation

X, Y real or complex Banach spaces

$\mathbb{K}$ base field, $\mathbb{R}$ or $\mathbb{C}$

B_X closed unit ball

S_X unit sphere

X^* topological dual

$L(X,Y)$ Banach space of all bounded linear operators from X to Y

$L(X)$ Banach algebra of all bounded linear operators from X to X

A walk through the “classical” numerical index

Definitions

Numerical range for Hilbert spaces (Toeplitz, 1918)

H Hilbert space, $(\cdot \mid \cdot)$ inner product, $T \in L(H)$

$$W(T) = \{(Tx \mid x) \colon x \in H,\, (x \mid x) = 1\}$$

- It is a convex subset of $\mathbb{K}$

Numerical range and numerical radius (Bauer, Lumer, early 1960's)

X Banach space, $T \in L(X)$

$$V(T) = \{x^*(Tx) \colon x \in S_X,\, x^* \in S_{X^*},\, x^*(x) = 1\}$$

$$\begin{aligned} v(T) &= \sup\{|\lambda| \colon \lambda \in V(T)\} \\ &= \sup\{|x^*(Tx)| \colon x \in S_X,\, x^* \in S_{X^*},\, x^*(x) = 1\} \end{aligned}$$

- $V(T)$ is connected not necessarily convex,
- $\overline{V(T)}$ contains the spectrum of T,
- obviously, $v(T) \leqslant \|T\|$ for every $T \in L(X)$.

Definitions

Numerical index (Lumer, 1968)

X Banach space

$$n(X) = \inf\{v(T)\colon T \in S_{L(X)}\} = \max\{k \geqslant 0\colon k\|T\| \leqslant v(T)\}$$

- $0 \leqslant n(X) \leqslant 1$
- v and $\|\cdot\|$ are equivalent norms iff $n(X) > 0$

Possible values of the numerical index

$$\{n(X)\colon X \text{ complex Banach space }\} = [\mathrm{e}^{-1}, 1]$$
$$\{n(X)\colon X \text{ real Banach space }\} = [0, 1]$$

Some known results

- H Hilbert space, $n(H)=0$ in real case and $n(H)=1/2$ in complex case.
- $n(C(K))=n(L_1(\mu))=1$ (Duncan-McGregor-Pryce-White, 1970)
- $n(X)=1$ iff $\max\limits_{|w|=1}\|\mathrm{Id}+w\,T\|=1+\|T\|\ \forall T\in L(X)$ (Duncan et al., 1970)
- Let $\{X_\lambda \colon \lambda\in\Lambda\}$ be an arbitrary family of Banach spaces. Then

$$n\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{c_0}\Big)=n\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{\ell_1}\Big)=n\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{\ell_\infty}\Big)=\inf_{\lambda\in\Lambda} n(X_\lambda)$$

$$n\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{\ell_p}\Big)\leqslant\inf_{\lambda\in\Lambda} n(X_\lambda)$$

(Martín-Payá, 2000)

Some known results

- X Banach space, K compact Hausdorff, μ positive measure

$$n\big(C(K,X)\big) = n\big(L_1(\mu,X)\big) = n(X) \qquad \text{(Martín-Payá, 2000)}$$
$$n\big(L_\infty(\mu,X)\big) = n(X) \qquad \text{(Martín-Villena, 2003)}$$

- $n(L_p(\mu)) = n(\ell_p)$ if $\dim L_p(\mu) = \infty$ (EdDari-Khamsi, 2006)
- $n(L_p(\mu)) > 0$ for $p \neq 2$ (Martín-Merí-Popov, 2011)
- $n(X^*) \leqslant n(X)$
  and the inequality can be strict (Boyko-Kadets-Martín-Werner,2007)
- X separable (WCG)

$$\Longrightarrow \quad \{n(X,|\cdot|) : |\cdot| \text{ equivalent norm}\} \supseteq \begin{cases} [0,1[& \text{real case} \\ [1/\mathrm{e},1[& \text{complex case} \end{cases}$$
(Finet-Martín-Payá, 2003)

- X real, $\dim(X) = \infty$, $n(X) = 1 \quad \Longrightarrow \quad X^* \supseteq \ell_1$
  (Avilés, Kadets, Martín, Merí, Shepelska, 2010)

Extending the concept of numerical range

Spatial numerical range

Bauer–Lumer (spatial) Numerical range

X Banach space, $T \in L(X)$,

$$V(T) = \{x^*(Tx) \colon x \in S_X,\ x^* \in S_{X^*},\ x^*(\mathrm{Id}\, x) = 1\}$$

★ $G \in L(X,Y)$ with $\|G\| = 1$, $T \in L(X,Y)$, how to define $V_G(T)$**?**
The first idea (not working):

$$V_G(T) = \{y^*(Tx) \colon x \in S_X,\ y^* \in S_{Y^*},\ y^*(Gx) = 1\}$$

(Approximate spatial) Numerical range with respect to G (Ardalani, 2014)

X, Y Banach spaces, $G \in L(X,Y)$ with $\|G\| = 1$, $T \in L(X,Y)$

$$V_G(T) = \bigcap_{\delta>0} \overline{\{y^*(Tx) \colon x \in S_X,\ y^* \in S_{Y^*},\ \operatorname{Re} y^*(Gx) > 1-\delta\}}$$

For $G = \mathrm{Id}$, by the Bishop–Phelps–Bollobás theorem

$$V_{\mathrm{Id}}(T) = \overline{V(T)} \qquad \text{for every } T \in L(X)$$

Intrinsic Numerical range

(Bonsall-Duncan, 1971)

Let X be a Banach space. Then for every $T \in L(X)$

$$\overline{\text{co}}\, V(T) = \{\Phi(T) \colon \Phi \in L(X)^*,\ \|\Phi\| = \Phi(\mathrm{Id}) = 1\}.$$

Consequently, $v(T) = \max\{|\Phi(T)| \colon \Phi \in L(X)^*,\ \|\Phi\| = \Phi(\mathrm{Id}) = 1\}$.

Intrinsic (or algebraic) numerical range

X Banach space, $T \in L(X)$,

$$\widetilde{V}(T) = \{\Phi(T) \colon \Phi \in L(X)^*,\ \|\Phi\| = \Phi(\mathrm{Id}) = 1\}$$

Intrinsic numerical range with respect to G

X, Y Banach spaces, $G \in L(X,Y)$ with $\|G\| = 1$, $T \in L(X,Y)$

$$\widetilde{V}_G(T) = \{\Phi(T) \colon \Phi \in L(X,Y)^*,\ \|\Phi\| = \Phi(G) = 1\}$$

The relationship

Two possible numerical ranges

X, Y Banach spaces, $G \in L(X,Y)$ with $\|G\| = 1$, $T \in L(X,Y)$

$$V_G(T) = \bigcap_{\delta>0} \overline{\{y^*(Tx)\colon x \in S_X,\ y^* \in S_{Y^*},\ \operatorname{Re} y^*(Gx) > 1-\delta\}}$$

$$\widetilde{V}_G(T) = \{\Phi(T)\colon \Phi \in L(X,Y)^*,\ \|\Phi\| = \Phi(G) = 1\}$$

Relationship (Martín, 2016)

X, Y be Banach spaces, $G \in L(X,Y)$ with $\|G\| = 1$, then

$$\widetilde{V}_G(T) = \operatorname{co} V_G(T) \qquad \text{for every } T \in L(X,Y)$$

Both concepts produce the same numerical radius:

Numerical radius with respect to G

X, Y Banach spaces, $G \in L(X,Y)$ with $\|G\| = 1$, $T \in L(X,Y)$

$$v_G(T) = \sup\{|\lambda|\colon \lambda \in V_G(T)\} = \sup\{|\lambda|\colon \lambda \in \widetilde{V}_G(T)\}$$

Numerical index with respect to an operator: definition

Numerical index with respect to an operator

Numerical index with respect to G

X, Y Banach spaces, $G \in L(X,Y)$ with $\|G\| = 1$,

$$n_G(X,Y) = \inf\{v_G(T) \colon T \in S_{L(X,Y)}\} = \max\{k \geqslant 0 \colon k\|T\| \leqslant v_G(T)\}$$

We recuperate the classical numerical index

$$n_{\mathrm{Id}}(X,X) = n(X)$$

Characterization

For $k \in [0,1]$, TFAE:

- $n_G(X,Y) \geqslant k$,
- $\inf\limits_{\delta>0} \sup\{|y^*(Tx)| \colon x \in S_X,\ y^* \in S_{Y^*},\ \operatorname{Re} y^*(Gx) > 1-\delta\} \geqslant k\|T\| \quad \forall\, T \in L(X,Y)$,
- $\max\limits_{|\theta|=1} \|G + \theta\, T\| \geqslant 1 + k\|T\| \quad \forall\, T \in L(X,Y)$.

Consequence

$$n_G(X,Y) > 0 \iff G \text{ is a (geometrically) unitary element of } L(X,Y)$$

Numerical index with respect to an operator: examples and properties

Some interesting examples I

Set of values

There exists X (real and complex versions) such that

$$\big\{n_G(X,X)\colon G\in L(X,X),\ \|G\|=1\big\}=[0,1].$$

Hilbert spaces

H_1, H_2 Hilbert spaces of dimension at least two,

- Real case: $n_G(H_1,H_2)=0$ for all $G\in L(H_1,H_2)$ with $\|G\|=1$,
- Complex case: $n_G(H_1,H_2)\in\{0,1/2\}$ for all $G\in L(H_1,H_2)$ with $\|G\|=1$.

Actually...

$G\in L(X,Y)$ with $\|G\|=1$, if X or Y is a real Hilbert space

$$\Longrightarrow\quad n_G(X,Y)=0.$$

★ There are more spaces with this property...

Some interesting examples II

L_p-spaces

$G \in L(X,Y)$ with $\|G\| = 1$, if X or Y is a $L_p(\mu)$-space $(1 < p < \infty)$,

$$\Longrightarrow \quad n_G(X,Y) \leqslant \begin{cases} \sup\limits_{t\in[0,1]} \dfrac{|t^{p-1} - t|}{1 + t^p} & \text{real case} \\ p^{-1/p} q^{-1/q} & \text{complex case} \end{cases}$$

Spaces of integrable functions

μ_1, μ_2 σ-finite measures,

$$n_G\big(L_1(\mu_1), L_1(\mu_2)\big) \in \{0,1\} \text{ for all } G \in L\big(L_1(\mu_1), L_1(\mu_2)\big) \text{ with } \|G\| = 1.$$

Spaces of essentially bounded functions

μ_1, μ_2 σ-finite measures,

$$n_G\big(L_\infty(\mu_1), L_\infty(\mu_2)\big) \in \{0,1\} \text{ for all } G \in L\big(L_\infty(\mu_1), L_\infty(\mu_2)\big) \text{ with } \|G\| = 1.$$

Sums of Banach spaces

Proposition

Let $\{X_\lambda \colon \lambda \in \Lambda\}$, $\{Y_\lambda \colon \lambda \in \Lambda\}$ be two families of Banach spaces and let $G_\lambda \in L(X_\lambda, Y_\lambda)$ with $\|G_\lambda\| = 1$ for every $\lambda \in \Lambda$. Let E be one of the Banach spaces c_0, ℓ_∞ or ℓ_1, let $X = \left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_E$ and $Y = \left[\bigoplus_{\lambda\in\Lambda} Y_\lambda\right]_E$ and define the operator $G \colon X \longrightarrow Y$ by

$$G\left[(x_\lambda)_{\lambda\in\Lambda}\right] = (G_\lambda x_\lambda)_{\lambda\in\Lambda}$$

for every $(x_\lambda)_{\lambda\in\Lambda} \in \left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_E$. Then

$$n_G(X, Y) = \inf_\lambda n_{G_\lambda}(X_\lambda, Y_\lambda).$$

Moreover, for $1 < p < \infty$

$$n_G\left([\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_p}, [\oplus_{\lambda\in\Lambda} Y_\lambda]_{\ell_p}\right) \leqslant \inf_\lambda n_{G_\lambda}(X_\lambda, Y_\lambda).$$

Composition operators

Theorem

Let X, Y be Banach spaces, and $G \in L(X,Y)$ with $\|G\| = 1$.

- K compact, consider $\widetilde{G}\colon C(K,X) \longrightarrow C(K,Y)$ given by $\widetilde{G}(f) = G \circ f$; then
$$n_{\widetilde{G}}\big(C(K,X), C(K,Y)\big) = n_G(X,Y).$$
- μ measure, consider $\widetilde{G}\colon L_1(\mu,X) \longrightarrow L_1(\mu,Y)$ given by $\widetilde{G}(f) = G \circ f$; then
$$n_{\widetilde{G}}\big(L_1(\mu,X), L_1(\mu,Y)\big) = n_G(X,Y).$$
- μ σ-finite, consider $\widetilde{G}\colon L_\infty(\mu,X) \longrightarrow L_\infty(\mu,Y)$ given by $\widetilde{G}(f) = G \circ f$; then
$$n_{\widetilde{G}}\big(L_\infty(\mu,X), L_\infty(\mu,Y)\big) = n_G(X,Y).$$

Besides, for vector-valued L_p-spaces one inequality holds:

$$n_{\widetilde{G}}\big(L_p(\mu,X), L_p(\mu,Y)\big) \leqslant n_G(X,Y)$$

for $1 < p < \infty$, $\widetilde{G}$ defined analogously.

Spear operators

Examples of spear operators

Spear operator (Ardalani, 2014; Kadets, Martín, Merí, Pérez, 2018)

G spear operator $\iff n_G(X,Y)=1 \iff \max\limits_{|\theta|=1}\|G+\theta\,T\| = 1+\|T\|\ \forall\,T\in L(X,Y)$.

Some interesting examples of spear operators

- Fourier transform (for example, $\mathcal{F} : L_1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$);
- The inclusion $A(\mathbb{D}) \hookrightarrow C(\mathbb{T})$;
- The identity operator on $C(K)$, $L_1(\mu)\ldots$
- $G : X \longrightarrow c_0$ spear iff $\left|x^{**}\left(G^*(e_n)\right)\right| = 1$ for $n\in\mathbb{N}$ and $x^{**}\in \operatorname{ext}\left(B_{X^{**}}\right)$;
- $G : \ell_1 \longrightarrow Y$ spear iff $\left|y^{*}\left(G(e_n)\right)\right| = 1$ for $n\in\mathbb{N}$ and $y^{*}\in \operatorname{ext}\left(B_{Y^{*}}\right)$;
- If $\dim(X)<\infty$, G spear iff $\left|y^{*}\left(Gx\right)\right| = 1$ for $y^{*}\in \operatorname{ext}\left(B_{Y^{*}}\right)$ and $x\in \operatorname{ext}\left(B_{X}\right)$;
- If $\dim(Y)<\infty$, G spear iff $\left|x^{**}\left(G^*(y^*)\right)\right| = 1$ for $x^{**}\in \operatorname{ext}\left(B_{X^{**}}\right)$ and $y^{*}\in \operatorname{ext}\left(B_{X^{*}}\right)$;

Studying spear operators

Spear operator (Ardalani, 2014; Kadets, Martín, Merí, Pérez, 2018)

G spear operator $\iff n_G(X,Y)=1 \iff \max\limits_{|\theta|=1}\|G+\theta\,T\| = 1+\|T\|\ \forall\, T\in L(X,Y).$

Remark

To work with spear operators, two other concepts are introduced:

- lush operator,
- the alternative Daugavet property (aDP),

★ Both are geometric properties (related to G)

★ They are related as follows:

lush operator $\Longrightarrow$ / $\not\Longleftarrow$ spear operator $\Longrightarrow$ / $\not\Longleftarrow$ operator with aDP

operator with aDP $\xLeftarrow{\text{SCD operator}}$ lush operator

$(X$ RNP, $X \not\supseteq \ell_1$, Y Asplund$\ldots)$

Spear operators: consequences

Some isomorphic and isometric consequences

X, Y Banach spaces, $G \in L(X,Y)$ spear operator,

- if $\dim(G(X)) = \infty$, then $X^* \supset \ell_1$,
- if X^* is strictly convex, then $X = \mathbb{K}$,
- if X^* is smooth, then $X = \mathbb{K}$,
- if B_X contains a WLUR point, then $X = \mathbb{K}$,
- if Y^* is strictly convex, then $Y = \mathbb{K}$,
- if B_Y contains a WLUR point, then $Y = \mathbb{K}$.

Norm attaintment

- If G is lush, G attains its norm; actually:

$$B_X = \overline{\text{co}}\Big\{x \in S_X : \|Gx\| = 1\Big\},$$

- There are examples of aDP operators which do not attain the norm,
- What about spear operators **?**

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