Strongly norm attaining Lipschitz maps

Miguel Martín

UNIVERSIDAD
DE GRANADA

(joint work with B. Cascales, R. Chiclana, L.C. García–Lirola, and A. Rueda)

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This talk is dedicated to Bernardo Cascales, *amigo* y *maestro*.

B. Cascales, R. Chiclana, L. García-Lirola, M. Martín, A. Rueda
On strongly norm attaining Lipschitz maps
Preprint `arXiv:1807.03363`.

Footnote: Sadly, Bernardo Cascales passed away in April, 2018. As this work was initiated with him, the rest of the authors decided to finish the research and to submit the paper with his name as coauthor. This is our tribute to our dear friend and master.

R. Chiclana, L.C. García-Lirola, M. Martín, A. Rueda

Contents

Preliminaries

Some notation

X, Y real Banach spaces

B_X closed unit ball

S_X unit sphere

X^* topological dual

$\mathcal{L}(X,Y)$ Banach space of all bounded linear operators from X to Y

$\mathcal{L}(X)$ Banach algebra of all bounded linear operators from X to X

Main definition and leading problem

Lipschitz function

M, N (complete) metric spaces. A map $F\colon M \longrightarrow N$ is Lipschitz if there exists a constant $k > 0$ such that

$$d(F(p), F(q)) \leqslant k\, d(p,q) \quad \forall\, p, q \in M$$

The least constant so that the above inequality works is called the Lipschitz constant of F, denoted by $L(F)$:

$$L(F) = \sup \left\{ \frac{d(F(p), F(q))}{d(p,q)} : p \neq q \in M \right\}$$

- If $N = Y$ is a normed space, then $L(\cdot)$ is a seminorm in the vector space of all Lipschitz maps from M into Y.
- F attain its Lipschitz number if the supremum defining it is actually a maximum.

Leading problem

Let M be a metric space, let Y be a Banach and let $F\colon M \longrightarrow Y$ be a Lipschitz map. Can F be approximated by Lipschitz functions from M to Y which attain their Lipschitz number?

First examples

Finite sets

If M is finite, obviously every Lipschitz map attains its Lipschitz number.
★ This characterizes finiteness of M.

Example (Kadets–Martín–Soloviova, 2016)

$M = [0,1]$, $A \subseteq [0,1]$ closed with empty interior and positive Lebesgue measure. Then, the Lipschitz function $f\colon [0,1] \longrightarrow \mathbb{R}$ given by

$$f(t) = \int_0^t \chi_A(s)\, ds,$$

cannot be approximated by Lipschitz functions which attain their Lipschitz number.

Objective

To extend those results (to more interesting ones).

More definitions

Pointed metric space

M is *pointed* if it carries a distinguished element called base point.

Space of Lipschitz maps

M pointed metric space, Y Banach space.
$\mathrm{Lip}_0(M, Y)$ is the Banach space of all Lipschitz maps from M to Y which are zero at the base point, endowed with the Lipschitz number as norm.

Strongly norm attaining Lipschitz map

M pointed metric space. $F \in \mathrm{Lip}_0(M, Y)$ strongly attains its norm, writing $F \in \mathrm{SNA}(M, Y)$, if there exist $p \neq q \in M$ such that

$$L(F) = \|F\| = \frac{\|F(p) - F(q)\|}{d(p,q)}.$$

Our objective is then

to study when $\mathrm{SNA}(M, Y)$ is norm dense in the Banach space $\mathrm{Lip}_0(M, Y)$

Some more definitions

Evaluation functional, Lipschitz-free space, molecule

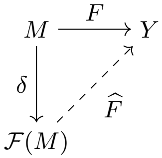
M pointed metric space.

- $p \in M$, $\delta_p \in \mathrm{Lip}_0(M,\mathbb{R})^*$ given by $\delta_p(f) = f(p)$ is the evaluation functional at p;
- $\mathcal{F}(M) := \overline{\mathrm{span}}\{\delta_p \colon p \in M\} \subseteq \mathrm{Lip}_0(M,\mathbb{R})^*$ is the Lipschitz-free space of M;
- For $p \neq q \in M$, $m_{p,q} := \frac{\delta_p - \delta_q}{d(p,q)} \in \mathcal{F}(M)$ is a molecule;
- $\mathrm{Mol}\,(M) := \{m_{p,q} \colon p,q \in M,\ p \neq q\}$.
- $B_{\mathcal{F}(M)} = \overline{\mathrm{conv}}(\mathrm{Mol}\,(M))$.

Very important property (Arens-Eells, Kadets, Godefroy-Kalton, Weaver...)

M pointed metric space.

- $\delta : M \rightsquigarrow \mathcal{F}(M)$, $p \longmapsto \delta_p$, is an isometric embedding;
- $\mathcal{F}(M)^* \cong \mathrm{Lip}_0(M,\mathbb{R})$;
- Actually, Y Banach space, $F \in \mathrm{Lip}_0(M,Y)$, $\exists$ (a unique) $\widehat{F} \in \mathcal{L}(\mathcal{F}(M),Y)$ such that $F = \widehat{F} \circ \delta$, and so $\|\widehat{F}\| = \|F\|$.



★ In particular, $\mathrm{Lip}_0(M,Y) \cong \mathcal{L}(\mathcal{F}(M),Y)$.

Two ways of attaining the norm

We have two ways of attaining the norm

M pointed metric space, Y Banach space, $F \in \mathrm{Lip}_0(M,Y) \cong \mathcal{L}(\mathcal{F}(M),Y)$.

- $\widehat{F} \in \mathrm{NA}(\mathcal{F}(M),Y)$ if exists $\xi \in B_{\mathcal{F}(M)}$ such that $\|F\| = \|\widehat{F}\| = \|\widehat{F}(\xi)\|$;
- $F \in \mathrm{SNA}(M,Y)$ if exists $m_{p,q} \in \mathrm{Mol}\,(M)$ such that

$$\|F\| = \|\widehat{F}\| = \|\widehat{F}(m_{p,q})\| = \frac{\|F(p)-F(q)\|}{d(p,q)}.$$

Clearly, $\mathrm{SNA}(M,Y) \subseteq \mathrm{NA}(\mathcal{F}(M),Y)$.

- Therefore, if $\mathrm{SNA}(M,Y)$ is dense in $\mathrm{Lip}_0(M,Y)$, then $\mathrm{NA}(\mathcal{F}(M),Y)$ is dense in $\mathcal{L}(\mathcal{F}(M),Y)$;
- But the opposite direction is NOT true:

Example

- $\overline{\mathrm{NA}(\mathcal{F}(M),\mathbb{R})} = \mathcal{L}(\mathcal{F}(M),\mathbb{R})$ for every M by the Bishop–Phelps theorem,
- But $\overline{\mathrm{SNA}([0,1],\mathbb{R})} \neq \mathrm{Lip}_0([0,1],\mathbb{R})$.

A little of geometry of the unit ball of $\mathcal{F}(M)$ (A–G–GL–P–P–R–W)

Preserved extreme point

$\xi \in B_{\mathcal{F}(M)}$, TFAE:

- ξ is extreme in $B_{\mathcal{F}(M)^{**}}$,
- ξ is a denting point,
- $\xi = m_{p,q}$ and for every $\varepsilon > 0 \; \exists \; \delta > 0$ s.t. $d(p,t) + d(t,q) - d(p,q) > \delta$ when $d(p,t), d(t,q) \geqslant \varepsilon$.

★ M boundedly compact, it is equivalent to:

- $d(p,q) < d(p,t) + d(t,q) \; \forall t \notin \{p,q\}$.

Strongly exposed point

$\xi \in B_{\mathcal{F}(M)}$, TFAE:

- ξ strongly exposed point,
- $\xi = m_{p,q}$ and $\exists \; \rho = \rho(p,q) > 0$ such that
$$\frac{d(p,t) + d(t,q) - d(p,q)}{\min\{d(p,t), d(t,q)\}} \geqslant \rho$$
when $t \notin \{p,q\}$.

Concave metric space

M is concave if $m_{p,q}$ is a preserved extreme point for all $p \neq q$.

★ Examples: $y = x^3$, S_X if X unif. convex...

Uniform Gromov rotundity

$\mathcal{M} \subset \mathrm{Mol}\,(M)$ is uniformly Gromov rotund if $\exists \rho_0 > 0$ such that
$$\frac{d(p,t) + d(t,q) - d(p,q)}{\min\{d(p,t), d(t,q)\}} \geqslant \rho_0$$
when $m_{p,q} \in \mathcal{M}$, $t \notin \{p,q\}$.

$\Longleftrightarrow$ M is a set of uniformly strongly exposed points (same relation $\varepsilon - \delta$)

★ $\mathrm{Mol}\,(M)$ uniformly Gromov rotund when:

- $M = ([0,1], |\cdot|^\theta)$,
- M finite and concave,
- $1 \leqslant d(p,q) \leqslant D < 2 \; \forall p,q \in M$, $p \neq q$.

★ $\mathbb{T}$ is concave but $\mathrm{Mol}\,(\mathbb{T})$ not u. Gromov r.

Negative results

Negative results

Previous result (Kadets–Martín–Soloviova, 2016)

If M is metrically convex (or "geodesic"), then $\mathrm{SNA}(M,\mathbb{R})$ is not dense in $\mathrm{Lip}_0(M,\mathbb{R})$.

Definition (length space)

Let M be a metric space. We say that M is length if $d(p,q)$ is equal to the infimum of the length of the rectifiable curves joining p and q for every pair of points $p,q \in M$.

★ Equivalently (Avilés, García, Ivankhno, Kadets, Martínez, Prochazka, Rueda, Werner)

- M is local (i.e. the Lipschitz constant of every function can be approximated in pairs of arbitrarily closed points);
- The unit ball of $\mathcal{F}(M)$ has no strongly exposed points;
- $\mathrm{Lip}_0(M,\mathbb{R})$ (and so $\mathcal{F}(M)$) has the Daugavet property.

Theorem

Let M be a length pointed metric space. Then,

$$\overline{\mathrm{SNA}(M,\mathbb{R})} \neq \mathrm{Lip}_0(M,\mathbb{R})$$

Other type of negative results

Observation

All the previous examples of M's such that $\mathrm{SNA}(M, \mathbb{R})$ is not dense in $\mathrm{Lip}_0(M, \mathbb{R})$ are arc-connected metric spaces and "almost convex".

Let's present two different kind of examples:

Example

M "fat" Cantor set, then $\overline{\mathrm{SNA}(M, \mathbb{R})} \neq \mathrm{Lip}_0(M, \mathbb{R})$ and M is totally disconnected.

Example

$M = \mathbb{T}$, then $\overline{\mathrm{SNA}(M, \mathbb{R})} \neq \mathrm{Lip}_0(M, \mathbb{R})$.

Positive results

Possible sufficient conditions

Observation (previously commented)

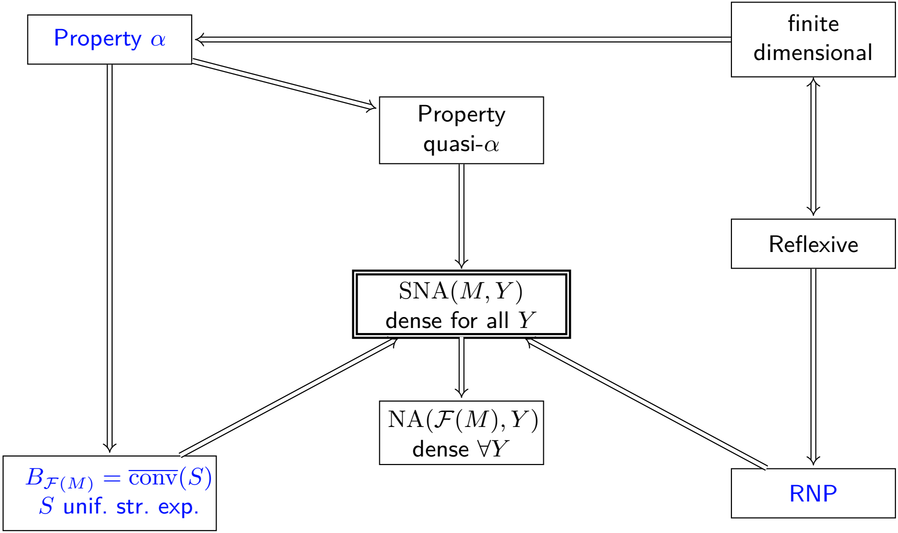
$\mathrm{SNA}(M,Y)$ dense in $\mathrm{Lip}_0(M,Y) \implies \mathrm{NA}(\mathcal{F}(M),Y)$ dense in $\mathcal{L}(\mathcal{F}(M),Y)$.

Therefore, it is reasonable to discuss the known sufficient conditions for a Banach space X to have $\overline{\mathrm{NA}(X,Y)} = \mathcal{L}(X,Y)$ for every Y:

- RNP,
- Property α,
- Property quasi-α,
- the existence of a norming set of uniformly strongly exposed points.

In the next slice we will relate all these properties for Lipschitz-free spaces:

Sufficient conditions for the density of $\mathrm{SNA}(M, Y)$ for every Y: relations



The RNP

Theorem (García-Lirola–Petitjean–Procházka–Rueda-Zoca, 2018)

Let M be a pointed metric space. Assume that $\mathcal{F}(M)$ has the RNP. Then,

$$\overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y) \quad \text{for every Banach space } Y$$

Proof

- Bourgain, 1977:
  X RNP $\implies \{T \in \mathcal{L}(X,Y) \colon T$ strongly exposes $B_X\}$ is dense in $\mathcal{L}(X,Y)$;
- T strongly exposing operator, then T attains its norm at a strongly exposed point;
- Weaver, 1999: strongly exposed points of $B_{\mathcal{F}(M)}$ are molecules.

$\mathcal{F}(M)$ has the RNP when...

- $M = (N, d^\theta)$ for (N,d) boundedly compact and $0 < \theta < 1$ (Weaver, 1999 - 2018);
- M is uniformly discrete (Kalton, 2004);
- M is countable and compact (Dalet, 2015);
- $M \subset \mathbb{R}$ with Lebesgue measure 0 (Godard, 2010).

Property alpha

Property alpha

X Banach space. X has property α if there exist a balanced subset $\{x_\lambda\}_{\lambda\in\Lambda} \subseteq X$ and a subset $\{x^*_\lambda\}_{\lambda\in\Lambda} \subseteq X^*$ such that

- $\|x_\lambda\| = \|x^*_\lambda\| = |x^*_\lambda(x_\lambda)| = 1 \quad \forall\, \lambda \in \Lambda$;
- There exists $0 \leqslant \rho < 1$ such that $|x^*_\lambda(x_\mu)| \leqslant \rho \quad \forall\, x_\mu \neq \pm x_\lambda$;
- $\overline{\text{co}}(\{x_\lambda\}_{\lambda\in\Lambda}) = B_X$.

- Introduced by Schachermayer in 1983 as a sufficient condition for X to get $\overline{\text{NA}(X,Y)} = \mathcal{L}(X,Y)$ for every Y;
- Every separable Banach space X can be renormed with property α;
- (Godun–Troyanski, 1993): this result extends to Banach spaces with long biorthogonal systems.
- (Schachermayer, 1983): If X has property α, then

$$\{T \in \mathcal{L}(X,Y) \colon T \text{ attains its norm at one } x_k \}$$

is dense in $\mathcal{L}(X,Y)$ for every Y.

Property alpha and density of $\mathrm{SNA}(M, Y)$

Theorem

M metric space such that $\mathcal{F}(M)$ has property α. Then,

$$\overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y) \quad \text{for every Banach space } Y.$$

Examples of M's such that $\mathcal{F}(M)$ has property alpha

- M finite,
- $M \subset \mathbb{R}$ with Lebesgue measure 0,
- $1 \leqslant d(p,q) \leqslant D < 2$ for all $p, q \in M$, $p \neq q$.

Characterization in the case of concave metric spaces

M concave metric space. TFAE:

- $\mathcal{F}(M)$ has property α.
- M is uniformly discrete, bounded, and $\mathrm{Mol}\,(M)$ is uniformly Gromov rotund.

A norming uniformly Gromov rotund set of molecules

Theorem

M pointed metric space, $A \subset \mathrm{Mol}\,(M)$ uniformly Gromov rotund (i.e. A is a set of uniformly strongly exposed points) such that $\overline{\mathrm{co}}(A) = B_{\mathcal{F}(M)}$.

$$\implies \quad \overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y) \quad \text{for every Banach space } Y.$$

Examples

- $\mathcal{F}(M)$ with property α (with $A = \{\pm x_\lambda \colon \lambda \in \Lambda\}$);
- $M = ([0,1], |\cdot|^\theta)$ (with $A = \mathrm{Mol}\,(M)$). This one does not have property α.

Particular case (uniformly Gromov concave metric spaces)

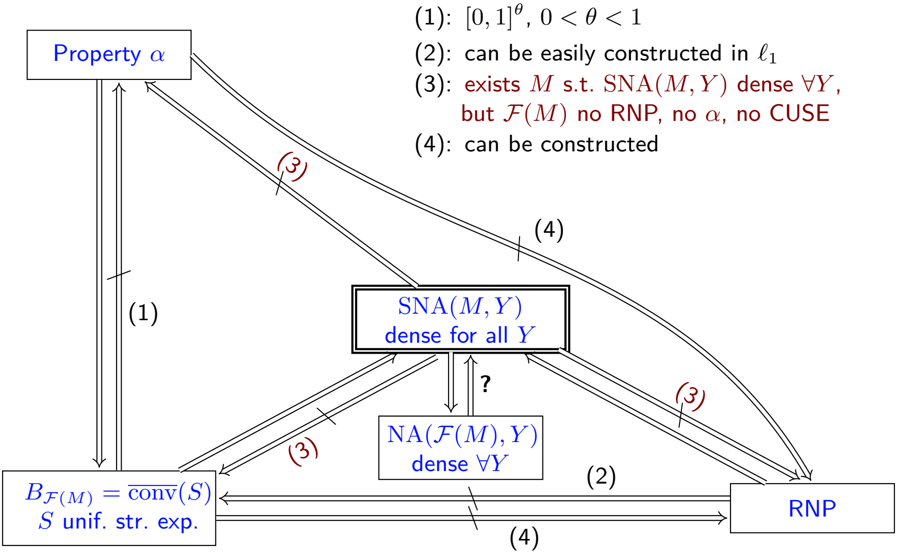
M pointed metric space. Suppose that

$$\frac{d(p,t)+d(t,q)-d(p,q)}{\min\{d(p,t),d(t,q)\}} \geqslant \rho_0 > 0 \qquad \forall p \neq q \neq t.$$

Then, $\overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y)$ for every Banach space Y.

★ We will see that something stronger happens.

Let us summarize the relations



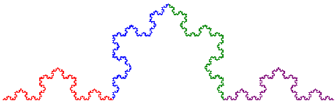
Two paradigmatic examples

Koch curve

Let $M_1 = ([0,1], |\cdot|^\theta)$, $0 < \theta < 1$.

- $\mathcal{F}(M_1)$ has RNP, so $\overline{\mathrm{SNA}(M_1, Y)} = \mathrm{Lip}_0(M_1, Y)$ $\forall Y$.
- Every molecule is strongly exposed,
- even more, $\mathrm{Mol}\,(M_1)$ is uniformly Gromov rotund.

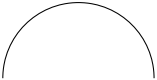
★ For $\theta = \log(3)/\log(4)$, M_1 is bi-Lipschitz equivalent to the Koch curve:



Microscopically, an small piece of M_1 is equivalent to M_1 itself.

The unit circle

Let M_2 be the upper half of the unit circle:



- We know that $\mathrm{SNA}(M_2, \mathbb{R})$ is not dense in $\mathrm{Lip}_0(M_2, \mathbb{R})$.
- So, $\mathcal{F}(M_2)$ has NOT the RNP.
- However, every molecule is strongly exposed...
- but NO subset $A \subset \mathrm{Mol}\,(M_2)$ which is uniformly Gromov rotund can be norming for $\mathrm{Lip}_0(M_2, \mathbb{R})$.

Microscopically, an small piece of M_2 is very closed to be an interval.

Weak density

Weak density

Theorem

M metric space $\implies$ $\mathrm{SNA}(M,\mathbb{R})$ is weakly sequentially dense in $\mathrm{Lip}_0(M,\mathbb{R})$.

Previously known

- $\mathcal{F}(M)$ RNP;
- Kadets–Martín–Soloviova, 2016: when M is length.

The tool

$\{f_n\} \subset \mathrm{Lip}_0(M,\mathbb{R})$ bounded with pairwise disjoint supports $\implies$ $\{f_n\}$ weakly null.

Observations

- The linear span of $\mathrm{SNA}(M,\mathbb{R})$ is always norm-dense in $\mathrm{Lip}_0(M,\mathbb{R})$;
- $\mathcal{F}(M)$ RNP $\implies$ $\mathrm{Lip}_0(M,\mathbb{R}) = \mathrm{SNA}(M,\mathbb{R}) - \mathrm{SNA}(M,\mathbb{R})$.

A by-product of our construction

Theorem

If M' is infinite or M is discrete but no uniformly discrete or M is compact (infinite) $\Longrightarrow$ then the norm of $\mathcal{F}(M)^{**}$ is octahedral.

Octahedral norm

The norm of X is octahedral iff $\forall\, Y \leqslant X$ finite-dimensional, $\forall \varepsilon > 0$, $\exists x \in S_X$ s.t.

$$\|y + \lambda x\| \geqslant (1-\varepsilon)\big(\|y\| + |\lambda|\big) \qquad \big(y \in Y,\ \lambda \in \mathbb{R}\big).$$

Equivalently

If M' is infinite or M is discrete but no uniformly discrete or M is compact (infinite) $\Longrightarrow$ every convex combination of slices of $B_{\mathrm{Lip}_0(M,\mathbb{R})}$ has diameter two.

Further results

From scalar-valued to vector-valued and viceversa

From vector-valued to scalar-valued

M metric space, $\mathrm{SNA}(M, Y)$ dense in $\mathrm{Lip}_0(M, Y)$ for some Y
$\implies \mathrm{SNA}(M, \mathbb{R})$ dense in $\mathrm{Lip}_0(M, \mathbb{R})$

★ We do not know if the density for scalar functions implies the density for all vector-valued maps, but there are some cases in which this happens:

From scalar-valued to vector-valued

M metric space such that $\overline{\mathrm{SNA}(M, \mathbb{R})} = \mathrm{Lip}_0(M, \mathbb{R})$, Y Banach space.

- If Y has property β (e.g. $c_0 \leqslant Y \leqslant \ell_\infty$), then $\overline{\mathrm{SNA}(M, Y)} = \mathrm{Lip}_0(M, Y)$.
- For compact Lipschitz maps, the same is true for $Y = C(K)$.

★ These results are proved using the concepts of ACK$_\rho$-spaces and Γ-flat operators from Cascales–Guirao–Kadets–Soloviova, 2018.

The strongly Lipschitz BPB property

The strongly Lipchitz Bishop-Phelps-Bollobás

M metric space, Y Banach space. (M, Y) has the Lip-BPBp if for every $\varepsilon > 0$ there is $\eta > 0$ such that for $F_0 \in \mathrm{Lip}_0(M, Y)$ with $\|F_0\| = 1$, $p \neq q \in M$ s.t.

$$\frac{\|F_0(p) - F_0(q)\|}{d(p, q)} > 1 - \eta,$$

there exist $F \in \mathrm{Lip}_0(M, Y)$ and $x \neq y \in M$ such that

$$1 = \|F\| = \frac{\|F(x) - F(y)\|}{d(x, y)}, \quad \|F_0 - F\| < \varepsilon \text{ and } \frac{d(p, x) + d(q, y)}{d(p, q)} < \varepsilon.$$

★ It is the Lipschitz version of the so-called BPBp for linear operators:

The BPB property for linear operators (Acosta–Aron–García–Maestre, 2008)

X, Y Banach spaces. (X, Y) has the BPBp if for every $\varepsilon > 0$ there is $\eta > 0$ such that whenever $T \in \mathcal{L}(X, Y)$, $\|T\| = 1$, $x \in S_X$ satisfy $\|Tx\| > 1 - \eta$, there exist $S \in \mathcal{L}(X, Y)$, $y \in S_X$ verifying that

$$1 = \|S\| = \|Sy\| \qquad \text{and} \qquad \|x - y\|, \|T - S\| < \varepsilon.$$

The strongly Lipschitz BPB property. II

Positive result (uniformly Gromov concave metric spaces)

$\mathrm{Mol}\,(M)$ uniformly Gromov rotund, Y arbitrary $\implies$ (M, Y) has Lip-BPBp.

★ Some particular cases:

- $M = [0,1]^\theta$ for $0 < \theta < 1$,
- M finite and concave,
- $1 \leqslant d(p,q) \leqslant D < 2$ for every $p, q \in M$, $p \neq q$,
- M concave such that $\mathcal{F}(M)$ has property α.

Partial result

M finite, Y finite-dimensional $\implies$ (M, Y) has the Lip-BPBp.

Negative examples

- Exists M finite and Y (infinite-dimensional) such that (M, Y) fails Lip-BPBp.
- $M = \mathbb{N}$ with the distance inherited from $\mathbb{R}$, then $(M, \mathbb{R})$ fails Lip-BPBp.

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