Vector space structure in the set of norm attaining functionals

(V. Kadets, G. López, M. Martín, and D. Werner)

Workshop on Infinite Dimensional Analysis
to celebrate the 60th birthday of Domingo García

Bibliography

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C. Read
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Roadmap of the talk

Norm attaining functionals

Norm attaining functionals

$x^* \in X^*$ attains its norm when

$$\exists\, x \in X,\ \|x\| = 1\ :\ x^*(x) = \|x^*\|$$

★ $\mathrm{NA}(X) = \{x^* \in X^* \colon x^*$ attains its norm$\}$

First results

- $\dim(X) < \infty \implies \mathrm{NA}(X) = X^*$ (Heine-Borel),
- X reflexive $\implies \mathrm{NA}(X) = X^*$ (Hahn-Banach),
- X non-reflexive $\implies \mathrm{NA}(X) \neq X^*$ (James),
- $\mathrm{NA}(X)$ is always norm dense in X^* (Bishop-Phelps).

Examples

- $\mathrm{NA}(c_0) = c_{00} \leqslant \ell_1$,
- $\mathrm{NA}(\ell_1) = \left\{x \in \ell_\infty \colon \|x\|_\infty = \max_n \{|x(n)|\}\right\}$.

Lineability

Examples

- $\mathrm{NA}(c_0) = c_{00} \leqslant \ell_1$,
- $\mathrm{NA}(\ell_1) = \left\{x \in \ell_\infty \colon \|x\|_\infty = \max_n\{|x(n)|\}\right\}$.

- Note that $\mathrm{NA}(c_0)$ is a linear space, but $\mathrm{NA}(\ell_1)$ is not.
- However, $\mathrm{NA}(\ell_1)$ contains the infinite-dimensional space c_0.

Lineability

Recall that a subset S of a vector space V is called lineable if $S \cup \{0\}$ contains an infinite-dimensional subspace.

Lineability of $\mathrm{NA}(X)$

Main question

Lineability of $\mathrm{NA}(X)$**?**

More concretely,

Problems (G. Godefroy, 2001)

(G_∞) Does $\mathrm{NA}(X)$ always contain an infinite-dimensional linear subspace?

(G) Does $\mathrm{NA}(X)$ always contain a linear subspace of dimension 2?

The case of dimension 1 is taken care of by the Hahn-Banach theorem!

Note that (G_∞) holds in all classical spaces.

Proximinality

Proximinal subspace

$Y \leqslant X$ is proximinal iff

$$\forall x \in X \ \exists y_0 \in Y \ : \quad \|x - y_0\| = \inf\{\|x-y\| : y \in Y\} = \operatorname{dist}(x, Y)$$

- Y proximinal iff $Q(B_X) = B_{X/Y}$ $\qquad$ ($Q : X \longrightarrow X/Y$ quotient map)
- $x^* \in \mathrm{NA}(X) \iff \ker x^*$ proximinal.

Problem (I. Singer, 1974)

(S) Is there always a proximinal subspace of codimension 2?

Proximinality and norm attaining functionals

The two main problems

(S) Does there always exist a proximinal subspace of codimension 2?

(G) Does $\mathrm{NA}(X)$ always contain a linear subspace of dimension 2?

Important result (Garkavi, 1967)

$$Y \leqslant X \text{ proximinal of finite codimension } \implies Y^{\perp} \subset \mathrm{NA}(X).$$

Therefore. . .

If (S) is true, then (G) is true.

The converse result is not true

There exist X and finite codimensional Y such that $Y^{\perp} \subset \mathrm{NA}(X)$ but Y is not proximinal (Phelps, 1963)

Read's and Rmoutil's theorems

Theorem (Read, 2013)

There is a counterexample X_R to (S).

As (S) $\Rightarrow$ (G), X_R is a natural candidate for a counterexample to (G).

Actually,

Theorem (Rmoutil, 2015)

- X/Y strictly convex and $Y^\perp \subset \mathrm{NA}(X) \implies Y$ proximinal.
- $\dim X_R/Y = 2 \implies X_R/Y$ strictly convex.
- Consequently, X_R is also a counterexample to (G).

A simplification of Rmoutil's proof by Kadets/López/Martín:

Proposition

X_R^{**} is strictly convex; hence *all* quotients of X_R are strictly convex.

Read's construction

X_R is a renorming of c_0:

Let $\Omega = \{(s_n)\colon (s_n)$ has finite support, all $s_n \in \mathbb{Q}\} \subset \ell_1$.
Enumerate $\Omega = \{u_1, u_2, \ldots\}$ so that every element is repeated infinitely often.
Take a sequence of integers (a_n) such that

$$a_k > \max \operatorname{supp} u_k, \quad a_k \geqslant \|u_k\|_{\ell_1}.$$

Renorm c_0 by

$$p(x) = \|x\|_\infty + \sum_k 2^{-a_k^2} |\langle u_k - e_{a_k}, x\rangle|.$$

Then Read shows that (c_0, p) fails (S), and Rmoutil shows, relying on Read's work, that (c_0, p) fails (G).

The proof of Read's theorem is not trivial at all!!!!!

A new, direct approach to (G)

We four are more used to norm-attainment than to proximinality,
so we changed the point of view:

We want to show directly that certain Banach spaces have a renorming failing (G) and *hence* have a renorming failing (S).

Let $R\colon X \longrightarrow \ell_1$ be continuous; we renorm X by

$$p(x) = \|x\| + \|Rx\|_{\ell_1}.$$

More precisely, let $[Rx](n) = 2^{-n} v_n^*(x), \quad (v_n^*) \subset B_{X^*}$, so

$$p(x) = \|x\| + \sum_{n=1}^{\infty} \frac{v_n^*(x)}{2^n}.$$

(Note that Read's renorming is of this type.)

Aim

Under suitable assumptions, the v_n^* can be chosen so that (X, p) fails (G) (and hence fails (S)).

A tentative calculation

$p(x) = \|x\| + \sum 2^{-n}|v_n^*(x)|$. Then $B_{(X^*,p^*)} = B_X + \sum 2^{-n}[-v_n^*, v_n^*]$ (Minkowski sum).

Let $x^* \in \mathrm{NA}_1(X,p)$ be norm attaining at x; then

$$x^* = x_0^* + \sum 2^{-n} t_n v_n^*$$

for some $x_0^* \in \mathrm{NA}_1(X)$ and $t_n = \operatorname{sign} v_n^*(x)$ whenever $v_n^*(x)$ is nonzero.
Write the same decomposition for $y^* \in \mathrm{NA}_1(X,p)$, norm attaining at y:

$$y^* = y_0^* + \sum 2^{-n} t_n' v_n^*.$$

Let's try to prove that $x^* + y^* \notin \mathrm{NA}(X,p)$: Otherwise we would have a similar decomposition for $z^* = (x^* + y^*)/\|x^* + y^*\|$:

$$z^* = z_0^* + \sum 2^{-n} s_n v_n^*.$$

Sort the items, setting $\lambda = \|x^* + y^*\|$:

$$0 = x^* + y^* - \lambda z^* = [x_0^* + y_0^* - \lambda z_0^*] + \left[\sum (t_n + t_n' - \lambda s_n) v_n^*\right]$$

Wish list

$$0 = [x_0^* + y_0^* - \lambda z_0^*] + \left[\sum (t_n + t_n' - \lambda s_n) v_n^*\right]$$

We now **wish** to select the v_n^* to be sort of "orthogonal" to $\operatorname{span}(\mathrm{NA}(X))$ (which contains the first bracket) so that both brackets vanish.

In addition we **wish** the v_n^* to have some Schauder basis character so that we can deduce from $\sum (t_n + t_n' - \lambda s_n) v_n^* = 0$ that all $t_n + t_n' - \lambda s_n = 0$.

Finally we **wish** the support points x and y to be distinct, and we **wish** the span of the v_n^* to be dense enough to separate x and y for many n, i.e., $v_n^*(x) < 0 < v_n^*(y)$ and thus $t_n + t_n' = 0$ fairly often, while at the same time $s_n \neq 0$ for at least one of those n.

This contradiction would show that $x^* + y^* \notin \mathrm{NA}(X, p)$.

Modest subspaces

Definition: operator range, (weak*) modest subspace

- V, W Banach spaces, $T\colon V \longrightarrow W$ injective. Then $T(V)$ is called an operator range.
- $Z \leqslant W$ is modest if there is a separable dense operator range Y with $Y \cap Z = \{0\}$.
- If W is a dual space, then $Z \leqslant W$ is weak* modest if there is a separable weak* dense operator range Y with $Y \cap Z = \{0\}$.

Note that the choice of V in the definition of a modest subspace is at our discretion since

$$E, F \text{ separable} \implies \exists \text{ continuous injection } S\colon E \longrightarrow F \text{ with dense range.}$$

Example

$$\{(s_n)\colon (s_n) \text{ has finite support}\} \text{ is modest in } \ell_1.$$

Indeed, let $A_r(\mathbb{D})$ the real Banach space of those function of the disk algebra which takes real valued on the real axis;
define $T\colon A_r(\mathbb{D}) \longrightarrow \ell_1$ by $[Tf](n) = 2^{-n} f(2^{-n})$; then T has dense range and every non-null sequence in $T(A_r(\mathbb{D}))$ can only take the value 0 finitely many times.

Main Theorem

Theorem

If $\operatorname{span}(\mathrm{NA}(X))$ is weak* modest, then X has a renorming that fails (G) and, consequently, fails (S). (We call such an equivalent norm a Read norm.)

Recall ansatz: $p(x) = \|x\| + \sum 2^{-n}|v_n^*(x)|$; how to choose the v_n^*?

Lemma

Let $Y \leqslant X^*$ be a separable operator range. Then there is an injective operator $S\colon \ell_1 \longrightarrow X^*$ such that, for $v_n^* = S(e_n)$, the set $\{v_n^*/\|v_n^*\|\}$ is dense in S_Y.

With this choice of v_n^* it is possible to fulfill our wishes: the v_n^* are "orthogonal" to $\mathrm{NA}(X)$ (wish #1), they are an injective image of a Schauder basis (wish #2) and sufficiently dense (wish #4). As for wish #3, if $x = y$, then $x \neq -y$ and one should look at $x^* - y^*$!

Thus we can show that for linearly independent $x^*, y^* \in \mathrm{NA}(X,p)$ of norm 1, at most one of $x^* \pm y^*$ can be in $\mathrm{NA}(X,p)$.

First consequence

Example (we recuperate Read's and Rmoutil's results)

c_0 admits a Read norm, that is, a norm failing (G) and hence failing (S).

Indeed, $\mathrm{NA}(c_0) = c_{00}$ is modest in ℓ_1.

Note

The original construction by Read is NOT a particular case of ours:

Indeed, both norms are of the form $p(x) = \|x\| + \sum 2^{-n}|v_n^*(x)|$, but

- in the original Read's construction, the v_n^*'s belong to $\mathrm{NA}(c_0)$,
- in our construction, the v_n^*'s are "orthogonal" to $\mathrm{NA}(c_0)$.

More consequences I

Proposition

A separable Banach space containing a copy of c_0 admits a Read norm.

Indeed, renorm X so that $X = c_0 \oplus_\infty E$; then $X^* = \ell_1 \oplus_1 E^*$ and $\mathrm{NA}(X) \subset \mathrm{NA}(c_0) \oplus_1 E^*$.The latter can be shown to be contained in a weak* modest subspace.

Example

$C[0,1]$ admits an equivalent Read norm.

Norms with additional properties

X separable containing c_0. Then for each $0 < \varepsilon < 2$ there is a Read norm p_ε on X with the following properties:

- p_ε is strictly convex and smooth,
- p_ε^* is strictly convex,
- p_ε^* is $(2-\varepsilon)$-rough; i.e., every slice of $B_{(X,p_\varepsilon)}$ has diameter $\geqslant 2-\varepsilon$,
- If moreover X^* is separable, then p_ε^{**} is strictly convex.

More consequences II

Theorem

A Banach space containing a copy of c_0 which has a countable system of norming functionals admits a Read norm.

$\{x_n^*\}$ is a norming system if $x \longmapsto \sup_n |x_n^*(x)|$ is an equivalent norm. Such a space is isomorphic to a closed subspace of ℓ_∞ and vice versa.

Example

ℓ_∞ admits an equivalent Read norm.

Norms with additional properties

X containing c_0 which a countable system of norming functionals. Then for each $0 < \varepsilon < 2$ there is a Read norm p_ε on X so that

- p_ε is strictly convex,
- p_ε^* is $(2-\varepsilon)$-rough; i.e., every slice of $B_{(X,p_\varepsilon)}$ has diameter $\geqslant 2-\varepsilon$,
- actually, every convex combination of slices (hence every relatively weakly open subset) of $B_{(X,p_\varepsilon)}$ has diameter $\geqslant 2-\varepsilon$.

Open problems

Open problem

Does every separable non-reflexive Banach space admit an equivalent Read norm **?**

- $\ell_\infty(\Gamma)$ with Γ uncontable does not admit a Read norm

Some remarks

- Our construction needs $\operatorname{span}(\mathrm{NA}(X))$ to be “small” (weak-star modest).
- This is not always possible: if X RNP, then $\operatorname{span}(\mathrm{NA}(X)) = X^*$ (Bourgain).
- Actually, if $\mathrm{NA}(X)$ is residual, then $\operatorname{span}(\mathrm{NA}(X)) = X^*$.

An stronger result

X separable, $\operatorname{span}(\mathrm{NA}(X))$ second category $\implies$ $\operatorname{span}(\mathrm{NA}(X)) = X^*$.

Two concrete problems

- Does ℓ_1 admit a Read norm? (observe that $\operatorname{span}(\mathrm{NA}(X)) = X^*$ for every $X \simeq \ell_1$)
- Does $L_1[0,1]$ admit a norm such that $\operatorname{span}(\mathrm{NA}(X))$ is weak-star modest? (observe that $\mathrm{NA}(L_1[0,1])$ is first category but $\operatorname{span}(\mathrm{NA}(L_1[0,1])) = L_1[0,1]^*$)