Spear operators on Banach spaces

Miguel Martín

http://www.ugr.es/local/mmartins

UNIVERSIDAD DE GRANADA

The 13$^{\mathrm{th}}$ ILJU School of Mathematics

Banach Spaces and Related Topics
January 26 – February 2, 2018
National Institute for Mathematical Sciences, Daejeon, Korea

Our objective

Our objective is to study spear operators on Banach spaces that is, (bounded linear) operators $G : X \longrightarrow Y$ satisfying that

$$\max_{|\omega|=1} \big\| G + \omega\, T \big\| = 1 + \|T\|$$

for every other operator $T : X \longrightarrow Y$.

Where is the material contained?

V. Kadets, M. Martín, J. Merí, A. Pérez
Spear operators on Banach spaces
Lecture Notes in Mathematics, **2205**, Springer Verlag, 2018

What will we study?

- Many examples,
- characterizations in some environments,
- isomorphic and isometric consequences. . .

What is the main motivation for this?

The case when G is the identity operator

Roadmap of the talk

Motivation: numerical range

Section 1

The case of the identity

When $G = \mathrm{Id}$

X Banach space. The identity operator is a spear operator iff

$$\max_{|\omega|=1} \|\mathrm{Id}_X + \omega T\| = 1 + \|T\| \qquad \text{(aDE)}$$

for every bounded linear operator $T : X \longrightarrow X$.

When does this happen?

- when $X = C_0(L)$ for any Hausdorff locally compact space,
- when $X = L_1(\mu)$ for any measure μ,
- when X is the disk algebra $A(\mathbb{D})$,
- and many more cases...

But...

What is the origin and the meaning of equality (aDE)?

Self-adjoint operators on Hilbert spaces

Old result

H Hilbert space, $T : H \longrightarrow H$ self-adjoint, then

$$\|T\| = \sup\Big\{|\langle Tx, x\rangle| : x \in H,\ \|x\| = 1\Big\}$$

(by polarization).

Consequence

This implies that for $T : H \longrightarrow H$ self-adjoint, there is ω with modulus one such that

$$\|\mathrm{Id} + \omega\, T\| = 1 + \|T\|.$$

Numerical radius for Hilbert space operators

Numerical range and numerical radius

For arbitrary $T : H \longrightarrow H$, define

$$V(T) := \big\{\langle Tx, x\rangle \colon x \in H,\ \|x\| = 1\big\}, \qquad v(T) := \sup\big\{|\lambda| \colon \lambda \in V(T)\big\}$$

to be, respectively, the numerical range and numerical radius of T.

Proposition

- T self-adjoint $\implies v(T) = \|T\|$;
- For general T, $v(T) = \|T\| \iff \max\limits_{|\omega|=1} \|\mathrm{Id} + \omega T\| = 1 + \|T\|$.

Some properties

H Hilbert space, $T : H \longrightarrow H$

- $V(T)$ is convex.
- $\mathrm{Spec}(T) \subseteq \overline{V(T)}$ (complex case).
- If T is normal, then $\overline{V(T)} = \overline{\mathrm{conv}}\,\mathrm{Spec}(T)$ (complex case).

Numerical range: Hilbert spaces. Motivation

Some reasons to study numerical ranges

- It gives a "picture" of the operator which allows to "see" many properties (algebraic or geometrical) of it.
- It is a comfortable way to study the spectrum.
- It is useful to work with some concepts like hermitian operator, skew-hermitian operator, dissipative operator. . .

What happens for general Banach spaces?

Let first present the needed notation

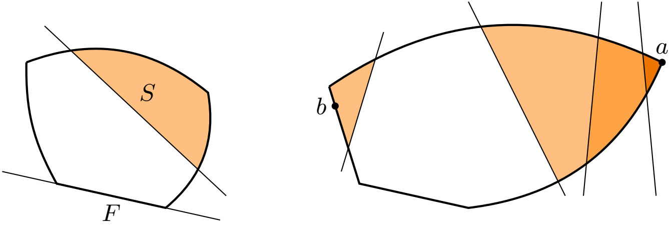
Basic notation I

- $\mathbb{K}$ base field ($\mathbb{R}$ or $\mathbb{C}$):
    - $\mathbb{T}$ modulus-one scalars,
    - $\operatorname{Re} z$ real part of z ($\operatorname{Re} z = z$ if $\mathbb{K} = \mathbb{R}$).
- X, Y Banach spaces:
    - S_X unit sphere, B_X unit ball,
    - X^* dual space,
    - $L(X,Y)$ bounded linear operators,
    - $L(X) := L(X,X)$.
- $T \in L(X,Y)$:
    - $T^* \in L(Y^*,X^*)$ adjoint operator of T.

Basic notation II

X Banach space, $B \subset X$:

- $\|B\| = \sup\{\|b\| \colon b \in B\}$,
- B is rounded if $\mathbb{T}B = B$,
- $\operatorname{conv}(B)$ convex hull of B, $\overline{\operatorname{conv}}(B)$ closed convex hull of B,
- $\operatorname{aconv}(B) = \operatorname{conv}(\mathbb{T}\,B)$ absolutely convex hull of B, $\overline{\operatorname{aconv}}(B) = \overline{\operatorname{conv}}(\mathbb{T}\,B)$,
- $\operatorname{Slice}(B, x^*, \alpha) := \big\{x \in B \colon \operatorname{Re} x^*(x) > \sup \operatorname{Re} x^*(B) - \alpha\big\}$,
  where $x^* \in X^*$ and $\alpha > 0$,
- $\operatorname{Face}(B, x^*) := \big\{x \in B \colon \operatorname{Re} x^*(x) = \sup \operatorname{Re} x^*(B)\big\}$,
  where $x^* \in X^*$ attains its supremum on B.
- $\operatorname{ext}(B)$ extreme points of B,
- $\operatorname{dent}(B)$ denting points of B (i.e. those belonging to arbitrarily small slices).



Banach space numerical range and numerical radius

Definition (Bauer 1962; Lumer, 1961)

X Banach space, $T \in L(X)$,

$$V(T) := \left\{x^*(Tx) \colon x^* \in S_{X^*},\ x \in S_X,\ x^*(x) = 1\right\} \quad v(T) := \sup\left\{|\lambda| \colon \lambda \in V(T)\right\}$$

are the numerical range and numerical radius of T.

Some properties

- $v(\cdot)$ is a seminorm,
- $\operatorname{Spec}(T) \subset \overline{V(T)}$ (complex case),
- if $X = H$, both definitions coincide.
- This concept allows to carry to arbitrary Banach spaces the concept of hermitian, skew-hermitian, dissipative. . . operators.

Important result (Duncan-McGregor-Price-White, 1970)

$$v(T) = \|T\| \quad \Longleftrightarrow \quad \|\mathrm{Id} + \mathbb{T}\,T\| = 1 + \|T\|.$$

Numerical index of a Banach space

Definition (Lumer, 1968)

X Banach space, its numerical index is

$$\begin{aligned} n(X) &:= \inf \left\{ v(T)\colon\ T \in L(X),\ \|T\| = 1 \right\} \\ &= \max \left\{ k \geqslant 0\colon k\,\|T\| \leqslant v(T)\ \ \forall\, T \in L(X) \right\}. \end{aligned}$$

Some basic properties

- $n(X) = 1$ iff v and $\|\cdot\|$ coincide iff $\|\mathrm{Id} + \mathbb{T}\,T\| = 1 + \|T\|\ \ \forall T \in L(X)$ iff Id is a spear operator,
- $n(X) = 0$ iff v is not an equivalent norm in $L(X)$,
- X complex $\ \Rightarrow\ n(X) \geqslant 1/\mathrm{e}$.

  (Bohnenblust–Karlin, 1955; Glickfeld, 1970)
- Actually,

$$\{n(X)\colon X \text{ complex},\ \dim(X) = 2\} = [\mathrm{e}^{-1}, 1]$$
$$\{n(X)\colon X \text{ real},\ \dim(X) = 2\} = [0, 1]$$

(Duncan–McGregor–Pryce–White, 1970)

Numerical index of Banach spaces: examples (I)

Some examples

1. H Hilbert space, $\dim(H) > 1$,

$$n(H) = 0 \quad \text{if } H \text{ is real}$$
$$n(H) = 1/2 \quad \text{if } H \text{ is complex}$$

2. $n\big(L_1(\mu)\big) = 1$ $\quad\mu$ positive measure
   $n\big(C(K)\big) = 1$ $\quad K$ compact Hausdorff space

   (Duncan et al., 1970)

3. If A is a C^*-algebra $\Rightarrow \begin{cases} n(A) = 1 & A \text{ commutative} \\ n(A) = 1/2 & A \text{ not commutative} \end{cases}$

   (Huruya, 1977; Kaidi–Morales–Rodríguez, 2000)

4. If A is a function algebra $\Rightarrow n(A) = 1$

   (Werner, 1997)

Numerical index of Banach spaces: some examples (II)

More examples

5. For $n \geqslant 2$, the unit ball of X_n is a $2n$ regular polygon:

$$n(X_n) = \begin{cases} \tan\left(\frac{\pi}{2n}\right) & \text{if } n \text{ is even,} \\ \\ \sin\left(\frac{\pi}{2n}\right) & \text{if } n \text{ is odd.} \end{cases}$$

(M.–Merí, 2007)

6. Every finite-codimensional subspace of $C[0,1]$ has numerical index one

(Boyko–Kadets–M.–Werner, 2007)

Numerical index of Banach spaces: some examples (III)

Even more examples

7. Numerical index of L_p-spaces, $1 < p < \infty$:
   - $n\big(L_p[0,1]\big) = n(\ell_p) = \lim_{m\to\infty} n\big(\ell_p^{(m)}\big)$.

     (Ed-Dari, 2005 & Ed-Dari-Khamsi, 2006)
   - In the real case,

$$\max\left\{\frac{1}{2^{1/p}},\ \frac{1}{2^{1/q}}\right\} M_p \leqslant n\big(\ell_p^{(2)}\big) \leqslant M_p$$

and $M_p = v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \max_{t\in[0,1]} \dfrac{|t^{p-1} - t|}{1+t^p}$

(M.–Merí, 2009)

   - In the real case, $n\big(L_p(\mu)\big) \geqslant \dfrac{M_p}{8e}$.
   - In particular, $n\big(L_p(\mu)\big) > 0$ for $p \neq 2$.

(M.–Merí–Popov, 2009)

Motivation: Banach spaces with numerical index one

Section 2

Banach spaces with numerical index one

Numerical index one

Recall that X has numerical index one ($n(X) = 1$) iff

$$\|T\| = \sup\Big\{|x^*(Tx)| : x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1\Big\}$$

(i.e. $v(T) = \|T\|$) for every $T \in L(X)$.
Equivalently, Id is a spear operator.

Examples

$C(K)$, $L_1(\mu)$, $A(\mathbb{D})$, H^∞, finite-codimensional subspaces of $C[0,1]$...

This is a property of X which is very complicated to work with as one has to deal with **all** the operators on the space.

Leading open question

X Banach space with numerical index one $\implies$ $X \supset c_0$ or $X \supset \ell_1$ **?**

How to deal with numerical index one property?

One the one hand: weaker properties

- In a general Banach space, we only can construct compact (nuclear) operators.
- Actually, we only may easily calculate the norm of rank-one operators.
- Most of the results we know for Banach spaces with numerical index one are actually true for Banach spaces with the alternative Daugavet property (ADP), that is, those Banach spaces satisfying:
  - $v(T) = \|T\|$ for every rank-one T,
  - equivalently, $\|\mathrm{Id} + \mathbb{T}\, T\| = 1 + \|T\|$ for every T rank-one.

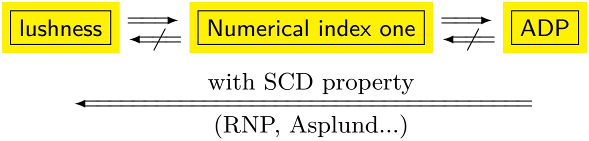
One the other hand: stronger properties

- We do not know any operator-free characterization of Banach spaces with numerical index one.
- When we know that a Banach space has numerical index one (or that it can be renormed with numerical index one), we actually prove more.
- There are some sufficient geometrical conditions.
- The weakest property is called lushness.

How to deal with numerical index one property?

Relationship between the properties

- One of the key ideas to get interesting results for Banach spaces with numerical index one is to study when the three properties below are equivalent.
- A very interesting property appears: the slicely countably determination.



The numerical index one has isomorphic consequences

Question

Does every Banach space admit an equivalent norm with numerical index one **?**

Negative answer (López–M.–Payá, 1999)

Not every Banach space can be renormed to have numerical index one.
Concretely:

- If X is real, reflexive, and $\dim(X) = \infty$, then $n(X) < 1$.
- Actually, if X is real, X^{**}/X separable and $n(X) = 1$, then X is finite-dimensional.
- Moreover, if X is real, RNP, $\dim(X) = \infty$, and $n(X) = 1$, then $X \supset \ell_1$.

Proving the 1999 results (I)

Lemma

X Banach space, $n(X) = 1$
$\Longrightarrow$ $|x_0^*(x_0)| = 1$ for every $x_0^* \in \operatorname{ext}(B_{X^*})$ and every $x_0 \in \operatorname{dent}(B_X)$.

Proof:

- Fix $\varepsilon > 0$. As x_0 denting point, $\exists y^* \in S_{X^*}$ and $\alpha > 0$ such that

$$\|z - x_0\| < \varepsilon \quad \text{whenever } z \in B_{X^*} \text{ satisfies } \operatorname{Re} y^*(z) > 1 - \alpha.$$

- (Choquet's lemma): $x_0^* \in \operatorname{ext}(B_{X^*})$, $\exists y \in S_X$ and $\beta > 0$ such that

$$|z^*(x_0) - x_0^*(x_0)| < \varepsilon \quad \text{whenever } z^* \in B_{X^*} \text{ satisfies } \operatorname{Re} z^*(y) > 1 - \beta.$$

- Let $T = y^* \otimes y \in L(X)$. $\|T\| = 1 \implies v(T) = 1$.
- We may find $x \in S_X$, $x^* \in S_{X^*}$, such that

$$x^*(x) = 1 \qquad \text{and} \qquad |x^*(Tx)| = |y^*(x)||x^*(y)| > 1 - \min\{\alpha, \beta\}.$$

- By choosing suitable $s, t \in \mathbb{T}$ we have

$$\operatorname{Re} y^*(sx) = |y^*(x)| > 1 - \alpha \qquad \& \qquad \operatorname{Re} tx^*(y) = |x^*(y)| > 1 - \beta.$$

- It follows that $\|sx - x_0\| < \varepsilon$ and $|tx^*(x_0) - x_0^*(x_0)| < \varepsilon$, and so

$$1 - |x_0^*(x_0)| \leqslant |tx^*(sx) - x_0^*(x_0)| \leqslant |tx^*(sx) - tx^*(x_0)| + |tx^*(x_0) - x_0^*(x_0)| < 2\varepsilon. \checkmark$$

Proving the 1999 results (II)

Proposition

X real, $A \subset S_X$ infinite with $|x^*(a)| = 1$ $\forall x^* \in \operatorname{ext}(B_{X^*})$, $\forall a \in A$.
$\Longrightarrow$ $X \supseteq c_0$ or $X \supseteq \ell_1$.

Proof:

- $X \supseteq \ell_1$ ✓
- (Rosenthal ℓ_1-theorem): Otherwise, $\exists \{a_n\} \subseteq A$ non-trivial weak Cauchy.
- Consider Y the closed linear span of $\{a_n : n \in \mathbb{N}\}$.
- $\|a_n - a_m\| = 2$ if $n \neq m \implies \dim(Y) = \infty$.
- (Krein-Milman theorem): every $y^* \in \operatorname{ext}(B_{Y^*})$ has an extension which belongs to $\operatorname{ext}(B_{X^*})$.
- So, $|y^*(a_n)| = 1$ $\forall y^* \in \operatorname{ext}(B_{Y^*})$, $\forall n \in \mathbb{N}$.
- $\{a_n\}$ weak Cauchy $\implies$ $\{y^*(a_n)\}$ is eventually 1 or -1.
- Then $\operatorname{ext}(B_{Y^*}) = \bigcup_{k\in\mathbb{N}} (E_k \cup -E_k)$ where

$$E_k = \{y^* \in \operatorname{ext}(B_{Y^*}) : y^*(a_n) = 1 \text{ for } n \geqslant k\}.$$

- $\{a_n\}$ separates points of Y^* $\implies$ E_k finite, so $\operatorname{ext}(B_{Y^*})$ countable.
- (Fonf): $Y \supseteq c_0$. So, $X \supseteq c_0$. ✓

Proving the 1999 results (III)

Lemma

X Banach space, $n(X) = 1$
$\Longrightarrow$ $|x_0^*(x_0)| = 1$ for every $x_0^* \in \operatorname{ext}(B_{X^*})$ and every $x_0 \in \operatorname{dent}(B_X)$.

Proposition

X real, $A \subset S_X$ infinite with $|x^*(a)| = 1$ $\forall x^* \in \operatorname{ext}(B_{X^*})$, $\forall a \in A$.
$\Longrightarrow$ $X \supseteq c_0$ or $X \supseteq \ell_1$.

Main consequence

X real, RNP, $\dim(X) = \infty$, and $n(X) = 1$ $\Longrightarrow$ $X \supseteq \ell_1$.

Proof.

- X RNP, $\dim(X) = \infty$ $\Longrightarrow$ $\exists$ infinitely many denting points of B_X.
- Therefore, $X \supseteq c_0$ or $X \supseteq \ell_1$.
- If X RNP, then $X \not\supseteq c_0$. ✓

Sufficient conditions for numerical index one

Some sufficient conditions

Let X be a Banach space. Consider:

(a) **Lindenstrauss, 1964:** X has the 3.2.I.P. if the intersection of every family of three mutually intersecting balls is not empty.

(b) **Fullerton, 1961:** X is a CL-space if B_X is the absolutely convex hull of every maximal face of S_X.

(c) **Lima, 1978:** X is an almost-CL-space if B_X is the closed absolutely convex hull of every maximal face of S_X.

$$\boxed{\text{(a)}} \underset{\not\Longleftarrow}{\Longrightarrow} \boxed{\text{(b)}} \underset{\not\Longleftarrow}{\Longrightarrow} \boxed{\text{(c)}} \underset{\not\Longleftarrow}{\Longrightarrow} \boxed{n(X)=1}$$

Showing that (c) $\Longrightarrow$ $n(X) = 1$, one realizes that (c) is too much.

Lushness (Boyko–Kadets–M.–Werner, 2007)

X is lush if given $x, y \in S_X$, $\varepsilon > 0$, there is $x^* \in S_{X^*}$ such that

$$x \in \operatorname{Slice}(B_X, x^*, \varepsilon) \quad \text{and} \quad \operatorname{dist}\Big(y, \operatorname{aconv}\big(\operatorname{Slice}(B_X, x^*, \varepsilon)\big)\Big) < \varepsilon.$$

Definition and first property

Lushness (Boyko–Kadets–M.–Werner, 2007)

X is lush if given $x, y \in S_X$, $\varepsilon > 0$, there is $x^* \in S_{X^*}$ such that

$$x \in \operatorname{Slice}(B_X, x^*, \varepsilon) \quad \text{and} \quad \operatorname{dist}\Big(y, \operatorname{aconv}\big(\operatorname{Slice}(B_X, x^*, \varepsilon)\big)\Big) < \varepsilon.$$

Theorem (Boyko–Kadets–M.–Werner, 2007)

X lush $\implies$ $n(X) = 1$.

Proof.

- $T \in L(X)$ with $\|T\| = 1$, $\varepsilon > 0$. Find $y_0 \in S_X$ which $\|Ty_0\| > 1 - \varepsilon$.
- Use lushness for $x_0 = Ty_0/\|Ty_0\|$ and y_0 to get $x^* \in S_{X^*}$ and

$$v = \sum_{i=1}^{n} \lambda_i \theta_i x_i \quad \text{where } \; x_i \in \operatorname{Slice}(B_X, x^*, \varepsilon),\ \lambda_i \in [0,1],\ \sum \lambda_i = 1,\ \theta_i \in \mathbb{T},$$

$$\text{with} \qquad \operatorname{Re} x^*(x_0) > 1 - \varepsilon \qquad \text{and} \qquad \|v - y_0\| < \varepsilon.$$

- Then $|x^*(Tv)| = \left| x^*(x_0) - x^*\left(T\left(\frac{y_0}{\|Ty_0\|} - v\right)\right)\right| \sim \|T\|$.
- By a convexity argument, $\exists\, i$ such that $|x^*(Tx_i)| \sim \|T\|$ and $\operatorname{Re} x^*(x_i) \sim 1$.
- Then $\|\mathrm{Id} + \mathbb{T}\,T\| \sim 1 + \|T\| \implies v(T) \sim \|T\|$. ✓

Reformulations of lushness and applications

Proposition (Boyko–Kadets–M.–Merí, 2009)

X Banach space. TFAE:

- X is lush,
- Every separable $E \subset X$ is contained in a separable lush Y with $E \subset Y \subset X$.

Separable lush spaces (Kadets–M.–Meri–Payá, 2009; Lee–M., 2012)

X separable. TFAE:

- X is lush.
- There is $G \subseteq S_{X^*}$ norming for X such that

$$B_X = \overline{\operatorname{aconv}}\,(\operatorname{Face}(B_X, x^*)) \qquad (x^* \in G).$$

Therefore, $|x^{**}(x^*)| = 1 \ \forall x^{**} \in \operatorname{ext}(B_{X^{**}}) \ \forall x^* \in G$.

An important consequence

Showed in the previous slide...

X lush separable, $\dim(X) = \infty \implies$ there is $G \in S_{X^*}$ infinite such that

$$|x^{**}(x^*)| = 1 \qquad \big(x^{**} \in \operatorname{ext}(B_{X^{**}}),\ x^* \in G\big).$$

Proposition (López–M.–Payá, 1999)

X real, $A \subset S_X$ infinite with $|x^*(a)| = 1$ $\forall x^* \in \operatorname{ext}(B_{X^*})$, $\forall a \in A$.
$\implies X \supseteq c_0$ or $X \supseteq \ell_1$.

Main consequence

X real lush, $\dim(X) = \infty \implies X^* \supseteq \ell_1$.

Proof.

- There is $E \subseteq X$ infinite-dimensional, separable, and lush.
- Then $E^* \supseteq c_0$ or $E^* \supseteq \ell_1 \implies E^* \supseteq \ell_1$.
- By the "lifting" property of $\ell_1 \implies X^* \supseteq \ell_1$. ✓

Lushness is not equivalent to numerical index one

Example (Kadets–M.–Merí–Shepelska, 2009)

There is a separable Banach space $\mathcal{X}$ such that

- $\mathcal{X}^*$ is lush but $\mathcal{X}$ is not lush.
- Since $n(\mathcal{X}^*) = 1$, also $n(\mathcal{X}) = 1$.
- But the set

$$\{x^* \in S_{\mathcal{X}^*} : |x^{**}(x^*)| = 1 \text{ for every } x^{**} \in \operatorname{ext}(B_{\mathcal{X}^{**}})\}$$

is empty.

Remark

We cannot expect to show that $X^* \supseteq \ell_1$ when $n(X) = 1$ using the previous ideas for lush spaces.

Slicely countably determined Banach spaces

Section 3

Two classical concepts: Radon-Nikodým property and Asplund spaces

The Radon-Nikodým property or RNP (1930's)

- X has the RNP iff the Radon-Nikodým theorem is valid for X-valued meassures;
- Equivalently [1960's], every bcc subset contains a denting point.

X Asplund $\Longleftrightarrow$ X^* RNP

Reflexive (say) $\Longrightarrow$ (RNP and Asplund)

(RNP or Asplund) $\Longrightarrow$???

Asplund spaces (1960's)

- X is an Asplund space if every continuous convex real-valued function defined on an open subset of X is Frechet-differentiable on a dense subset;
- Equivalently [1970's], every separable subspace has separable dual.

SCD sets and spaces: Definitions and examples

SCD sets

$A \subset X$ bounded convex is slicely countably determined (SCD) if there is a sequence $\{S_n : n \in \mathbb{N}\}$ of slices of A satisfying one of the following equivalent conditions:

- every slice of A contains one of the S_n's,
- $A \subseteq \overline{\operatorname{conv}}(B)$ if $B \subseteq A$ satisfies $B \cap S_n \neq \emptyset \ \forall n$,
- given $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in S_n \ \forall n \in \mathbb{N}$, $A \subseteq \overline{\operatorname{conv}}\big(\{x_n : n \in \mathbb{N}\}\big)$.

SCD spaces

X is Slicely Countably Determined (SCD) if so are all its bounded convex subsets.

Avilés–Kadets–M.–Merí–Shepelska, 2010

Remarks

- A is SCD iff $\overline{A}$ is SCD.
- If A is SCD, then it is separable.

Main examples of SCD sets and spaces

Examples of sets

$A \subset X$ **separable** bounded and convex.

1. (Easy): A RNP $\Longrightarrow$ A is SCD,
2. (Easy): A Asplund $\Longrightarrow$ A is SCD,
3. (Main): $A \not\supseteq \ell_1$ $\Longrightarrow$ A is SCD,
4. $B_{C[0,1]}$ and $B_{L_1[0,1]}$ are not SCD.

Examples of spaces

X **separable** Banach space.

1. X RNP $\Longrightarrow$ X is SCD,
2. (Easy): X Asplund $\Longrightarrow$ X is SCD,
3. (Main): $X \not\supseteq \ell_1$ $\Longrightarrow$ X is SCD,
4. $C[0,1]$ and $L_1[0,1]$ are not SCD.

SCD sets: Proving the elementary examples I

Example

A separable and $A = \overline{\text{conv}}(\text{dent}(A)) \implies A$ is SCD.

Proof.

- Take $\{a_n : n \in \mathbb{N}\}$ denting points with $A = \overline{\text{conv}}\big(\{a_n : n \in \mathbb{N}\}\big)$.
- For every $n, m \in \mathbb{N}$, take a slice $S_{n,m}$ containing a_n and of diameter $1/m$.
- If $B \cap S_{n,m} \neq \emptyset \implies a_n \in \overline{B}$.
- Therefore, $A = \overline{\text{conv}}\big(\{a_n : n \in \mathbb{N}\}\big) \subseteq \overline{\text{conv}}(\overline{B}) = \overline{\text{conv}}(B)$.

Example

In particular, A RNP separable $\implies A$ SCD.

Corollary

- If X is separable LUR $\implies B_X$ is SCD.
- So, every separable space can be renormed such that $B_{(X,|\cdot|)}$ is SCD.

SCD sets: proving the elementary examples II

Example

If X^* is separable $\Longrightarrow$ A is SCD.

Proof.

- Take $\{x_n^* \colon n \in \mathbb{N}\}$ dense in S_{X^*}.
- For every $n, m \in \mathbb{N}$, consider $S_{n,m} = \operatorname{Slice}(A, x_n^*, 1/m)$.
- It is easy to show that any slice of A contains one of the $S_{n,m}$

SCD sets: proving the main example I

Convex combination of slices

$W = \sum_{k=1}^{m} \lambda_k S_k \subset A$ where $\lambda_k \geqslant 0$, $\sum \lambda_k = 1$, S_k slices.

Proposition

In the definition of SCD we can use a sequence $\{S_n : n \in \mathbb{N}\}$ of convex combination of slices.

Bourgain's lemma

Every relative weak open subset of A contains a convex combination of slices.

Corollary

In the definition of SCD we can use a sequence of relative weak open subsets: the set A is SCD if there is a sequence $\{V_n : n \in \mathbb{N}\}$ of relative weak open subsets of A such that every slice of A contains one of the V_n's.

SCD sets: proving the main example II

Corollary

In the definition of SCD we can use a sequence of relative weak open subsets: the set A is SCD if there is a sequence $\{V_n : n \in \mathbb{N}\}$ of relative weak open subsets of A such that every slice of A contains one of the V_n's.

π-bases

A π-base of the weak topology of A is a family $\{V_i : i \in I\}$ of weak open sets of A such that every weak open subset of A contains one of the V_i's.

Proposition

If $(A, \sigma(X, X^*))$ has a countable π-base $\Longrightarrow$ A is SCD.

SCD sets: proving the main example III

Theorem

A separable without ℓ_1-sequences $\Longrightarrow$ $(A, \sigma(X, X^*))$ has a countable π-base.

Proof.

- We see $(A, \sigma(X, X^*)) \subset C(T)$ where $T = (B_{X^*}, \sigma(X^*, X))$.
- By Rosenthal ℓ_1 theorem, $(A, \sigma(X, X^*))$ is a relatively compact subset of the space of first Baire class functions on T.
- By a result of Todorčević, $(A, \sigma(X, X^*))$ has a σ-disjoint π-base.
- $\{V_i : i \in I\}$ is *σ-disjoint* if $I = \bigcup_{n\in\mathbb{N}} I_n$ and each $\{V_i : i \in I_n\}$ is pairwise disjoint.
- A σ-disjoint family of open subsets in a separable space is countable. ✓

Main example

A separable without ℓ_1-sequences $\Longrightarrow$ A is SCD.

SCD is a link between ADP and lushness

Theorem

X (separable) SCD,

$$n(X) = 1 \text{ (actually ADP)} \quad \Longrightarrow \quad X \text{ lush.}$$

Main consequence

X (arbitrary) such that $X \not\supseteq \ell_1$,

$$n(X) = 1 \text{ (actually ADP)} \quad \Longrightarrow \quad X \text{ lush.}$$

Corollary

X real $+ \dim(X) = \infty +$ ADP $\Longrightarrow X^* \supseteq \ell_1$.

Proof.

- If $X \supseteq \ell_1 \implies X^*$ contains ℓ_∞ as a quotient, so X^* contains ℓ_1 as a quotient, and the lifting property gives $X^* \supseteq \ell_1$ ✓
- If $X \not\supseteq \ell_1 \implies X$ is SCD + ADP, so X is lush.
- Lush $+ \dim(X) = \infty \implies X^* \supseteq \ell_1$ ✓